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A novel interpretable model of bathtub hazard rate based on system hierarchy

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ABSTRACT

The degradation of a functioning system can be characterized using the hazard rate functions (HRFs) in the context of reliability theory. The bathtub hazard rate (BHR) is a widely-seen form of HRF for many practical systems when considering the system as a whole. Many previous studies attempt to propose different predictable models at system level for BHRs. However, since these models are usually not interpretable, they can only deal with the system-level lifetime data. If the data could not bring explicit HRF shape due to the limitation of its amount, then it is hard to select a proper model. To offset the shortcoming of system-level models, we study the underlying mechanism in a multiple-component scenario and provide a model based on hierarchy–structure systems to infer the appearance of BHR via their structure induced failure modes. The novel model is interpretable and renders a steady BHR at system level which could be a theoretical support for the BHR in reliability engineering. The proposed model provides several applications in different practical conditions, such as the prediction of the HRF's shape with only information of structure, inference of structure with lifetime data and fusing information from different levels.

1. Introduction

The hazard rate function (HRF) is one of the most fundamental means to characterize the degradation of functioning systems in the reliability theory [1], allowing for predicting the risk of failure of a functioning system. The BHR has been used to describe the degradation behaviors of many systems [2–4]. A standard BHR possess three periods: early failure, random failure and wear-out period. The earliest BHR is reported in 1693 [5]. Later, BHRs appear in many classic reliability analysis texts. As follows the literature on BHR is briefly reviewed, and one could find a more exhaustive survey about BHR in Refs. [6–8].

A number of previous studies construct new families of distribution function. For example, Refs. [9,10] proposed the mixture and extensions of Weibull distributions to construct BHRs, which is recently generalized by Ref. [11]; Ref. [12] proposed log-normal modified Weibull distribution to model different shaped HRFs including BHRs. Apart from these studies, the idea for constructing parametric distribution functions that render BHRs can be also found in Refs. [13–19]. Ref. [20] presents the application of the fuzzy approach for BHR and estimate the maintenance costs. Ref. [21] shows that the BHRs could also be obtained with applying maximum entropy principle to whole of the system if two or more accessed moments fall in a proper region. Ref. [22] analyze the data in the wear-out phase with q-Exponential likelihood. Additionally, the HRF can also be modeled based on some other mechanisms, e.g., Refs. [23,24] present two different phenomenological degrading model at system level to render BHRs.

However, due to the lack of support by a rational and rigorous modeling of the underlying mechanism in microscopic levels, the BHRs are largely regarded as an empirical law, causing several criticism and argument over its justification [6,25,26]. On one hand, the parameterized-distribution models are efficient and reliable if the priori information contains the shape of HRF. If the system-level information is limited to construct a precise and convinced shape for HRF, then it is hard to select the proper failure modes to be mixed. On the other hand, the mechanics based models usually require the knowledge on physical parameters that depends on the types of system, such as temperature, pressure, voltage and so on. Since complex systems usually involve hybrid components, to construct a mechanics based model is generally difficult. This brings a question that how can we predict the shape of the system-level HRF of complex system (such as BHR which is widely concerned), when the lifetime data is limited and mechanics based physical model is hard to construct.

In this work, we narrow this question into a smaller one, i.e., make prediction for the appearance of BHR via the systems' structure function for the system of systems. That is analyzing the degradations

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Notations

L	Number of the levels.
$\sigma^{(k)}_{(i_L,i_{L-1},i_{L-2},\ldots,i_{k+1},i_k)}$	State of the element at the kth level with the location $(i_L, i_{L-1}, i_{L-2}, \ldots, i_{k+1}, i_k)$.
Σ	State of the system.
$X(t)$	Hazard rate function.
$\tilde{X}$	Hazard rate function with finite time step.
$\phi^{(k)}$	Structure function of each element at the kth level.
$n^{(k)}$	Number of the elements at the kth level
$f[\phi^{(k)};\cdot] \equiv f_{\phi^{(k)}}(\cdot)$	Survival probability of each element at the kth level as a function of that of the $k-1$th level.
$R^{(k)}$	Survival probability of the elements at the kth level.
r	Component-level survival probability.
s	Characterized sequence of a deterministic structure.
R_s	System-level survival probability corresponding to the characterized sequence s.
$\psi(s)$	Probability of the sequence s.

of the complex systems with levels "hierarchy", namely the systems involve multiple components in different hierarchical levels following the corresponding structure functions. The core idea here is to consider the system-level degradation (the degradation of the system as a whole) as the effect of *emergence*, which is one of the two fundamental paradigms in the modern physics while the other is *reductionism*. The two paradigms from physics have not been emphasized in some recent studies of the system reliability. In the terminology of reliability theory, reductionism is closely associated with the concept of modular decomposition. And it means that a system can be decomposed into a number of subsystems, and a subsystem can be further decomposed into a number of sub-subsystems and so on until down to the fundamental elements, which are called components and are considered as the basic elements. In such a treatment, these systems exhibit hierarchical structures when viewed at different levels. The reliability engineering for such a system has attracted increasing attention in recent years, e.g., optimal problem for hierarchical system [27] the redundancy-allocation problem for multi-level system [28,29], risk assessment of system of systems [30], the related repair optimization problems [31, 32], the multi-layer urban rail transit network [33] and so on. For such a complex system, we show in the proceeding proof that the system-level (macroscopic) behavior is less sensitive to the component-level (microscopic) behavior. In other word, the system-level degradation depends hardly on the details of the component-level degradation; consequently, it is an emergent behavior of the complex structure. One typical example is the percolation with a classic model widely studied in statistical physics [34]. It was recently developed to analyze the network reliability [35,36] since the connectivity of complex networks appears if the probability of edges is greater than some critical values, which is called percolations. Another example regarding the reliability of complex systems is presented in Ref. [30], where the central limit theorem plays an important role. The central limit theorem claims that the collection of the sufficient large numbers of identical independent random variables converges to a normal distribution, which is independent of the identical distribution for each variable.

The effect of hierarchy invokes a novel model that possesses the advantages of both parametric-distribution based models and mechanics based models. In our proposed model, the BHR is rendered mainly by structure functions. The system-level hazard rate is formulated according to the structure functions at different levels in the system hierarchy. In the current study, the hierarchy is loosely referred to the fact that the system can be decomposed into subsystems iteratively down to the basic elements which cannot be further decomposed. This type of system is called the system of systems and such structure could be rendered by, for example, the modular decomposition. Each element has two possible states, namely, the failed state and the survival state. If the state of an element at a level depends on those of the lower-level elements, then each of the lower-level elements is said to belong to the higher-level element. Mathematically this dependence is described by the structure function, i.e., the Boolean function. Since the number of structure functions involved at all levels is usually prohibitively large for the direct calculation of the hazard rate in the system-level, typical types of structure functions with the following specific features are chosen for the convention: (1) There is no intersection for the elements at the same level. The systems satisfy this condition are called "the system of systems", which is same with those considered in Ref. [30]; (2) The structure functions of the elements at same level are identical; and (3) The states of components are assumed to be identically independent.

It shows that, the hierarchical model brings many differences compared to the system-level models. It can be used to fuse the information from different levels and make inference from components to system. The structure and the lifetime data are closely linked by the model. These bring the direct application on all-level reliability analysis. All the parameters in the model has physical meanings, thus the model is interpretable. There are more extensions of the model, e.g., model other type HRFs, predict the HRFs of hierarchical structure functions and so on. In addition, the emergent hazard rate of a complex system renders a limited number of shapes depending on the structure and its uncertainty. Furthermore, the BHR is universal in the sense of a appropriate type of structures. These results show that the proposed model goes beyond the conventional mixture based methods [13]. Additionally, the real data application shows that the Akaike Information Criterion (AIC) value for this model is close to that of the recent high-performance system-level models [12,15,17,37].

This paper is organized as follows. In Section 2, the literatures of BHRs are briefly reviewed. In Section 3, the mathematical model of the systems is built. It is shown that the structure function is sorted into three types, which dominate the key feature of the system-level degradation. Sections 4–6 show different examples for the system structure which render BHRs. A sufficient condition of mixed structure functions for BHR is obtained. Section 7 present the real data application. Finally discussions and conclusions are presented.

2. Model

2.1. Relevant definitions and basic notations

To build the model, some relevant definitions and basic notations are reviewed. We consider a system involving multiple components. The system-level state Σ can be expressed using the states of the constituent components via the structure function of the system as,

$$\Sigma = \phi(\sigma_1, \sigma_2, \sigma_3, \ldots, \sigma_n), \tag{1}$$

where σ_i denotes the state of the ith component and the function ϕ is called the structure function. Specifically, the state of the system and the components has two possible states $\{0,1\}$; therefore, the structure function $\phi : \{0,1\}^n \to \{0,1\}$ is a Boolean function, where $\sigma_i = 1$ denotes that the ith component is in the survival state and $\sigma_i = 0$ represents the failed state. For binary systems, the structure functions is the Boolean functions. For general multi-state system (MSS) [38], the structure function is extended from Boolean algebra to the multi-valued cases [39]. This study is focused on the monotonically increasing structure function which satisfies $\phi(0,0,0,\ldots,0) = 0$, $\phi(1,1,1,\ldots,1) = 1$. The monotonically increasing condition is given such that $\forall i$ if $\sigma_i \le \sigma_i'$, then $\phi(\sigma_1, \sigma_2, \ldots) \le \phi(\sigma_1', \sigma_2', \ldots)$. The monotonic feature implies that a lower reliability of the components always results in a lower reliability of the system.

Suppose that the states of the individual components $\sigma_i, i = 1, 2, \ldots$ are independent and identically distributed random variables with $\Pr(\sigma_i = 1) = r(t)$. The system-level survival probability R is the average value of Σ, namely

$$R \equiv \langle \phi(\sigma_1, \sigma_2, \ldots, \sigma_n) \rangle = f[\phi; r], \tag{2}$$

where the notation $\langle \cdots \rangle$ means the average of each the possible states of all the components. The HRF, $X(t)$, of the system is closely associated with $f[\phi; r]$ by,

$$X(t) = -\frac{\mathrm{d} \ln f[\phi; r]}{\mathrm{d}r} \times \frac{\mathrm{d}r}{\mathrm{d}t}. \tag{3}$$

Eq. (3) shows that the system-level HRF is the product of two parts. The first part $\mathrm{d}f[\phi; r]/\mathrm{d}r$ depends on the structure function, and the second one depends on the time evolution of the survival probability of the components. The second part, $\mathrm{d}r/\mathrm{d}t$, is considered as the detailed information of the component-level degradation in the current study. It will be shown in the later sections that for a hierarchical system the first part dominate the shape of the HRF due to the complexity of the structure, leading to the emergence of the system-level degradation behavior.

2.2. Classification of increasing structure functions

The properties for the function $f[\phi; r]$ with an increasing ϕ are studied as follows. Since $\phi(0, 0, 0, \ldots, 0) = 0$, $\phi(1, 1, 1, \ldots, 1) = 1$, the structure function ϕ is increasing, one has $f[\phi; 1] = 1, f[\phi; 0] = 0$ and $\mathrm{d}f[\phi; r]/\mathrm{d}r \ge 0$.

In the current study, coherent structure functions [29,40] are considered. The coherent structure function is defined as an increasing structure function whose states rely on all the components, i.e., $\forall i, \exists (\sigma_1, \sigma_2, \ldots, \sigma_{i-1}, \sigma_{i+1}, \sigma_{i+2}, \ldots)$ such that $\phi(\sigma_1, \sigma_2, \ldots) = \sigma_i$. For a coherent structure function ϕ, there exist a number of non-negative integers $a_{\phi,i}, i = 1, 2, \ldots, n$ such that

$$f[\phi; r] = \sum_{i=0}^{n} a_{\phi,i} (1-r)^{n-i} r^i, \tag{4}$$

with $a_{\phi,n} = 1, a_{\phi,0} = 0$. Here, the coefficients $a_{\phi,i}$ is associated with the survival signature [41–43]. Namely, the survival signature is defined as $a_{\phi,i}/C_n^i$, where C_n^i is the binomial coefficient [42].

The coherent structure function could be divided into the three types as illustrated in Fig. 1. The three types are given as follows:

Type I, $f[\phi; r] < r$ with $\forall r, 0 < r < 1$;

Type II, $f[\phi; r] > r$ with $\forall r, 0 < r < 1$; and

Type III, the equation $f[\phi; r] = r$ has one solution $r = r_*$ with $0 < r_* < 1$ and $f[\phi; r] > r$ for $r > r_*$, $f[\phi; r] < r$ for $r < r_*$.

It is found that (1) for any coherent structure function ϕ in Type I and ϕ' in Type II, $f[\phi; f[\phi', r]] = r$ has one solution $r = r_*$ with $0 < r_* < 1$, and thus the structure function $\tilde{\phi}$ with $f[\tilde{\phi}; r] \equiv f[\phi; f[\phi', r]]$ is in Type III. A similar result hold true for the function $f[\phi'; f[\phi, r]]$, and (2) for any coherent function ϕ in Type III and a structure function ϕ' in an arbitrary type, the structure function $\tilde{\phi}$ with $f[\tilde{\phi}; r] \equiv f[\phi; f[\phi', r]]$ and $f[\tilde{\phi}; r] \equiv f[\phi'; f[\phi, r]]$ are also in Type III. These three types of structure functions in a hierarchical system can lead to drastically different system-level degradation behaviors.

In the main body of text, we briefly present the idea of proof to explain why there are only tree types and the complete proof is presented in Appendix A. Firstly, it is clear that the line $f[\phi; r] = r$ corresponds to a kind of structure functions that the output only depend on one of inputs and this kind of structure functions is increasing but not coherent by the definition. We label such structure function with ϕ_0 and without loss of generality, we set $\phi_0(\sigma_1, \sigma_2, \ldots, \sigma_n) = \sigma_1$. Secondly, each increasing structure function ϕ could mapping to a simplicial complex (the definition is given in Ref. [44]), which contains a number of k-dimensional simplexes ($k = 1, 2, \ldots, n$). The number of k-dimensional simplexes is just the value of $a_{\phi,n-k}$. Thirdly, according to the basic property of simplicial complexes, any two involving $k+1$-dimensional simplexes could intersect with at most one k-dimensional simplex. The single-input-dependent structure function ϕ_0 is a very special case. Any two of its $i+1$-dimensional simplexes intersect with one i-dimensional simplexes. This implies that compared with ϕ_0, any other increasing structure functions must satisfy the condition as follows: if the number of $k+1$-dimensional simplexes for its corresponding simplicial complex is greater than that for ϕ_0, then the number of k-dimensional simplexes is also greater than that for ϕ_0. Finally, with this condition, one could find there are only three situations for coherent structure functions: situation I: for all k, the number of k-dimensional simplexes for its corresponding simplicial complex is greater than that for ϕ_0; situation II: for all k, the number of k-dimensional simplexes for its corresponding simplicial complex is less than that for ϕ_0; situation III: for $n > k > k_c \ge 1$, the number of k-dimensional simplexes for its corresponding simplicial complex is less than that for ϕ_0. A straightforward analysis shows that these three situations are corresponding to the three types, respectively.

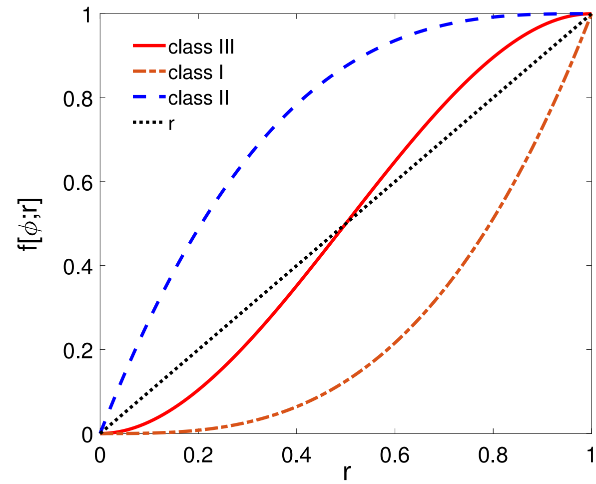


Fig. 1. Illustration of three types of the coherent structure function.

2.3. The hierarchical model

In the current paper, the system with hierarchical structure is mainly considered. As the modular composition could be applied to many realistic systems. The structure functions of these systems are with hierarchy. It will show that the hierarchy could dominate the system-level HRF's shape in many specific cases. This property is significant and it is helpful for reduce information requirement for the estimation on HRF in reliability engineering.

For a system with $L+1$ levels, let $k = 0, 1, 2, \ldots, L$ be the labels of the levels. The basic component (which cannot be further decomposed) level is $k = 0$, and the system level is $k = L$. It can be seen that the quantity L determines the number of the levels of the system, and L is called the system size in this paper.

With the three features for the systems listed in the introduction section, it is convenient to define the location of each element at each level. For example, the location of the system is $i_L = 1$; the location of the i_{L-1}th subsystem at level $L-1$ is $(1, i_{L-1})$; and the location for the i_{L-2}th sub-subsystem belonging to the i_{L-1}th subsystem is $(1, i_{L-1}, i_{L-2})$, and so on until the basic component level, as illustrated in Fig. 2.

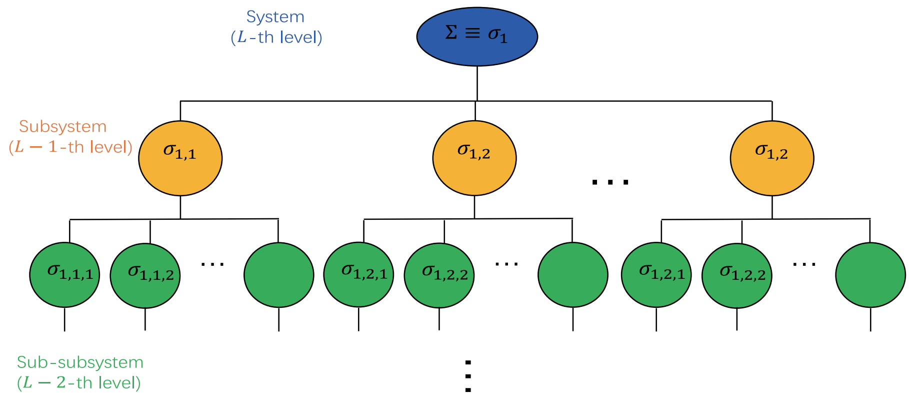


Fig. 2. Schematic representation of a hierarchical model studied in this work. Each node represents an element in the model. The link between a low-level element with a high-level element represents the low-level element is a constituent part of the high-level element.

The states of the component at the $(k+1)$th level depend on those of the components at the kth level, i.e., $\forall i \in \{1,2,3,\ldots,n^{(k+1)}\}$

$$\sigma_{(i_L,i_{L-1},\ldots,i_{k+1})} = \phi^{(k)}\left(\sigma^{(k)}_{(i_L,i_{L-1},\ldots,i_{k+1},1)}, \sigma^{(k)}_{(i_L,i_{L-1},\ldots,i_{k+1},2)}, \sigma^{(k)}_{(i_L,i_{L-1},\ldots,i_{k+1},3)}, \cdots\right). \tag{5}$$

Since we consider the elements at the same level are identical, combining Eq. (2) with Eq. (5) to obtain the survival probability of the $(k+1)$th level as

$$R^{(k+1)} = f[\phi^{(k+1)}; R^{(k)}]. \tag{6}$$

Let $\circ$ be the function composition, which is explicitly defined as $f \circ g(x) \equiv f(g(x))$. With the symbolic simplification $f[\phi;\cdot] \equiv f_\phi(\cdot)$, the system-level survival probability is

$$R^{(L)} = f_{\phi^{(L)}} \circ f_{\phi^{(L-1)}} \circ f_{\phi^{(L-2)}} \circ \cdots \circ f_{\phi^{(1)}}(r), \tag{7}$$

and generally, a k-level element's survival probability is

$$R^{(k)} = f_{\phi^{(k)}} \circ f_{\phi^{(k-1)}} \circ f_{\phi^{(k-2)}} \circ \cdots \circ f_{\phi^{(1)}}(r). \tag{8}$$

2.4. Probabilistic structure function

The aforementioned discussions have not incorporated the uncertainty of the structure function. The structure function at each level interprets the logical dependencies between the elements. In practice uncertainties of the environment could influence structure functions, such as the uncertainty lead by manufacturing process, the uncertainty of the hardly monitored components, the error caused by the finite number of levels, mixture of multiple tasks, etc. All these may lead to break down of a deterministic structure function, thus instead of using the deterministic structure function, the probabilistic structure function is a reasonable extension to model a more kind of realistic systems. The similar consideration was also presented in some of previous studies, such as Ref. [41] where the structure function is extended to $\bar{\phi}$: $\{0,1\}^n \to [0,1]$. In this subsection, we consider a general scheme to define a probabilistic structure function as follows. For a system with uncertainty of both the components' states and the structure function, the system-level reliability R could be defined with

$$R = \sum_\phi p_\phi \langle \phi(\sigma_1,\sigma_2,\sigma_3,\ldots)\rangle, \tag{9}$$

where $\langle\cdots\rangle$ denote the average over all the components' possible states, and p_ϕ is the probability that the structure function takes ϕ. For example, for a double-component system, the coherent structure function could be either series or parallel, whose probability are p_{series} and p_{parallel}, then the system-level reliability R is

$$R = p_{\text{series}}\langle\sigma_1\sigma_2\rangle + p_{\text{parallel}}\langle[1-(1-\sigma_1)(1-\sigma_2)]\rangle. \tag{10}$$

If the components' states obey independent identical distribution (whose reliability is r), Eq. (9) can be simplified as

$$R = \sum_\phi p_\phi f[\phi; r]. \tag{11}$$

The definition in (9) is equivalent to the conditional probability of system state with given components' states. This is usually considered in dynamical Bayesian network as the conditional probability table [45–47].

In the current paper, several kinds of specific cases will be studied to obtain a emergent BHR.

3. Self-similarity structure as function composition

In this section, the self-similarity structure function as the simplest case of Eq. (8) is studied. The self-similarity of the structure function at each level means that $\forall i, \phi^{(i)} = \phi$. It follows that

$$R^{(L)}(r) = f_\phi \circ f_\phi \circ f_\phi \circ \cdots \circ f_\phi(r). \tag{12}$$

3.1. Pure (deterministic) self-similarity structure

Suppose that ϕ is in Type III. Fig. 3(a) presents the resulting $R^{(L)}(r)$ with different Ls, and Fig. 3(b) presents the function $-\mathrm{d}\ln R^{(L)}(r)/\mathrm{d}r$. Fig. 3(c) and 3(d) present system-level HRFs with the various system size L under different component-level HRFs $\alpha = 1/5$ and $\alpha = 1/2$, respectively. Here, α is the parameter in Weibull distribution $r(t) = \exp[-(\kappa t)^\alpha]$. Note that $-\mathrm{d}\ln R^{(L)}(r)/\mathrm{d}r$ can be also regarded as the HRF for $r(t) = 1-t$. The condition $r(t) = 1-t$ implies that the component-level HRF is $1/(1-t)$, which is monotonically increasing.

Fig. 3(a) shows that for $L \gg 1$, the system-level survival probability (i.e. the system-level reliability) decrease sharply at a specific component-level reliability. In fact, it has been already pointed out in the classic reliability theory textbook [40] that the repeated composition of the function $f[\phi; r]$ in Type III should lead to a step function. This phenomenon is called percolation. By comparing Fig. 3(c) with 3(d), it is observed that as the component-level HRFs (curves associated with $L=1$) changes from $\alpha = 1/5$ to $\alpha = 1/2$, the HRFs of the $L=2$ system behave drastically different; however, the HRFs of the systems with a sufficient large size ($L=8$ in this numerical case) with different α have similar behaviors. It implies that as system size L increases the shapes of the system-level HRFs become less sensitive to the HRFs of individual components.

In the numerical example Type III is chosen. In general, the shape of HRFs for self-similarity-structure system is dominated by the type of the structure function at each level, since a fast decreasing of the system-level reliability occurs at the unstable fix point $r = r_*$ with $f[\phi, r_*] = r_*$ and the value of the solution is closely related to the type of the structure function: for Type I $r_* = 1$; for Type II $r_* = 0$; for Type III $0 < r_* < 1$.

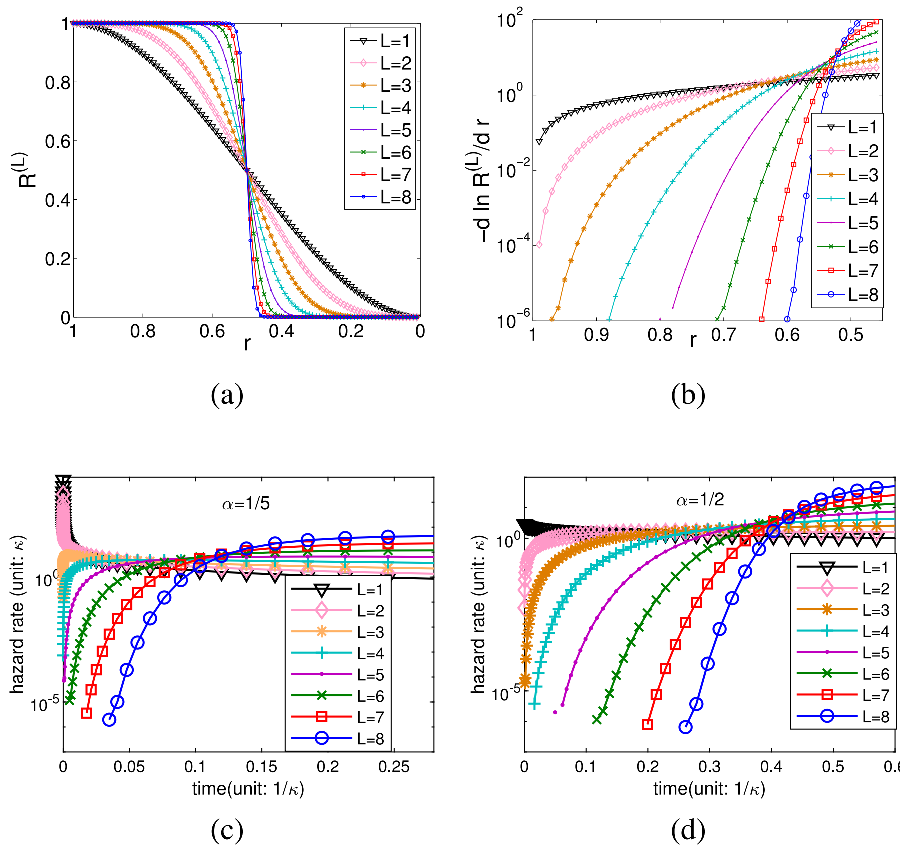


Fig. 3. (a) Presents $R^{(L)}(r)$ with different Ls, Fig. 3(b) presents the function $\mathrm{d}\ln R^{(L)}(r)/\mathrm{d}r$, Fig. 3(c) (d) presents system-level HRFs for different the component-level degradation law. The components' lifetimes are modeled by Weibull distribution $r(t) = \exp[-(\kappa t)^{\alpha}]$, where κ is the parameter to characterize the arbitrary decay rate of $r(t)$. At each level the structure is set to be the 2-out-of-3:G system [48], is a specific structure function in Type III.

3.2. Mixed self-similarity structure

The results above show that the self-similar structure only with the structure function of Type III usually leads to a monotonically increasing HRF. The similar analysis could also be performed to the structure function of Type I and Type II. A monotonically decreasing HRF can be obtained for Type I and a monotonically increasing HRF for Type II. The resulting shapes of the HRFs are shown to be less sensitive to those of the components for a large L. It can be inferred that the BHR could be obtained with a "mixture" of the structure functions of Type I and Type III.

To demonstrate this, we consider the mixed survival probability $f(r) \equiv \sum_{\phi} p_{\phi} f[\phi; r]$ follows the *sufficient condition*

Condition 1. *$f(r) < r$ for $r_* < r < 1$, $f(r) > r$ for $r'_* < r < r_*$ and $f(r) < r$ for $0 < r < r'_*$.*

Clearly, this condition cannot be achieved if the structure function is deterministic since it does not belong to any types.

Proper mixtures of Type I and Type III satisfy Condition 1. For example, consider the mixture of 5-out-of-5:G, 3-out-of-5:G and 2-out-of-5:G, whose probability is $1 - p_1 - p_2$, p_1 and p_2, respectively. The necessary condition for proper mixtures is $p_1 + p_2 < 4/5$. Fig. 4(a) present the image of $p_1 + p_2 = 3/4$ and $p_2 = 2/5$ and explicitly shows condition is satisfied.

The hazard rate generated by the system-level reliability $R = f \circ f \circ \cdots \circ f(r)$ becomes a BHR if the number of the levels is large.

Since the initial and the final values of $-\mathrm{d}\ln R/\mathrm{d}t$ is divergent as $L \to \infty$, the observable HRF need to be carefully dealt with. Here, the initial value means $-\mathrm{d}\ln R/\mathrm{d}t|_{t\to 0}$ and the final value means $-\mathrm{d}\ln R/\mathrm{d}t|_{t=t_*}$, where $r(t_*)$ is the smallest fix point that is unstable. For $L \to \infty$, due to the sudden decreasing of system reliability $R(t)$, these two values both go infinite. This divergence is caused by infinite large L and infinitesimal time step $\mathrm{d}t$. These two conditions cannot be achieved in practice, so we consider a finite time step and finite Ls. For finite time step Δt the observable HRF $\tilde{X}$ is simply defined as

$$\tilde{X}(t) = -\frac{1}{R(t)} \frac{R(t+\Delta t) - R(t)}{\Delta t}. \tag{13}$$

Note that the observable HRF recovers the idealized HRF at the limiting state of $\Delta t \to 0$.

Fig. 4(b) and (c) show the hazard rate of the systems with different number of levels L and with the increasing of L, the HRF become bathtub. And compare Fig. 4(b) with (c), one can see that a larger L lead to more steady of the HRF's shape. For example, for $L = 2$, with the increasing of α, the HRF changes from approximately decreasing (exactly a bathtub with very flat wear-out curve) to approximately increasing, while for $L = 8$, different values of α both render the BHRs. This is the emergence of BHRs.

As it is shown by Eq. (3), the system-level HRF relies on both the structure-dependent part $\mathrm{d}\ln R/\mathrm{d}r$ and the component-dependent part $-\mathrm{d}r/\mathrm{d}t$. For large L, the structure-dependent part is dominant. To show this, recall the relation $R = f \circ f \circ \cdots \circ f(r)$, where $f(r)$ is the function presented in Fig. 4(a) and the number of compositions is L. There are totally four fix points, i.e., the horizontal coordinates of the intersection points between the solid line and the dash line. We label them as $r = 0, r = r_*, r = r'_*, r = 1$, where $r_* < r'_*$. It found that $r = r_*, r = 1$ are unstable. To see this, taking $r = r_*$ as an example, the small deviation between r and r_* are amplified by the mapping composition $f \circ f \circ \cdots \circ f(r)$, thus $|R - r_*| > |r - r_*|$. The points $r = 0$ and $, r = r'_*$ are stable, because the mapping composition leads to smaller deviation. In fact, for $L \to \infty$, R becomes a piecewise function, which could be

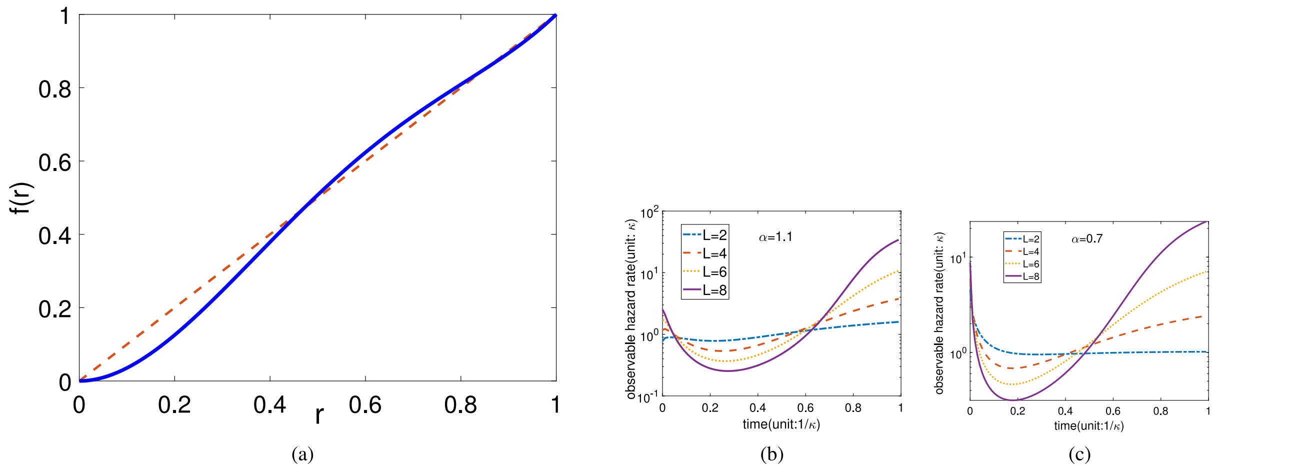


Fig. 4. (a) The blue solid is the $f(r)$ for the mixture of 1-out-of-3:G and 2-out-of-3:G with $p_1 + p_2 = 3/4$ and $p_2 = 2/5$; the red dash line is $f(r) = r$. (b) (c) The hazard rate function for such a hierarchical system. The components' lifetimes are modeled by Weibull distribution $r(t) = \exp[-(\kappa t)^\alpha]$, where κ is the parameter to characterize the arbitrary decay rate of $r(t)$.

written as

$$\lim_{L\to\infty} R(r) = \begin{cases} 1 & r = 1 \\ r'_* & r_* < r < 1 \\ r_* & r = r_* \\ 0 & 0 \le r < r_*. \end{cases} \tag{14}$$

It follows from the above equation that the structure-dependent part $\mathrm{d}\ln R/\mathrm{d}r$ divergence at $t = 0$ and $t = r^{-1}(r_*)$ where r^{-1} denotes the inverse function of $r(t)$, and for other value of t, $\mathrm{d}\ln R/\mathrm{d}r$ equals to zero. Therefore, if the component-dependent part $-\mathrm{d}r/\mathrm{d}t$ is nonzero for any t, a large L always renders BHR.

If the component-dependent part $-\mathrm{d}r/\mathrm{d}t$ become zero for $t = 0$, the shape of HRF could be analyzed as follows. For convenience, let the derivation of $f(r)$ at $r = 1$ be denoted by $a \equiv f'(1)$. For large but finite L and finite Δt, the observable HRF's initial value $\tilde{X}(0) \sim a^L(\kappa\Delta t)^{\alpha-1}$ for $a^L(\kappa\Delta t)^\alpha \ll 1$ and $\tilde{X}(0) \sim (\kappa\Delta t)^{-1}$ for $a^L(\kappa\Delta t)^\alpha \approx 1$. Thus, there exist a region for the parameters that the HRF changes from bathtub (the latter case) to an approximately increasing function (the former case) due to the vanishing of the HRF's initial value when $\alpha > 1$. If given time step Δt, one can evaluated that the BHR appears when $\alpha < 1 + L|\ln a/\ln \kappa\Delta t|$. Note that the same analysis can be extended to deal with a vanishing component-dependent part for another fix point $t = r^{-1}(r_*)$. This evaluation directly shows that a larger L renders a more steady BHR.

The theoretical analysis shows the low sensitivity of the system-level HRF due to the hierarchy of the system of systems. This property could be applied to the practice that if the component-level datum is limited to access or the amount of system-level lifetime datum is not enough to determine the HRF precisely, the information of structure function, such as fault tree, block diagram, could be applied to the systems with hierarchy to infer the shape of the system-level HRF. This implies that the information of structure function could be used to alleviate the difficulties brings by the lack of lifetime data. These difficulties are common in reliability engineering. Additionally, the mixture of structure function is relevant to that the information about some of the components are missing or the system involving other inaccessible uncertain inputs, such as the environment. Thus, a mixed structure function could be more efficient to model a realistic system that with these properties in reliability engineering.

4. An example of two mixed structures render BHR

BHRs can emergent in more other mixed structures that need not to be self-similar. And a proper mixture for two-type structures is enough. In this section, one kind of hierarchical systems consisted of the structures of two types is studied. Let the two structure functions ϕ and ϕ' belong to Type I and Type III, respectively.

The system-level survival probability is given as

$$\begin{aligned} R_{\mathbf{s}}(r) = &\overbrace{f_\phi\circ f_\phi\circ\cdots f_\phi}^{s_n}\circ f_{\phi'}\circ\overbrace{f_\phi\circ f_\phi\circ\cdots f_\phi}^{s_{n-1}}\circ f_{\phi'}\circ \\ &f_\phi\circ\cdots f_\phi\circ f_{\phi'}\circ\overbrace{f_\phi\circ f_\phi\circ\cdots f_\phi}^{s_k}\circ f_{\phi'}\circ\cdots \\ &f_{\phi'}\circ\overbrace{f_\phi\circ f_\phi\circ\cdots f_\phi}^{s_0}(r), \end{aligned} \tag{15}$$

where $s_0, s_1, \ldots, s_n$ denote the number of f_ϕs in each interval of the two adjacent $f_{\phi'}$s. Let the sequence $(s_0, s_1, \ldots, s_n)$ denote a specific deterministic structure.

4.1. The uncertain structures and BHR

In this model, consider the structure functions of the elements at level l have a probability $h(l/L)$ in Type I and $1 - h(l/L)$ in Type III, where $h(x)$ is assumed to be monotonically increasing with $h(1) = 1$. Note that the normalized quantity l/L is introduced for convenience to perform further calculation without introducing any further assumption. We consider that all the elements at a same level are with a same structure function. For the element at the lth level ($l = 0, 1, \ldots, L$), the structure function could be either 3-out-of −3:G (Type I) or 2-out-of −3:G (Type III), which is randomly chosen with the probability $h(l/L)$ and $1 - h(l/L)$, respectively. And the probabilities for different levels are independent. This constructs a probability space of the random structure function for the system with totally L levels. It follows that the survival probability is the average of the random instances of the specific sequences,

$$R(r) = \sum_{\mathbf{s}} \psi(\mathbf{s}) R_{\mathbf{s}}(r), \tag{16}$$

where $\psi(\mathbf{s})$ is the probability for the sequence $\mathbf{s}$, reads

$$\begin{aligned} \psi(\mathbf{s}) = &\prod_{j=0}^{n-1}\left\{\left[1 - h\left(\frac{j + \sum_{k=0}^{j} s_k}{L}\right)\right] \prod_{i_j = j + \sum_{k=0}^{j-1} s_k}^{j + \sum_{k=0}^{j} s_k - 1} h\left(\frac{i_j}{L}\right)\right\} \\ &\times \prod_{i=L-s_n+1}^{L} h\left(\frac{i}{L}\right) \end{aligned} \tag{17}$$

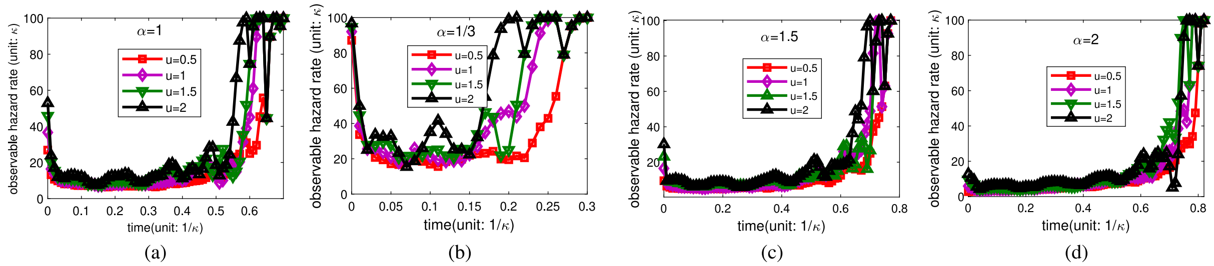


Fig. 5. The observable HRF with $\Delta t = 0.01/\kappa$, $r = \exp[-(\kappa t)^{\alpha}]$, and $L = 12$. BHRs appear when α is not large.

For example, we consider a non-ideal series system described by the model. Since any series structures can be naturally decomposed into a number of levels, if all the structure functions in Eq. (15) are in type I, there is no redundancy at each level and the system become an ideal series structure. The structural uncertainty, that is the mixture with the type III structure, brings redundancy. Explicitly, we consider the structural uncertainty of higher levels is smaller than that of lower levels.

The numerical simulation results of the HRF using the above setup of the level-dependent probability are presented in Fig. 5, where the specific condition $h(x) = 1 - (1-x)^{u}/2$ is chosen. In the simulation the structure functions ϕ and ϕ' are set to be 3-out-of-3:G and 2-out-of-3:G, respectively.

It shows that with the increasing of α, the shapes of the observable HRFs gradually change from the bathtub shape to an monotonically increasing one. Since α is the parameter to characterize the component-level degradation, the results further emphasize that the shape of the system-level HRF is less sensitive to that of the components.

4.2. Early-stage degradation analysis

Same as the self-similar case, we study the region for the parameters that BHRs hold or change to other shapes via the analysis of the degradation at early stage. Explicitly, the relation among the initial observable HRF, the size of the system L, the observation time interval Δt, and component degradation parameter α is studied.

For any Type I structure function ϕ, $a_{\phi,n-1} \le a_{\phi_0,n-1} \equiv n-1$, where ϕ_0 denote the single-input structure function. Thus, the derivative $\mathrm{d}f[\phi;r]/\mathrm{d}r|_{r=1} > 0$ equal to a positive integer denoted by a. For any Type III structure function ϕ', $a_{\phi',n-1} > a_{\phi_0,n-1} \equiv n-1$. Since $a_{\phi',n-1}$ must be a integer and not greater than n, one has $a_{\phi',n-1} = n$ and thus, $\mathrm{d}f[\phi;r]/\mathrm{d}r|_{r=1} = 0$. Without loss of generality, the following equation holds for $\epsilon \ll 1$:

$$\begin{aligned} &f[\phi; 1-\epsilon] = 1 - a\epsilon, \\ &f[\phi'; 1-\epsilon] = 1 - b\epsilon^{c}, \end{aligned} \tag{18}$$

where the parameter a, b, c are positive integers greater than 1. For example, with the structure functions ϕ and ϕ' being the 3-out-of-3:G and 2-out-of-3:G, one has $a = 3, b = 3, c = 2$.

Let the component-level survival probability be $r(\Delta t) \approx 1 - (\kappa \Delta t)^{\alpha}$ with $a^{-L/\alpha} \ll \kappa \Delta t \ll 1$. We estimated that the initial observable HRF $\tilde{X}(0)$ becomes

$$\begin{aligned} \tilde{X}(0) &\equiv -\frac{1}{R(0)} \frac{R(\Delta t) - R(0)}{\Delta t} \\ &= \frac{1}{\Delta t} \int_{r(\Delta t)}^{1} \frac{\mathrm{d}R}{\mathrm{d}r} \mathrm{d}r \\ &\sim \frac{1}{\Delta t} \int_{0}^{(\kappa \Delta t)^{\alpha}} x^{-\log_a h(0) - 1} \mathrm{d}x \\ &\sim \Delta t^{-\alpha \log_a h(0) - 1}. \end{aligned} \tag{19}$$

where $0 < h(0) < 1$. Eq. (19) shows that the sign of $-\alpha \log_a h(0) - 1$ changes from negative to positive with the increasing α; consequently, the shape of HRF changes from the bathtub to the monotonically increasing. The details of the estimation is given in Appendix B.

This analytical result is consistent with the numerical results. The transition point of α is approximately $\alpha = -1/\log_a h(0)$ in this case. This result shows that for a complex system, the system-level initial HRF might not become zero even with the components having zero initial HRFs.

The early-stage analysis in this section and the above section shows there exists a threshold for component reliability at early stage that related to the infant mortality of system. This threshold has realistic significance, that is, to suppress or remove the system-level infant mortality one could improve the quality of components such that their early-stage HRFs are greater than the threshold or ameliorate the structure to decrease the value of threshold to adapt the current quality of components. Thus, the evaluation of threshold could be a reference for updating the components or the system in practice.

5. An example based on realistic system

The examples introduced in previous section are purely theoretical and idealized. In this section, we put them in an example with the applications in the microelectronic field, which is similar with one example given in Ref. [30].

Suppose there are 10000 transistors and they are divided into 5 levels. Each level has 10 elements. Thus the system contains totally $10 \times 10 \times 10 \times 10 \times 10 = 10000$ transistors. At each level, there are circuits connecting the elements which have several states and fail much slower than element. The elements have a number of failure modes according to the states of the circuits, where the failure modes is given by deterministic structure function. For example, there are z failure modes with respect to the z states of circuits labeled with $1, 2, \ldots, z$. The probabilities that the circuits at the lth level are in the ith state are $q_i^{(l)}$ ($\sum_i q_i^{(l)} = 1$), where the probability for the state of each circuit at same level is identical independent. Explicitly, every element at level l involve ten $l-1$-level elements and one l-level circuit. For instance, let $\sigma^{(l)}$ denote the state of a element at lth level, then $\sigma^{(l)} = \sum_i \delta_{\tilde{\sigma},i} \phi_i(\sigma_1^{(l-1)}, \sigma_2^{(l-1)}, \ldots, \sigma_{10}^{(l-1)})$, where ϕ_i is a ten-variables coherent structure function, $\sigma_1^{(l-1)}, \sigma_2^{(l-1)}, \ldots, \sigma_{10}^{(l-1)}$ are the state of the ten elements at level $l-1$ which is involved in the l-level element, δ_{ij} is the Kronecker delta function and $\tilde{\sigma} = 1, 2, \ldots, z$ is the state of the circuit. Now, we average over the states of the circuit to yield a mixed structure function of the involved elements at level $l-1$, namely $\langle \sigma^{(l)} \rangle = \sum_i q_i^{(l)} \phi_i(\sigma_1^{(l-1)}, \sigma_2^{(l-1)}, \ldots, \sigma_{10}^{(l-1)})$.

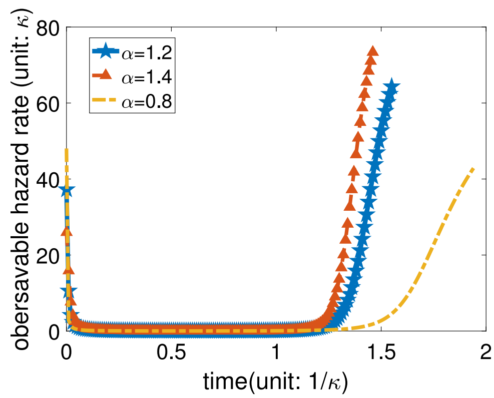
We test the systems with the different values of $q_i^{(l)}$ given in the Table a and Table b. And in each case, the values of $q_i^{(l)}$ ensure that the mixed survival probabilities $f^{(l)}(r)$ at each level satisfy the condition: $f^{(l)}(r) < r$ for $r_* < r < 1$, $f^{(l)}(r) > r$ for $r'_* < r < r_*$ and $f^{(l)}(r) < r$ for $0 < r < r'_*$.

Case 1.: Consider three failure modes: 10-out-of-10:G, 8-out-of-10:G and 3-out-of-10:G. $q_i^{(l)}$ which are given in Table a.

Case 2.: Consider ten failure modes: 1-out-of-10:G, 2-out-of-10:G, ⋯, 10-out-of-10:G with respect to the states for circuit $1, 2, \ldots, 10$. The corresponding $q_i^{(l)}$ are given in Table b.

Table a
$q_i^{(l)}$ for case 1.

i	l				
	1	2	3	4	5
1	0.2	0.32	0.41	0.3	0.39
2	0.2	0.13	0.1	0.23	0.09
3	0.6	0.55	0.49	0.47	0.52

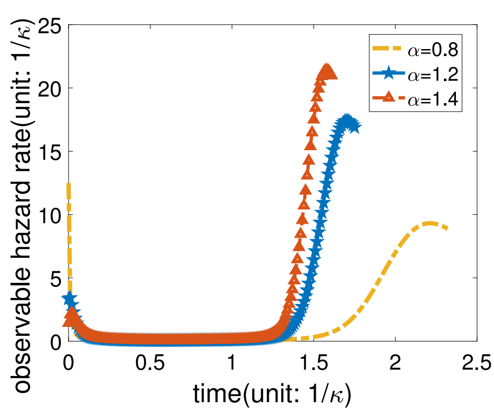


(a)

Fig. 6. The observable HRF with $\Delta t = 0.01/\kappa$, $r = \exp[-(\kappa t)^{\alpha}]$ for the system in case 1.

Table b
$q_i^{(l)}$ for case 2.

i	l				
	1	2	3	4	5
1	0.074	0.0158	0.0014	0.0053	0.0421
2	0.0436	0.1528	0.2586	0.0914	0.2785
3	0.5090	0.2633	0.3461	0.3111	0.1949
4	0.0054	0.0356	0.0472	0.1190	0.1920
5	0.1011	0.1508	0.0209	0.0072	0.0593
6	0.0047	0.0955	0.0009	0.0008	0.0037
7	0.0201	0.0001	0.1135	0.0557	0.0353
8	0.0484	0.1365	0.0437	0.0016	0.0035
9	0.1754	0.0011	0.0189	0.1426	0.0002
10	0.0849	0.1485	0.1486	0.2653	0.1906



(a)

Fig. 7. The observable HRF with $\Delta t = 0.01/\kappa$, $r = \exp[-(\kappa t)^{\alpha}]$ for the system in case 2.

In all the two cases the BHRs appear, which shows that the condition for mixture indeed lead to the steady BHRs (See Figs. 6 and 7.). The two tables are randomly generated by Monte Carlo algorithm for illustrating that the sufficient condition indeed holds in numerical calculation. And

Table c
Lifetime data for 50 samples of devices.
Source: For the data source, see Refs. [17,49].

Lifetime datum									
0.1	0.2	1	1	1	1	1	2	3	6
7	11	12	18	18	18	18	18	21	32
36	40	45	46	47	50	55	60	63	63
67	67	67	67	72	75	79	82	82	83
84	84	84	85	85	85	85	85	86	86

this system could be an experimental platform to verify or falsify the theoretical prediction.

5.1. Comparison with traditional methods

The above cases show that, with Condition 1, one could construct parameterized probability distributions for system lifetimes whose HRFs are in bathtub shapes. For case 1, the generated model has 12 independent parameters ($q_i^{(l)}$ matrix has 15 parameters however 10 of them are independent due to the normalization $\sum_i q_i^{(l)} = 1$ and the other two parameters are in Weibull distribution $r = \exp[-(\kappa t)^{\alpha}]$). Similarly, for case 2, the generated model has 47 independent parameters. These are the two cases, generally the method can generate parameterized probability distributions with different number of parameters. And the hierarchy–structure (the number of levels and the number of elements at each level) could be determined according to the decomposition of the system. This is different with traditional methods for BHR.

Traditional methods for BHR generate models with mixture of proper probability distributions. The probability distributions for mixture are artificially selected according to experience. Some of most recent models are listed here for comparison, for example, the generalization the mixed Weibull distribution [11] has 5 parameters, the log-normal modified Weibull distribution [12] has 4 parameters. The traditional methods could not generate different models. Thus, our model is more flexible and require less experience. Additionally, apart from the two parameters for Weibull distribution $r(t)$ which we have shown are insensitive due to the hierarchy, all other the parameters represent the probabilities. This implies that our model is more interpretable.

6. Model fitting: illustration with real data

From the above sections, it could be found that the model is mathematically equivalent to a multi-level Bayesian network. A deep network and proper conditional probability table at each level render the emergence of BHRs. Thus, model-fitting process is nothing but a training of the network. In the purpose of illustration, we adopt least square curve fitting as cost function to train the Bayesian network.

The real data is from Refs. [17,49], which is the lifetime data for 50 samples of devices. The lifetime data is listed in the Table c.

The adopted model for illustration is the 4-level model with and without self-similarity. The possible failure modes at each level are 5-out-of-5:G, 3-out-of-5:G and 2-out-of-5:G. The self-similarity model involve two independent parameters: the probabilities for 5-out-of-5:G and for 3-out-of-5:G which are labeled by q_1 and q_2, respectively. The 4-level model without self similarity involve eight independent parameters: the probabilities for 5-out-of-5:G and for 3-out-of-5:G at each level, which are labeled by $q_1^{(l)}$ and $q_2^{(l)}$ with $l = 1, 2, 3, 4$, respectively. Explicitly, the system survival probability for the self-similarity model is

$$R_{\mathrm{SF4}} = f_{\mathrm{SF}} \circ f_{\mathrm{SF}} \circ f_{\mathrm{SF}} \circ f_{\mathrm{SF}}(r(t)), \quad (20)$$

where $f_{\mathrm{SF}}(x) = x^5 + (q_1+q_2)[x^4(1-x)+x^3(1-x)^2]+q_2 x^2(1-x)^3$. Similarly, the system survival probability for the model without self similarity

$$R_{\mathrm{NSF4}} = f^{(4)}_{\mathrm{NSF}} \circ f^{(3)}_{\mathrm{NSF}} \circ f^{(2)}_{\mathrm{NSF}} \circ f^{(1)}_{\mathrm{NSF}}(r(t)), \quad (21)$$

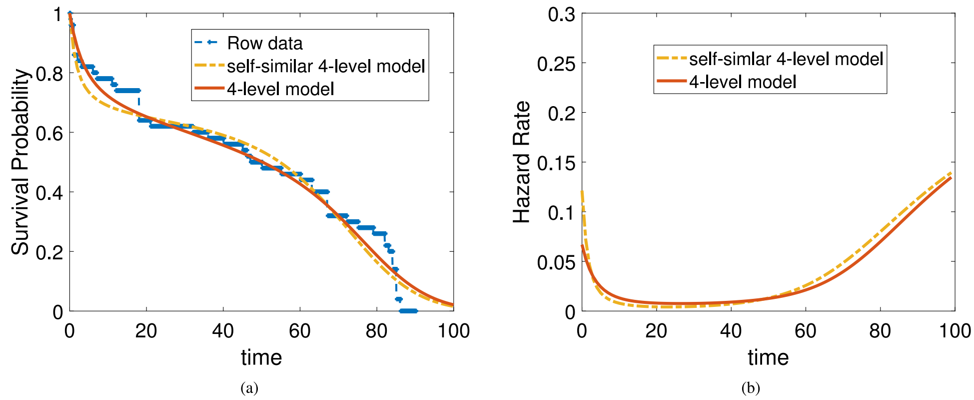


Fig. 8. (a) The survival probability and (b) the hazard rate (with $\Delta t = 1$) for fitting model.

Table d
The optimal value of $q_i^{(l)}$ for the 4-level model.

	l			
	1	2	3	4
5-out-of-5:G	0.19	0.25	0.34	0.49
3-out-of-5	0.11	0.14	0.14	0.09
2-out-of-5:G	0.70	0.61	0.52	0.42

where $f^{(l)}_{\text{NSF}}(x) = x^5 + (q_1^{(l)} + q_2^{(l)})[x^4(1-x) + x^3(1-x)^2] + q_2^{(l)} x^2(1-x)^3$, with $l = 1, 2, 3, 4$. For both two models, $r(t)$ is set to be $r(t) = \exp(-\lambda t)$ with fixed value of $\lambda = 1/60$ and λ is not to be optimized. The functions R_{SF4} and R_{NSF4} are used for least square curve fitting with the raw datum.

The performance of two fitting models are presented in Fig. 8. Refs. [12,16,17] summarize the performances of their proposed models and other different models in the historical researches to fit the raw data, which could be used to compare the previous models with ours.

The mixed structures without self-similarity obtained by model fitting are presented in Table d, where level-2 and level-3 satisfy the sufficient condition. For self-similarity model, the optimized parameters are $q_1 = 0.36, q_2 = 0.02$. This means for the mixed structure function, the probabilities for 2-out-of-5:G, 3-out-of-5 and 5-out-of-5:G are 0.62, 0.02 and 0.36, respectively, which satisfy Condition 1.

We also compute that the AIC for the 4-level self-similarity model is approximately 444.07, which is greater than four- and five-parameter log-normal modified Weibull [12], FIRE [15], Competing risk model [37] and approximately equal to the additive Burr XII [17]. Here the logarithmic likelihood is $\ln(\mathrm{d}R_{SF4}/\mathrm{d}t)$ and to compute AIC this likelihood is optimized. However, each data is from system level while the model is constructed based on system structure. If there is data for other levels it can fuse the information to bring a higher performance. This property is not included in system-level models.

This part shows the main application of this model to the reliability engineering, since estimation of reliability and HRF with lifetime datum is an important issue in the practice of reliability engineering. The proposed model in the current study is able to fuse datum at each level and information of structure which, compared with the traditional models, is a main improvement.

7. Discussion and conclusion

In conclusion, we propose a novel model based on system hierarchy that could render a steady BHRs (and increasing HRFs). The model provides a parameterized distribution on system-level lifetime and a mechanics based on structure function. The new findings and contributions are listed as follows:

(1) The model build a direct bridge from structure function to system-level HRFs for the system of systems, which was not included in previous studies. This implies the application to the HRF-inference problem and structure-inference problem with model-fitting approaches. Apart from these applications, this could also provide not only an but also a valuable reference that could be helpful in via updating or designing the structure to improve the system-level reliability in reliability-engineering practice.

(2) The proposed model is interpretable. It is closely associated with the conventional "weak sister" explanation, since it leads to a mixture of the different hazard rate functions (HRFs). Moreover, this study shows that all the sensitive parameters describe the mixtures of failure modes with respect to pure structure functions. This is not included in the previous methods and models [9,10,12–14,16,17,21]. It casts a new light in understanding the underlying mechanism of the formation of the BHRs in many complex systems, that is, the BHR dominated by mixed structure functions.

(3) The model is extensible. Although the BHR is specially concerned in the current study, the HRFs with other shapes can be also generated with a certain mixture of failure modes at every level, e.g., the roller-coaster could be rendered if there are more than three solutions for the equations $f^{(l)}(r) = r$ for each level l and $f^{(l)}(r) \le r$ for $r \to 1^-$, $f^{(l)}(r) \ge r$ for $r \to 0^+$.

Meanwhile, the current model could be improved. The model is suitable for the system with structural hierarchy and large number of levels, but not generally efficient for the systems without explicit hierarchy. To alleviate difficulties brought by the limitations and achieve relative extensions, some recent approaches [50–52] might be useful, which need to be further studies. This study has not considered relevances among the component. For further considering the relevances, if for a sufficient high level so that the relevance between the components could be neglected, then the main results do not change. Or said, if relevance between the components is local, then it does not affect the main results, due to the hierarchy. Otherwise, the global relevance could significantly affect the system-level HRF. This needs to be further studied. Additionally, the study does not include components with multiple states. Note that the property that hierarchy lead to a structure-dominate-system-level degradation also holds for a general multi-state system. Thus, this could be a basic idea to extend the model to multi-state systems. This invokes further studies.

CRediT authorship contribution statement

Yi-Mu Du: Conceptualization, Methodology, Software, Writing – original draft, Editing. **C.P. Sun:** Conceptualization, Writing – review & editing.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

Data will be made available on request.

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Appendix A. Proofs of the claims in Section 2.2

An arbitrary increasing Boolean function is denoted by $\phi(\sigma) \equiv \phi(\sigma_1, \sigma_2, \ldots, \sigma_n)$. To discuss the mathematical structure for the increasing Boolean functions, one would define the sets as follows

$$\begin{aligned} a_\sigma &\equiv \{i|\forall i, \sigma_i = 1\}, \\ \bar{a}_\sigma &\equiv \{i|\forall i, \sigma_i = 0\}, \end{aligned} \tag{22}$$

with $\sigma \equiv (\sigma_1, \sigma_2, \ldots, \sigma_n)$, and

$$\begin{aligned} A_\phi &\equiv \{a_\sigma|\forall \sigma, \phi(\sigma) = 0\}, \\ \bar{A}_\phi &\equiv \{\bar{a}_\sigma|\forall \sigma, \phi(\sigma) = 1\}. \end{aligned} \tag{23}$$

Due to the increasing ϕ, the set A_ϕ must satisfy the condition: $\forall\ b \subseteq a$ if $a \in A_\phi$, then $b \in A_\phi$, which is called an abstract simplicial complex [44]. Similarly, $\bar{A}_\phi$ is also an abstract simplicial complex. If there are two increasing Boolean functions ϕ, ϕ' such that A_ϕ is isomorphic to $\bar{A}_{\phi'}$, $A_\phi \cong \bar{A}_{\phi'}$, then $\phi(\sigma) = 1 - \phi'(\mathbf{1} - \sigma)$, where $\mathbf{1} \equiv (1, 1, \ldots, 1)$. It can also be verified that if $\phi(\sigma) = 1 - \phi'(\mathbf{1} - \sigma)$, then $A_\phi \cong \bar{A}_{\phi'}$. Thus one has

$$A_\phi \cong \bar{A}_{\phi'} \Leftrightarrow \phi(\sigma) = 1 - \phi'(\mathbf{1} - \sigma) \tag{24}$$

The relation implies the intrinsic duality of the space for the increasing Boolean functions, which has been given in many previous literatures, e.g., Ref. [29].

The two sets in Eq. (23), implies two one-to-one surjections: $M, \bar{M}$, from the set consists of the labeled abstract simplicial complexes $\{B|\forall B, D(B) \leq n\}$ to the space of n-variable increasing Boolean functions, such that

$$A_{M(B)} = \bar{A}_{\bar{M}(B)} = B. \tag{25}$$

Thus, it is convenient to use the abstract simplicial complex B to study the corresponding Boolean functions.

Let $|a|$ be the number of the elements in set a. The dimension of an abstract simplicial complex B is defined as $D(B) \equiv \max_{\forall a \in B} |a|$. Let B_k be the subset of the abstract simplicial complex B which involves all the subsets of B with the number of elements k, i.e., $B_k \equiv \{a|\forall a \in B, |a| = k\}$. It follows that

$$\begin{aligned} a_{M(B),k} &= C_n^k - |B_k|, \\ a_{\bar{M}(B),k} &= |B_{n-k}|, \end{aligned} \tag{26}$$

where C_n^k is the binomial coefficients and $|B_k|$ is the elements' number of B_k.

Consider the case $f[\phi_0; r] = r = \sum_{k=1}^n C_{n-1}^{k-1} r^k (1-r)^{n-k}$. Without loss of generality, let $\phi_0(\sigma)$ be $\phi_0 = \sigma_1$. Since $\phi_0(\sigma) = 1 - \phi_0(\mathbf{1} - \sigma)$, it follows that

$$A_{\phi_0} \cong \bar{A}_{\phi_0}, \tag{27}$$

which implies that there is only one abstract simplicial complex corresponding to ϕ_0, i.e., A_{ϕ_0}.

Now, we are in the position to prove that for an arbitrary increasing Boolean function ϕ such that A_ϕ is not isomorphic to A_{ϕ_0}, there is at most one solution for the equation $f[\phi; r] = r$ with $0 < r < 1$. Clearly, $\forall 0 < k < n$ $A_{\phi_0,k}$ is composed of all the k-element subsets of the set $\{2, 3, 4, \ldots, n\}$. For an abstract simplicial complex B which is not isomorphic to A_{ϕ_0} if $\exists k$ such that $|B_k| > |A_{\phi_0,k}|$, then $|B_{k-1}| > |A_{\phi_0,k-1}|$.

There are four possible cases for an arbitrary abstract simplicial complex B:

(1) $\forall 0 < k < n$ $|B_k| > |A_{\phi_0,k}|$, then for $0 < r < 1$, $f[M(B); r] < r$ and $f[\bar{M}(B); r] > r$, there is no solution for the equations $f[M(B); r] = r$ and $f[\bar{M}(B); r] = r$ with $0 < r < 1$.

(2) $\forall 0 < k < n$ $|B_k| < |A_{\phi_0,k}|$, then for $0 < r < 1$, $f[M(B); r] > r$ and $f[\bar{M}(B); r] < r$, there is no solution for the equations $f[M(B); r] = r$ and $f[\bar{M}(B); r] = r$ with $0 < r < 1$.

(3) $\exists k_0$ such that $|B_k| > |A_{\phi_0,k}|$ for $k < k_0$ and $|B_k| \leq |A_{\phi_0,k}|$ for $k \geq k_0$, which implies

$$\begin{aligned} f[\bar{M}(B); r] - f[\phi_0; r] = &\sum_0^{k_0} (|B_k| - |A_{\phi_0,k}|) r^{n-k} (1-r)^k \\ &+ \sum_{k_0+1}^{n} (|B_k| - |A_{\phi_0,k}|) r^{n-k} (1-r)^k \\ \equiv &\sum_1^{n-1} z_k r^k (1-r)^{n-k}, \end{aligned} \tag{28}$$

where $z_k > 0$ for $k > k_0$ and $z_k \leq 0$ for $k \leq k_0$. It follows that the equation $f[\bar{M}(B); r] - f[\phi_0; r] = 0$ has only one solution with $0 < r < 1$.

Proof. It is clear that the solutions of the equation exist, since $\exists \epsilon > 0$ such that $f_{\bar{M}(B)}(\epsilon) < f_{\phi_0}(\epsilon)$ and $f[\bar{M}(B); 1-\epsilon] > f[\phi_0; 1-\epsilon]$. Suppose that there are more than one solutions denoted by $r_*, r_1, r_2, \ldots$ with $0 < r_* < r_1 < r_2 < \cdots < 1$. then one has

$$\sum_1^{n-1} z_k r_*^k (1-r_*)^{n-k} = \sum_1^{n-1} z_k g_k r_*^k (1-r_*)^{n-k} = 0, \tag{29}$$

where $g_k = (r_1/r_*)^k [(1-r_1)/(1-r_*)]^{n-k}$, and thus $g_{n-1} > g_{n-2} > \cdots > g_1$. Eq. (29) implies that $\forall u \in \mathbb{R}$,

$$\sum_1^{n-1} z_k (g_k - u) r_*^k (1-r_*)^{n-k} = 0. \tag{30}$$

However, it is found that for $g_{k_0+1} > u > g_{k_0}$, $\sum_1^{n-1} z_k (g_k - u) r_*^k (1-r_*)^{n-k} > 0$. Thus, the equation $f[\bar{M}(B); r] - f[\phi_0; r] = 0$ has only one solution with $0 < r < 1$. This completes the proof.

Similarly, this is also hold for $f_{M(B)}(r)$.

(4) $B \cong A_{\phi_0}$ and $f[\bar{M}(B); r] = f[M(B); r] = r$, where B is not coherent.

If $\mathrm{d}f[\phi; r]/\mathrm{d}r|_{r=0} = 0$ then the structure function ϕ is in Type I. If $\mathrm{d}f[\phi; r]/\mathrm{d}r|_{r=1} = 0$ then the structure function ϕ is in Type II. If $\mathrm{d}f[\phi; r]/\mathrm{d}r|_{r=1} = 0$ and $\mathrm{d}f[\phi; r]/\mathrm{d}r|_{r=0} = 0$, then the structure function ϕ is in Type III. It is straightforward that for any coherent structure functions ϕ in Type I and ϕ' in Type II, the structure functions $\tilde{\phi}$ with $f[\tilde{\phi}; r] \equiv f[\phi; f[\phi', r]]$ or is $f[\tilde{\phi}; r] \equiv f[\phi'; f[\phi, r]]$ in Type III; and for any functions ϕ in Type III and an arbitrary structure functions ϕ', the structure functions $\tilde{\phi}$ with $f[\tilde{\phi}; r] \equiv f[\phi; f[\phi', r]]$ or $f[\tilde{\phi}; r] \equiv f[\phi'; f[\phi, r]]$ are also in Type III.

Appendix B. Properties of $R_\mathbf{s}$

Let $R_{\mathbf{s},k} \equiv R_{(s_0, s_1, \ldots, s_k)}$ be

$$\begin{aligned} R_{\mathbf{s},k}(r) = &f_{\phi'} \circ \overbrace{f_\phi \circ f_\phi \circ \cdots f_\phi}^{s_k} \circ f_{\phi'} \circ \overbrace{f_\phi \circ f_\phi \circ \cdots f_\phi}^{s_{k-1}} \circ f_{\phi'} \circ f_\phi \circ \\ &\cdots f_\phi \circ f_{\phi'} \circ \overbrace{f_\phi \circ f_\phi \circ \cdots f_\phi}^{s_{k'}} \circ f_{\phi'} \circ \cdots f_{\phi'} \circ \overbrace{f_\phi \circ f_\phi \circ \cdots f_\phi}^{s_0}(r). \end{aligned} \tag{31}$$

It follows that

$$R_{\mathbf{s},k+1} = f_{\phi'} \circ \overbrace{f_\phi \circ f_\phi \circ \cdots f_\phi}^{s_k}(R_{\mathbf{s},k}). \tag{32}$$

To denote $\ln R_{\mathbf{s},k}$ by $y_{\mathbf{s},k}$, rewrite the above equation as follows

$$y_{\mathbf{s},k+1} = \ln f[\phi'; a^{s_k} y_{\mathbf{s},k}] \equiv \Lambda(a^{s_k} y_{\mathbf{s},k}). \quad (33)$$

Let $y_* < 0$ be the solution of the equation $\Lambda(y_*) = y_*$ and y_{*k} for the equation $y_{\mathbf{s},k}(y_{*k}) = y_*$. Since the function Λ has the property that $\Lambda(y) > y$ for $0 > y > y_*$ and $\Lambda(y) < y$ for $y < y_*$, one has

$$y_{*k} \le y_{*k+1}. \quad (34)$$

The function $R_{\mathbf{s}}$ rapidly decrease to zero for $r < \exp(y_{*n})$. With the approximation $f[\phi'; 1-\epsilon] \approx 1 - b\epsilon^c$,

$$y_{*n} \sim -a^{-\sum_{j=0}^{n} s_j/c_j}, \quad (35)$$

where $a^{-\sum_{j=0}^{n} s_j/c_j} \ll 1$. It follows that $1 - \exp(y_{*n}) \sim a^{-\sum_{j=0}^{n} s_j/c_j}$. Thus, we have following results.

The survival probability $R_{\mathbf{s}}$ depends on the characterized quantity $\chi_{\mathbf{s}} \equiv \sum_j s_j/c^j$. There is one peak for the function $\mathrm{d}R_{\mathbf{s}}(r)/\mathrm{d}r$ with $r = r_{\text{peak}}$. It follows that

$$\begin{aligned} &1 - r_{\text{peak}} \sim a^{-\chi_{\mathbf{s}}}, \\ &\frac{\mathrm{d}R_{\mathbf{s}}(r)}{\mathrm{d}r}\Big|_{r_{\text{peak}}} \sim a^{\chi_{\mathbf{s}}}, \end{aligned} \quad (36)$$

and the width of the peak is therefore proportional to $a^{\chi_{\mathbf{s}}}$.

The above relations are utilized to analyze the initial value of HRF. Eqs. (16) and (36) imply that the mixture of the $R_{\mathbf{s}}(r)$ with different s could be resorted according the peaks' locations r_{peak}. That is, for $\epsilon \ll 1$,

$$R(1-\epsilon) \approx \sum_{\mathbf{s}, \chi_{\mathbf{s}} = -\frac{\ln\epsilon}{\ln a}} \psi(\mathbf{s}) R_{\mathbf{s}}(1-\epsilon). \quad (37)$$

The derivative of $R(r)$ is

$$\begin{aligned} \frac{\mathrm{d}R}{\mathrm{d}r}\Big|_{r=1-\epsilon} &\sim \frac{1}{\epsilon} \sum_{\mathbf{s}, \chi_{\mathbf{s}} = -\frac{\ln\epsilon}{\ln a}} \psi(\mathbf{s}) \\ &\approx \frac{1}{\epsilon} \sum_{s_1, s_2, \dots} \psi(-\frac{\ln\epsilon}{\ln a}, s_1, s_2, \dots), \end{aligned} \quad (38)$$

where the approximation preserves the leading order term $\chi_{\mathbf{s}} \equiv \sum_j s_j/c^j \approx s_0$.

The marginal distribution for s_0 is

$$\psi_0(s_0) \equiv \sum_{s_1, s_2, \dots} \psi(s_0, s_1, s_2, \dots) = \left[1 - h\left(\frac{s_0}{L}\right)\right] \prod_{i=0}^{s_0-1} h\left(\frac{i}{L}\right). \quad (39)$$

If the size of the system is sufficiently large, i.e., $L \gg 1$, one has

$$\psi_0(s_0) \overset{L\gg1}{\approx} [1 - h(x_0)] \exp\left[L \int_0^{x_0} \ln h(x)\mathrm{d}x\right], \quad (40)$$

where $x_0 \equiv s_0/L$. For $-\log_a \epsilon/L \ll 1$, it follows that

$$\frac{\mathrm{d}R}{\mathrm{d}r}\Big|_{r=1-\epsilon} \sim \epsilon^{-\log_a h(0) - 1}. \quad (41)$$

References

[1] Gnedenko B, Belyayev Y, Solovyev A. Mathematical methods of reliability theory. Cambridge, MA, USA: Academic Press; 2014.
[2] Smith AM. Reliability-centered maintenance. New York, NY, USA: McGraw-Hill; 1992.
[3] Wasson CS. System engineering: analysis, design, and development. Hoboken, NJ, USA: Wiley; 2004.
[4] Nash FR. Reliability assessments: concepts, models, and case studies. Boca Raton, FL, USA: CRC Press; 2016.
[5] Halley E. An estimate of the degrees of the mortality of mankind, drawn from curious tables of the births and funerals at the city of Breslau; with an attempt to ascertain the price of annuities upon lives. Philos Trans R Soc Lond 17(1693):596.
[6] Gaonkar A, Patil RB, Kyeong S, Das D, Pecht MG. An assessment of validity of the Bathtub model hazard rate trends in electronics. IEEE Access 2021;9:10282.
[7] Scheidegger A, Leitao JP, Scholten L. Statistical failure models for water distribution pipes - a review from a unified perspective. Water Res 2015;83:237.
[8] R?stum J. Statistical modelling of pipe failures in water networks. 2000.
[9] Xie M, Lai CD. Reliability analysis using an additive Weibull model with bathtub-shaped failure rate function. Reliab Eng Syst Saf 1996;52:87.
[10] Xie M, Tang Y, Goh TN. A modified Weibull extension with bathtub-shaped failure rate function. Reliab Eng Syst Saf 2002;76:279.
[11] Ahmad AEA, Ghazal MGM. Exponentiated additive Weibull distribution. Reliab Eng Syst Saf 2020;193:106663.
[12] Shakhatreh Mohammed K, Lemonteb Artur J, Moreno-Arenasc G. The log-normal modified Weibull distribution and its reliability implications. Reliab Eng Syst Saf 2019;188:6.
[13] Wondmagegnehu ET, Navarro J, Hernandez Pedro J. Bathtub shaped failure rates from mixtures: A practical point of view. IEEE Trans Relib 2005;54:2.
[14] Jiang R. A new bathtub curve model with a finite support. IEEE Trans Relib 2013;119:44.
[15] Wang X, Yu C, Li Y. A new finite interval lifetime disbutribution model for fitting bathtub-shaped failure rate curve. Math Probl Eng 2015;2015.
[16] t. Zeng H, Lan T, Chen Q-M. Five and four-parameter lifetime distributions for bathtub-shaped failure rate using perks mortality equation. Reliab Eng Syst Saf 2016;152:307.
[17] Wang FK. A new model with bathtub-shaped failure rate using an additive Burr XII distribution. Reliab Eng Syst Saf 2000;70:305.
[18] Arshad MZ, Muhammad ZI, Asad A, Zahoor A, Oluwafemi SB. A new bathtube shaped failure rate model: properties and applications to engineering sector. Pak J Stat 2021;37(1):57.
[19] Prataciera F, Ortega Edwin MM, Cordeiro Gauss M, Pescim Rodrigo R, Verssani Bruna AW. A new generalized odd log–logistic flexible Weibull regression model with applications in repairable systems. Reliab Eng Syst Saf 2018;176:13.
[20] Romaniuk M, Hryniewicz O. Estimation of maintenance costs of a pipeline for a U-shaped hazard rate function in the imprecise setting. Eksploatacja I Niezawodnosc 2020;22(2):352.
[21] Du YM, Ma YH, Wei YF, Guan XF, Sun CP. Maximum entropy approach to reliability. Phys Rev E 2020;101:012106.
[22] de Negreirosa ACSV, Linsa ID, das C. Mouraa MJ, Droguettc EL. Reliability data analysis of systems in the wear-out phase using a (corrected) q-exponential likelihood. Reliab Eng Syst Saf 2020;197:106787.
[23] Wang KS, Hsu FS, liu PP. Modeling the bathtub shaped hazard rate function in terms of reliability. Reliab Eng Syst Saf 2002;75:397.
[24] Smith RW, Dietrich DL. The bathtub curve: An alternative explanation. In: Proceedings annual reliability and maintainability symposium. 1994.
[25] Wong KL. The bathtub does not hold water any more. Qual Reliab Eng Int 1988;4:279.
[26] Klutke G-A, Kiessler PC, Wortman MA. A critical look at the bathtub curve. IEEE Trans Relib 2003;52:125.
[27] Nishijima K, Maes M, Goyet J, Faber MH. Optimal reliability of components of complex systems using hierarchical system models. ETH Zurich: Institute of Structural Engineering; 2007.
[28] Hsieh T-J. Hierarchical redundancy allocation for multi-level reliability systems employing a bacterial-inspired evolutionary algorithm. Inform Sci 2014;288:174.
[29] Torrado N, Arriaza A, Navarro J. A study on multi-level redundancy allocation in coherent systems formed by modules. Reliab Eng Syst Saf 2021;213:107694.
[30] Ye X-F, Zhang Y, Harutoshi O, Kim C-W. Hierarchical probability and risk assessment for K-out-of-N system in hierarchy. Reliab Eng Syst Saf 2019;189:242.
[31] Louzada F, Cuminato JA, Rodriguez OMH, Tomazella VLD, Ferreira PH, Ramos PL, et al. A repairable system subjected to hierarchical competing risks: Modeling and applications. IEEE Access 2019;7:171707.
[32] Louzada F, Tomazella VLD, Gonzatto Junior OA, Bochio G, Milani EA, Ferreira PH, et al. Reliability assessment of repairable systems with series-parallel structure subjected to hierarchical competing risks under minimal repair regime. Reliab Eng Syst Saf 2022;222:108364.
[33] Yin D, Huang W, Shuai B, Liu H, Zhang Y. Structural characteristics analysis and cascading failure impact analysis of urban rail transit network: From the perspective of multi-layer network. Reliab Eng Syst Saf 2022;218:108161.
[34] Christensen K, Moloney NR. Complexity and criticality. Imperial College Press Advanced Physics Texts; 2005.
[35] Li D-Q, Zhang Q, Zio E, Havlin S, Kang R. Network reliability analysis based on percolation theory. Reliab Eng Syst Saf 2015;142:556.
[36] Behrensdorf J, Regenhardt T-E, Broggi M, Beer M. Numerically efficient computation of the survival signature for the reliability analysis of large networks. Reliab Eng Syst Saf 2021;216:107935.
[37] Zhang T, Dwight R. Choosing an optimal model for failure data analysis by graphical approach. Reliab Eng Syst Saf 2013;115:111.
[38] Lisnianski A, Levitin G. Multi-state system reliability: assessment, optimization and applications. Singapore: World Scientific; 2003.
[39] Zaitseva E, Levashenko V. Reliability analysis of multi-state system with application of multiple-valued logic. Int J Qual Reliab Manage 2017;34(6):862.
[40] Barlow RE, Proschan F. Statistical theory of reliability and life testing. In: International series in decision processes. New York: Holt, Rinehart and Winston, Inc. 1975.

[41] Frank PA. Coolen and tahani coolen-maturi, the structure function for system reliability as predictive(imprecise) probability. Reliab Eng Syst Saf 2016;154:180.
[42] Coolen FPA, Coolen-Maturi T. Generalizing the signature to systems with multiple types of components. Adv Intell Soft Comput 2012;170:115, AISC.
[43] Coolen FPA, Coolen-Maturi T, Al-Nefaiee AH. Nonparametric predictive inference for system reliability using the survival signature. Proc Inst Mech Eng O J Risk Reliab 2014;228(5):437.
[44] Munkres JR. Elements of algebraic topology. Addison-Wesley Publishing Company, Inc. 1984.
[45] Langseth H, Portinale L. Bayesian networks in reliability. Reliab Eng Syst Saf 2007;92:92.
[46] Zheng Y-X, Xiahou T, Liu Y, Xie C. Structure function learning of hierarchical multi-state systems with incomplete observation sequences. Reliab Eng Syst Saf 2021;216:107902.
[47] Yang L, Wang P, Wang Q, Bi S, Peng R, Behrensdorf J, Beer M. Reliability analysis of a complex system with hybrid structures and multi-level dependent life metrics. Reliab Eng Syst Saf 2021;209:107469.
[48] Kuo W, Zhang W, Zuo M. A consecutive-K-out-of-N: G system: the mirror image of a consecutive-K-out-of-N: F system. IEEE Trans Reliab 1990;39:244.
[49] Aarset MV. How to identify bathtub hazard rate. IEEE Trans Reliab 1987;36:106.
[50] Alkaff A, Qomarudin MN, Bilfaqih Y. Network reliability analysis: Matrix-exponential approach. Reliab Eng Syst Saf 2020;204:107192.
[51] Alkaff A. Discrete time dynamic reliability modeling for systems with multistate components. Reliab Eng Syst Saf 2021;209:107462.
[52] Guan X, Xiang Z, Bao Y, Li H. Structural dominant failure modes searching method based on deep reinforcement learning. Reliab Eng Syst Saf 2022;219:108258.