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Dynamic quantum decoherence in Stern-Gerlach experiment *

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Abstract The Stern-Gerlach (SG) experiment is considered as a quantum measurement process measuring the spin states of particles from their spatial distribution. The wave packet collapse or quantum decoherence can be described as a dynamical evolution governed by the interaction between the space degrees of freedom and the spin degrees of freedom. The analytic solution of this exactly solvable model shows that the decaying process of coherence with time in the SG experiment obeys no longer the linear exponential law.

Keywords: quantum decoherence, wave packet collapse, exponential law, quartum irreversible process.

It is well known that the microscopic dynamics of single particle described by the Schröedinger equation with the symmetry of time-inversion is basically reversible. However, a macroscopic object composed of such particles appears to be classically irreversible in macroscopic world. Until now, it is still an essential problem without final solution to make the macroscopic irreversible phenomena be consistent with the fundamental quantum mechanics. In fact, physicists of several generations have payed much attention to this problem[1, 2]. These irreversible phenomena concern many frontier fields and fundamental problems of modern physics, such as quantum dissipation of small system in bath[3], wave packet collapse in quantum measurement[4] and quantum decoherence caused by environment[2]. A comprehensive investigation for this problem not only is the necessity to throw light on the foundation of basic quantum mechanics, but also will enhance the further development of frontier topics in modern physics, such as the mesoscopic physics[5] and quantum computation[6].

Based on the systematic research on quantum dynamics theory for quantum measurement process[7—12], we reconsider a quantum irreversible phenomenon in the Stern-Gerlach(SG) experiment, the wave packet collapse in the quantum measurement of spin. It is shown that, as the SG experiment is understood as the quantum measurement process of spin state through spatial distribution of particles, the macroscopic distinction between "two spots" on screen for spatial distribution will result in an irreversible decoherence process for a superposition of spin states. Meanwhile the time evolution of decoherence deviates from the usual exponential decay law, and it appears to be the complex nonlinear behavior similar to the non-exponential decay in quantum tunnelling experiment[13].

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1 Decoherence in quantum measurement and its dynamic approach

According to the fundamental principles of quantum mechanics, when a quantum system is prepared in coherence superposition $|\Phi\rangle = \sum_n c_n |n\rangle$ of eigenstates $|n\rangle$ of observable A and we measure A, the obtained result is uncertain, but its probable value is only one λ_n of the eigen-values of A with probability $|c_n|^2$ accordingly. It is known from von Neumann's quantum measurement theory[4] that once we get a result λ_k in one measurement, the system will subside to the eigenstate $|k\rangle$ of A after this measurement. This is the so-called von Neumann's postulate of wave packet collapse. With the density matrix, we can describe this process as a coherence vanishing off-diagonal elements of pure state

$$\rho_s(0) = |\Phi\rangle\langle\Phi| = \sum_{m,n} c_n c_m^* |n\rangle\langle m|, \tag{1}$$

namely

$$\rho_s(0) \rightarrow \rho_s(t) = \sum_n |c_n|^2 |n\rangle\langle n|, \tag{2}$$

where t is the time of measurement.

In fact, we can consider the happening of decoherence by measuring the expectation value of another dynamic-observable B not commuting with A. In the pure state $\rho_s(0)$, the expectation value of B, $\overline{B} = T_r(\rho_s(0)B) = \sum_{m,n} c_n c_m^* B_{nm}$ concerns not only the diagonal elements B_{nn} of B, but also the off-diagonal elements $B_{nm}(n \neq m)$. This shows the system keeps its coherence. As to the mixed state, the expectation value of B, $\overline{B} = T_r(\rho_s(t)B) = \sum_n |c_n|^2 B_{nn}$ depends only on the diagonal element B_{nn}. The disappearance of off-diagonal elements B_{nm} reflects the complete vanishing of coherence. The density matrix $\rho_s(t)$ represents the classical state that the coherence vanishes completely.

We remark that the above decoherence making a transition from quantum to classical is certainly an irreversible process. This is because the density matrixes $\rho_s(0)$ and $\rho_s(t)$ have different ranks and thus they can not be transformed into each other through a unitary time-evolution matrix. Therefore, as a postulate with certain classical elements, the wave packet collapse can not be derived from Schröedinger equation and other laws in quantum mechanics. Since the quantum mechanics was founded, physicists wished to add this postulate into the system of axioms in quantum mechanics. Von Neumann and Wigner took the first attempt to consider the measurement detector D plus the measured system as a total system (called "universe") satisfying the Schröedinger equation. They hoped that, projected on the system S, the evaluation of the "universe" seemed to lead to wave packet collapse naturally. However, this approach brought to philosophical difficulty[14] because it did not take the macroscopic character of detector D into account.

In 1972, Hepp raised the dynamical description[15] of wave packet collapse caused by macroscopic character of detector via a simple exactly-solvable model. Later Namiki's group generalized this work to put forward various new models for quantum measurement[16]. In 1993, by analyzing those models and taking the classical limit of detector D into account, we found that an essence im-

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plied by these model was the factorization structure[7, 8]. Namely, the reduced density matrix of the system:

$$\rho_s(t) = \sum_n |c_n|^2 |n\rangle\langle n| + \sum_{m,n} [c_n c_m^* F_{mn}(t) |m\rangle\langle n| + hc] \tag{3}$$

obtained by tracing out the variables of detector possesses factorized decoherence factors

$$F_{mn}(t) = \prod_{j=1}^{N} F_{mn}^j(J_j, t) \tag{4}$$

that are the coefficients of off-diagonal elements of $\rho_s(t)$. Each factor $F_{mn}^j(J_j, t)$ depends only on dynamical variable of each particle constituting the detector, and $|F_{mn}^j(J_j, t)| \leqslant 1$. So it is possible that $F_{mn}(t) \to 0$, at the macroscopic limit that the number of particles composing the detector, $N \to \infty$. The factorization structure is universal and all the dynamical models of quantum measurement can be regarded as the concrete realizations of this structure[9—12, 16].

The goal of this work is to develop a new kind of dynamical model of quantum measurement, which does not possess this factorization structure. This model only associates the wave packet collapse-decoherence problem with the requirement that the result of measurement should be macroscopically observable.

2 State correlation in SG experiment

The process of quantum measurement is intuitionally an observing process that "reads out" the system states from the "macroscopically distinguished" states of detector. Under this circumstance, a state of the system will be correlated to one of the detector states through the interaction between system and detector. If the initial state of the "universe" before observation at time $t = 0$ is

$$|\Phi_T(0)\rangle = \sum_{n=1}^{d} c_n |n\rangle \otimes |D\rangle, \tag{5}$$

where $|D\rangle$ is an initial state of the detector. Then the state of "universe" at time t is

$$|\Phi_T(t)\rangle = \sum_{n=1}^{d} c_n(t) |n\rangle \otimes |D_n\rangle, \tag{6}$$

where $|D_n\rangle$ is the detector state entangled with a system state $|n\rangle$. Eq. (6) means a correlation between system states $|n\rangle$ and detector states $|D_n\rangle$:

$$|1\rangle \to |D_1\rangle, \cdots\cdots |n\rangle \to |D_n\rangle, \cdots\cdots |d\rangle \to |D_d\rangle. \tag{7}$$

If $|D_n\rangle$ is supposed to be "macroscopically distinguished" and once the detector is found on state $|D_n\rangle$, we can infer that the system is just on state $|n\rangle$. The measurement is thought to be ideal when $|D_n\rangle$ $(n = 1, 2, \cdots)$ are orthogonal to each other. In the following we will quantitatively show the dynamical realization of this correlation in SG experiment.

In SG experiment, Ag-atoms initially situated on ground state enter a magnetic field inhomoge-

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neous in Z direction. Because the atoms of spin states $|+\rangle$ and $|-\rangle$ separately suffer two forces from opposite directions along Z direction, the atoms in a superposition of two spin states will form two macroscopically-distinguishable spots on the detecting screen, each of which is correlated to one of the spin states. This process is the so-called spin-space correlation and enables people to pick out different spin states according to the spatial distribution. This process of correlation can be described with Hamiltonian

$$H = \frac{\hat{p}^2}{2m} + \mu \boldsymbol{B}(z) \cdot \boldsymbol{\sigma}, \tag{8}$$

where $\hat{p}$ is the momentum operator of atomic mass center, $\boldsymbol{\sigma}$ is the spin operator of atom on the ground state and $\boldsymbol{B}(z) = B(z)\boldsymbol{e}_z$. The initial state of atom beam just entering the magnetic field is

$$|\Phi(0)\rangle = (c_1|+\rangle + c_2|-\rangle) \otimes |\phi_s(z)\rangle,$$

where the spatial part $|\phi_s\rangle$ is a Gaussian wave packet distributed along direction Z centered at the origin and

$$|\phi_s(z)\rangle = \int \left(\frac{1}{2\pi a^2}\right)^{\frac{1}{4}} \mathrm{e}^{-z^2/4a^2} |z\rangle \mathrm{d}z,$$

where a is the width of atom beam.

Notice that the atoms in different spin states initially have the same spatial distribution at time $t=0$. Due to the space-spin interaction the atom beam will split into two wave-packets along Z direction. This beam-splitting process can be pictured with a wave function

$$|\Phi(t)\rangle = c_1|+\rangle \otimes U_+(t)|\phi_s(z)\rangle + c_2|-\rangle \otimes U_-(t)|\phi_s(z)\rangle.$$

The effective evolution matrixes acting on different spin states are

$$U_\pm(t) = \exp\left[-\frac{it}{\hbar}\left(\frac{\hat{p}^2}{2m} \pm \mu B(z)\right)\right]. \tag{9}$$

With linear approximation, we have $B(z) \simeq [\partial_z B(z=0)]z$. If we denote $f = \mu \partial_z B\ (z=0)$, $U_\pm(t)$ represents the evolution process of particles drived by external forces $\pm f$ of opposite directions. Accordingly, the effective Hamiltonians are

$$H_\lambda = \frac{\hat{p}^2}{2m} + \lambda f \hat{z},\ \lambda = \pm 1. \tag{10}$$

By the Wei-Noman Algebra method[17], the factorized forms of $U_\pm(t)$ are calculated as

$$U_\lambda(t) \equiv \mathrm{e}^{\alpha(t)\hat{p}^2} \mathrm{e}^{\beta(t)\hat{p}} \mathrm{e}^{\gamma(t)\hat{z}} \mathrm{e}^{\mu(t)}, \tag{11}$$

where

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$$\alpha(t) = -\frac{\mathrm{i}t}{2m\hbar} \equiv -\mathrm{i}\tilde{\alpha}(t);$$

$$\beta(t) = -\frac{\mathrm{i}t^2}{2m\hbar}\lambda f \equiv -\mathrm{i}\tilde{\beta}_\lambda(t);$$

$$\gamma(t) = -\frac{\mathrm{i}t}{\hbar}\lambda f \equiv -\mathrm{i}\tilde{\gamma}_\lambda(t)/\hbar;$$

$$\mu(t) = -\frac{\mathrm{i}(\lambda f)^2 t^3}{6m\hbar} = -\frac{\mathrm{i}f^2 t^3}{6m\hbar} \equiv -\mathrm{i}\tilde{\mu}(t).$$

Then we get the distribution of atoms along direction Z in coordinate representation at time t:

$$\Phi(z, t) = \langle z|\Phi(t)\rangle = c_1\Psi_+(z, t)|+\rangle + c_2\Psi_-(z, t)|-\rangle,$$

where

$$\Psi_\lambda(z, t) = \left(\frac{a^2}{2\pi^3\hbar^4}\right)^{\frac{1}{4}}\left(\frac{\pi\hbar^2}{a^2+\frac{\mathrm{i}\hbar t}{2m}}\right)^{\frac{1}{2}} \mathrm{e}^{-\mathrm{i}\theta(t)-\frac{\mathrm{i}\lambda t}{\hbar}fz}\exp\left[-\frac{\left(z+\frac{1}{2}\cdot\frac{\lambda f}{m}t^2\right)^2\left(a^2-\frac{\mathrm{i}\hbar t}{2m}\right)}{4a^4+\frac{t^2\hbar^2}{m^2}}\right].$$

$$\left(\theta(t) = \frac{f^2t^3}{6m\hbar}\right) \quad (12)$$

denotes the Gaussian wave packets centered in $z_\pm = \pm\frac{1}{2}\cdot\frac{f}{m}t^2$ with the same width $a(t) = a\left(1+\frac{t^2\hbar^2}{4m^2a^2}\right)^{\frac{1}{2}}$ and different group speeds $v_\pm = \mp ft$ along the opposite directions separately. It is obvious that the motions of the wave packet centers obey the classical motion law that the particle of mass m forced by $\pm f$ will move with acceleration $\pm f/m$. Notice that the quantum character of this motion appears in the spreading of wave-packet.

The macroscopic distinction of wave-packets in quantum measurement requires that the distance between the two wave-packets should be far larger than the width of each wave packet, i.e.

$$\frac{f}{m}t^2 \gg a\left(1+\frac{t^2\hbar^2}{4m^2a^2}\right)^{\frac{1}{2}}. \quad (13)$$

This condition will be satisfied when evolution time is long enough.

3 Non-exponential decoherence process

To analyze the relationship between measurement and decoherence quantitatively, we compute the reduced density matrix for spin degree of freedom:

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$$\rho_s(t) = \mathrm{Tr}_z(|\Phi(t)\rangle\langle\Phi(t)|) = \int_{-\infty}^{+\infty}\langle z|\Phi(t)\rangle\langle\Phi(t)|z\rangle\mathrm{d}z$$
$$= |c_1|^2|+\rangle\langle+| + |c_2|^2|-\rangle\langle-| + [c_1c_2^* F(t)|+\rangle\langle-| + hc], \tag{14}$$

where $F(t) = \int\Psi_+(z,t)\Psi_-^*(z,t)\mathrm{d}z$ is the decoherence factor. We can see from eq. (14) that the off-diagonal elements of reduced density matrix for spin degree of freedom depend on the overlapping part of spatial wave-packets. Therefore, the extent of coherence depends totally on the overlapping integral. Using the Gaussian formula, we can explicitly integrate it by obtaining its norm

$$|F(t)| = \exp\left[-\frac{\hbar^2 f^2}{8a^2m^2(4a^4m^2 + t^2\hbar^2)}t^6 - \frac{a^2f^2}{8a^4m^2 + 2t^2\hbar^2}t^4 - \frac{2a^2f^2}{\hbar^2}t^2\right]. \tag{15}$$

It is obvious that the decoherence process in the SG experiment does not obey the exponential law $e^{-\gamma t}$. The character time of this process can not be simply determined as it is defined to be $\tau_d = 1/\gamma$ for the usual exponential process. It can be seen from eq. (15) that on a very short time scale, i.e. $t \ll \frac{ma^2}{\hbar}$, we have

$$|F(t)| \sim \exp\left[-\frac{\hbar^2 f^2}{32a^4m^4}t^6 - \frac{f^2}{8a^2m^2}t^4 - \frac{2a^2f^2}{\hbar^2}t^2\right]$$
$$\sim \exp\left[-\frac{2a^2f^2}{\hbar^2}t^2\right] \sim 1 - \alpha t^2. \tag{16}$$

This Gaussian form just implies the quantum Zeno effect. On a large time scale, i.e. $t \gg \frac{ma^2}{\hbar}$, we have

$$F(t) \sim \exp\left[-\frac{f^2}{8a^2m^2}t^4 - \frac{f^2}{16\hbar^2}t^2 - \frac{2a^2f^2}{\hbar^2}t^2\right]$$
$$\sim -\frac{f^2}{8a^2m^2}t^4. \tag{17}$$

It shows that the temporal behavior of decoherence is rather complicated in many cases. Like that of quantum dissipation[16], it usually deviates from the ordinary exponential decay.

4 Decoherence in polarization measurement

The important effect of decoherence or wave packet collapse is its influence on the measurement of expectation value of an observable quantity. As to a system of complete decoherence, its quantum ensemble average has the feature of classical probability. Suppose the probabilities on states $|1\rangle$, $|2\rangle, \cdots, |n\rangle$ are $W_1, W_2, \cdots, W_M$ respectively, the quantum ensemble average of A,

$$\overline{A} = \sum_{n=1}^{M} W_n\langle n|A|n\rangle \tag{18}$$

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has no relation with the off-diagonal elements of A. This process can be described by a totally diagonalized density matrix $\rho = \sum W_n | n\rangle\langle n |$. However, if the system is in a pure state $|\Phi\rangle = \sum_{n=1}^{M} \sqrt{W_n} | n\rangle$, the probability is non-classical and the system has the same probability W_n in state $|n\rangle$ as above. This is because the quantum average of A,

$$\langle A\rangle = \langle\Phi | A |\Phi\rangle = \sum_{n=1}^{M} W_n \langle n | A | n\rangle + \sum_{m \neq n} \sqrt{W_n W_m} \langle m | A | n\rangle \tag{19}$$

has something to do with the off-diagonal elements.

Now we will show that the transition from quantum probability to classical probability appears to be in a dynamical way for the polarization measurement of the SG experiment. Its mathematical essence is due to the wave packet collapse described in section 1. We consider the polarization measurement on the state $\Phi(z, t) = c_1 \Psi_+(z,t) | +\rangle + c_2 \Psi_-(z,t) | -\rangle$ at time t. The spin operator of atom along direction $\boldsymbol{n} = (\sin\theta, 0, \cos\theta)$ is

$$\hat{P} = \boldsymbol{n} \cdot \boldsymbol{\sigma} = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$$

$$= \sin\theta(| +\rangle\langle - | + | -\rangle\langle + |) + \cos\theta(| +\rangle\langle + | - | -\rangle\langle - |). \tag{20}$$

In this experiment, all the atoms in spatial points should be accumulated in principle. So the expectation value $\overline{P} = \mathrm{Tr}(\hat{P} | \Phi(t)\rangle\langle\Phi(t) |)$ of $\hat{P}$ on state $|\Phi(t)\rangle$ can be derived from the reduced density matrix $\rho_s(t)$, i.e.

$$\overline{P} = \mathrm{Tr}_s(\rho_s(t)\hat{P})$$

$$= \cos\theta(|c_1|^2 - |c_2|^2) + \sin\theta(c_1 c_2^* + c_1^* c_2) F(t), \tag{21}$$

where Tr_s denotes the trace over the spin degree of freedom. The second term of eq. (21) reflects the contribution of off-diagonal elements of $\hat{P}$, which is proportional to the overlap integral $F(t)$ of spatial wave function. When the passing time of atoms through magnetic field is long enough, or the magnetic field is inhomogeneous sufficiently, i.e. $\partial B(z)/\partial t$ is large enough, we get $F(t) \to 0$, and in this sense the atom polarization

$$\overline{P} = |c_1|^2 \langle + | \hat{P} | +\rangle + |c_2|^2 \langle - | \hat{P} | -\rangle$$

$$= \cos\theta(|c_1|^2 - |c_2|^2) \tag{22}$$

depends only on the classical survival probability $|c_1|^2$ and $|c_2|^2$. This description strongly shows that an effective measurement about spin states will cause decoherence process for the polarization measurement and results in the transition from quantum average to classical average.

5 Discussion

In this paper, the decoherence in SG experiment is analyzed comprehensively. The deviation

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from the exponential decay law in temporal process of decoherence is shown through exactly solvable model. This kind of phenomenon reflects the complexity of a quantum irreversible process. The brief discussion on measuring the expectation value of polarization vector predicts the possible testing of von Neumann's postulate of wave packet collapse in a real experiment. Though the quantum Zeno effect verifies this postulate to some extend in both theory and experiment, there are still some disagreements and questions concerning the complex interaction. Therefore, it is sensible to seek a more concise and direct way of testing this postulate.

The factorization structure has been shown to be an essence of various dynamical models for quantum measurement process, but as illustrated in this paper, there still exists other dynamic realization of quantum measurement process beyond this structure. Therefore, perhaps the macroscopically observable property is a basic condition to cause wave packet collapse, which does not depend on the details, such as "factorization structure".

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