Cyclic Representations of q-Boson Algebra and their Applications to Quantum Algebras and Quantum Superalgebras (*).

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Summary. — In this paper the representation theory of q-boson algebra in the non-generic case that q is a root of unity is studied from a purely mathematical point of view. Emphasis is placed on its cyclic representations. Various cyclic representations of the q-boson algebra are thereby constructed in explicit form. Using them, we obtain the cyclic representations of quantum superalgebras $U_q\, osp(1, 2)$, $U_q\, osp(1, 4)$ and quantum algebra $(C_2)_q$ through the q-deformed-boson realization method.

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1. - Introduction.

At present much progress in the representation theory of quantum algebras and quantum superalgebras [1-4] has been made for both the generic case that q is not a root of unity [5, 6] and the non-generic case that q is a root of unity [7-15]. For the latter, a completely new class of representations, cyclic representations, of quantum algebras (superalgebras) is constructed by different authors. In the cyclic representation, the generators beyond the Cartan sector are not nilpotent and thus it is not a highest-weight or lowest-weight representation. More recently, we generalized the q-boson realization theory [16-18] in the generic case to construct cyclic representations for $sl_q(2)$ and $sl_q(3)$. However, up to now, there is not a quite general theory to answer the foundamental questions in the representation of the

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q-boson algebra B_q in the *non-generic* case: When does a finite-dimensional representation (FDR) of B_q exist? How many kinds of irreducible FDR are there for B_q? One of the main purposes in this paper is to build such a theory. Thereby, we also give the explicit constructions of irreducible FDRs for B_q. Using them, we explicitly obtain the cyclic representations of quantum superalgebras $A = U_q osp(1, 2)$, $A' = U_q osp(1, 4)$ and quantum algebra $(C_2)_q$ through the q-boson realization method for the first time.

Notice that our obtained FDRs of the q-boson algebra B_q include its cyclic representations and the obtained cyclic FDRs of A, A', and $(C_2)_q$ may be probably used to construct new R-matrices of the Yang-Baxter equation in connection with the generalized Potts models according to the recent works of Date *et al.* [13].

2. – Cyclic representation and other irreducible FDRs of the q-boson algebra.

The q-boson algebra B_q is an associative algebra over the complex-number field $\boldsymbol{C}$ by a^+, $a^- = a$ and $Q^\pm$ satisfying

$$(2.1)\qquad \begin{cases} a^+ a - q^{\mp 1} a^+ a = Q^\pm , & Q^+ Q^- = Q^- Q^+ = 1 , \\ Q^+ a^\pm = q^{\pm 1} a^\pm Q^+ , & Q^- a^\pm = q^{\mp 1} a^\pm Q^- , \end{cases} \qquad q \in \boldsymbol{C}.$$

Formally, if we write $Q^\pm = q^{\pm \hat{N}}$, then the above relations are rewritten as follows:

$$(2.2)\qquad aa^+ = [\hat{N} + 1], \qquad a^+ a = [\hat{N}], \qquad [\hat{N}, a^\pm] = \pm a^\pm ,$$

where $[f] = (q^f - q^{-f})/(q - q^{-1})$. Notice that the relations (2.1) or (2.2) are just satisfied by the original q-boson operators in ref. [16-20] *constrained on the* q-*Fock space.* Now, we are trying to prove

Proposition 1. Only when q is a root of unity, the FDR of B_q exists.

Proof. Let V be the carrier space of an FDR ρ of B, *i.e.* $\dim V < \infty$. As the field $\boldsymbol{C}$ is algebraically closed and Q^+ commutes with Q^-, there must exist a non-zero vector such that

$$Q^\pm f_0 = q^{\pm\lambda} f_0 , \qquad \lambda \in \boldsymbol{C},$$

where we formally denote $\rho(x)$ by x for $x \in B_q$. The vectors

$$f_0,\ a^+ f_0,\ a^{+2} f_0,\ \dots,\ a^{+k} f_0,\ \dots$$

are the eigenvectors of Q^+ corresponding to the eigenvalues

$$q^\lambda,\ q^{\lambda+1},\ q^{\lambda+2},\ \dots,\ q^{\lambda+k},\ \dots .$$

If q is not a root of unity, they are linearly independent when they are non-zero. This is because they correspond to different eigenvalues in this case. Due to $\dim V < \infty$, there must be $l \in \boldsymbol{Z}^+ = \{0, 1, 2, \dots\}$ such that $(a^+)^l f_0 = 0$ but $(a^+)^{l-1} f_0 \neq 0$. Similarly, for the vectors

$$u_0 = a^{+l-1} f_0, \qquad u_1 = au_0, \qquad u_2 = a^2 u_0, \qquad \dots \qquad u_k = a^k u_0,\ \dots,$$

there must exist l' such that $a^{l'}u_0 = 0$ but $a^{l'-1}u_0 \neq 0$. Then

$$0 = aa^{+l} f_0 = [N+1]a^{+l-1} f_0 = [\lambda + l]u_0 ,$$

$$0 = a^+ \, a^{l'} u_0 = [N] a^{l'-1} a^{+l-1} f_0 = [\lambda + l - l'] u_{l'-1} ,$$

that is to say

$$[\lambda + l - l'] = 0, \qquad [\lambda + l] = 0$$

or $q^{2l'} = 1$. So, q must be a root of unity. The proof ends.

According to the above proposition, the focus to study the FDRs of $B_{\rm q}$ must be placed on the non-generic case. In the following discussions, we suppose that q is a primitive p-th root of 1, *i.e.* $q^p = 1$. With the help of a direct calculation and eqs. (2.1) or (2.2), we have

Lemma 1. $(Q^{\pm})^{kp}$ and $(a^{\pm})^{kp}$ belong to the centre of $B_{\rm q}$ at $q^p = 1$, $k \in \boldsymbol{Z}^+ = \{0, 1, 2, \dots\}$.

Therefore, due to Schur's lemma, it is reasonable to find the cyclic representation ρ of $B_{\rm q}$ in which

$$(\rho(a^{\pm}))^{kp} = \xi_{\pm}(k) I ,$$

where I is a unit matrix and $\xi_{\pm}(k) \in \boldsymbol{C}$. When $k = 1$, we denote $\xi_{\pm}(1)$ by $\xi_{\pm}$. Let us state a central result in this paper.

Proposition 2. i) If ξ_+ or $\xi_- \neq 0$, there exist an irreducible FDR of $B_{\rm q}$ which is cyclic, *i.e.* $(a^+)^p$ or $(a^-)^p \neq 0$, where p is also the dimension of the representation.

ii) If $\xi_+ = \xi_- = 0$, there exists a p- or $(1/2)p$-dimensional irreducible FDR of $B_{\rm q}$.

Proof. i) Let V be the carrier space of an irreducible FDR of $B_{\rm q}$. If $\dim V > p$, $a^{+p} = \xi_+ I \neq 0$ implies that the vectors $f_n = a^{+n} f_0 (0 \leqslant n \leqslant p-1) \neq 0$ and so they are linearly independent. Let us try to prove that they span a $B_{\rm q}$-invariant subspace that is irreducible. In fact, due to

$$a^+ \, f_{p-1} = a^{+p} f_0 = \xi_+ f_0 \equiv \xi f_0 ,$$

$$\xi a f_0 = aa^+ \, f_{p-1} = [N+1] a^{+p-1} f_0 = [\lambda] f_{p-1} ,$$

we have the actions of $B_{\rm q}$ on $\{f_n\}$:

$$\begin{cases} a^+ f_n = f_{n+1} , & 0 \leqslant n \leqslant p-2; \\ a^+ f_{p-1} = \xi f_0 , & \\ a f_n = [n+\lambda] f_{n-1} , & 1 \leqslant n \leqslant p-1, \\ a f_0 = [\lambda] \xi^{-1} f_{p-1} , & \\ Q^{\pm} f_n = q^{\pm(\lambda+n)} f_n . & \end{cases} \tag{2.3}$$

They define a p-dimensional representation on an invariant subspace $V(\xi, \lambda)$: $\{f_m \mid 0 \leqslant m \leqslant p-1\}$. It is contradictory to the irreducibility of V.

If $\dim V < p$, there exists $l < p-1$ such that $f_0, f_1, f_2, \ldots, f_{l-1}$ are linearly independent, but $f_0, f_1, \ldots, f_{l-1}, f_l$ are linearly dependent. Since the eigenvalue of f_l is different from those of $f_n (0 \leqslant n \leqslant l-1 \leqslant p-1)$, $f_l = 0$. Thus,

$$\xi f_0 = a^{+p} f_0 = (a^+)^{p-l}(a^+)^l f_0 = 0$$

or $\xi = 0$ and the contradiction appears. Part i) of the proposition is proved.

ii) If $\dim V < p$, there exists $m(<p) \in \boldsymbol{Z}^+$ such that $a^{+m} f_0 = 0$, but $a^{+m-1} f_0 = u_0 \neq 0$. Similarly, there exists m' such that $a^{m'} u_0 = 0$ but $a^{m'-1} u_0 \neq 0$. Then, the following equations:

$$\text{(2.4)} \qquad \begin{cases} a^+ u_0 = 0; \qquad a^+ u_n = [\lambda + m - 1] u_{n-1}, & n \neq 0, \\ a u_n = u_{n+1}, \qquad (n \neq 0), \qquad a u_{m'-1} = 0, \\ Q^{\pm} u_n = q^{\pm(\lambda + m - n - 1)} u_n, \end{cases}$$

define an $m'(<p)$-dimensional subrepresentation on $\{u_n = a^n u_0 \mid 0 \leqslant n \leqslant m'-1\}$. However,

$$0 = a^+ a^{m'} u_0 = [\lambda + m - m'] u_{m'-1},$$

$$0 = a a^{+m} f_0 = [\lambda + m] u_0,$$

that is to say, $m = (1/2)p$ for even p or 0 for odd p. A similar discussion shows that there does not exist an irreducible representation with dimension larger than p.

In the irreducible FDR (2.3) constructed above, $a^{+p} = \xi_+ I$ and $a^p = \xi_- I = \xi^{-1}[\lambda][\lambda+1]\ldots[\lambda+p-1] I$. So we call the FDR (2.3) cyclic representation. However, the FDR (2.4) is a highest-weight representation, which is equivalent to that given in ref. [14].

3. – Higher-dimensional cyclic representation of B_q.

In this section we construct a class of cyclic representations, in which $(a^{\pm})^{\alpha p} = \xi_{\pm}(\alpha)$ $(\alpha \in \boldsymbol{Z}^+)$ but $(a^{\pm})^p$ are not multiples of a unit matrix. To this end, we consider a subspace W generated by an eigenvector $|\lambda\rangle$ of $Q^{\pm}$, which satisfies

$$\text{(3.1)} \qquad Q^{\pm} |\lambda\rangle = q^{\pm\lambda} |\lambda\rangle, \qquad \lambda \in \boldsymbol{C}.$$

If W carries the above-mentioned representations, its basis can be chosen as

$$|0(\lambda)\rangle \equiv |0\rangle \equiv |\lambda\rangle, \quad |1(\lambda)\rangle = a^+ |\lambda\rangle, \quad |2(\lambda)\rangle = a^{+2} |\lambda\rangle, \ldots, |m(\lambda)\rangle = a^{+m} |\lambda\rangle, \ldots.$$

Obviously, they are degenerate as the eigenvectors of $Q^{\pm}$ when $\alpha \geqslant 2$. In the following discussion, we denote W by $W(\xi(\alpha), \lambda)$ for $\alpha \geqslant 2$.

Suppose that $\xi(\alpha) \equiv \xi_+(\alpha) \neq 0$. Considering the following relations:

$$[\lambda]\,|\alpha p - 1\rangle = aa^+\,|\alpha p - 1\rangle = \xi(\alpha)\,a\,|\alpha p - 1\rangle,$$

we write down an αp-dimensional cyclic representation $\pi_{\xi(\alpha),\lambda}$ of B_q:

$$\begin{cases} a^+\,|n(\lambda)\rangle = |n+1(\lambda)\rangle, & 0 \leq n \leq \alpha p - 2, \\ a^+\,|p-1(\lambda)\rangle = \xi(\alpha)\,|0(\lambda)\rangle, & \\ a\,|n(\lambda)\rangle = [n+\lambda]\,|n-1(\lambda)\rangle, & \\ a\,|0(\lambda)\rangle\,\xi(\alpha)^{-1}[\lambda]\,|\alpha p - 1(\lambda)\rangle, & \\ Q^{\pm}\,|n(\lambda)\rangle = q^{\pm(\lambda+n)}\,|n(\lambda)\rangle. & \end{cases} \tag{3.2}$$

It follows from eqs. (3.3) that

$$(a^+)^{\alpha p} = \xi(\alpha)\,I,$$

$$(a^-)^{\alpha p} = \xi_{-1}(\alpha)\,I \equiv \xi(\alpha)^{-1}[\lambda][\lambda+1]\dots[\lambda+\alpha p-1]\,I.$$

According to Schur's lemma, this representation is reducible because the central elements $(a^{\pm})^p$ are not multiples of the unit matrix. In fact, with the help of a direct calculation, we observe that the p linearly independent vectors

$$f(m) = \sum_{l=0}^{\alpha-1} \eta^{-l}\,|lp + m(\lambda)\rangle, \qquad 0 \leq m \leq p-1 \tag{3.3}$$

span a B_q-invariant subspace for $\eta^\alpha = \xi(\alpha)$, *on which the irreducible representation*

$$\begin{cases} a^+ f(m) = f(m+1), & 0 \leq m \leq p-2; \\ a^+ f(p-1) = \eta f(0), & \\ af(m) = [m+\lambda]\,f(m+1), & m \neq 0, \\ af(0) = \eta^{-1}[\lambda]\,f(p-1), & \\ Q^{\pm} f(m) = q^{\pm(\lambda+m)}\,f(m), & \end{cases}$$

is isomorphic to that carried by $V = W(\xi(\alpha = 1), \eta)$.

4. – Elaborations on the q-boson realization.

The q-boson realization theory of quantum algebras is a q-analogue of the boson realization of Lie algebras that is also called Jordan-Schwinger mapping (for Lie algebra of $SU(2)$) [21]. In order to construct cyclic representations of quantum algebras and superalgebras, we reformulate it in a general framework and then give its elaboration for the cyclic-representation case.

Let A and S be two given associative algebras, and ρ: $S \to \mathrm{End}(V)$ representation of S. If there exists a homomorphism φ: $A \to S$, then a representation $\bar{\rho}$: $A \to \mathrm{End}(V)$ of A can be constructed through the following commutative diagram:

$$\begin{tikzcd} A \arrow[r, "\varphi"] \arrow[dr, "\bar{\rho}"'] & S \arrow[d, "\rho"] \\ & \mathrm{End}\,(V) \end{tikzcd}$$

from the representation ρ of S, namely

$$\bar{\rho}(g)\,x = \rho(\varphi(g))\,x\,, \qquad \forall g \in A\,, \quad x \in V\,. \tag{4.1}$$

Usually, S is chosen to be «simpler» than A. We call $\varphi(A)$ and $\bar{\rho} = \rho \circ \varphi$ S-realizations of A and its representation, respectively. When S is taken to be Heisenberg-Weyl (HW) algebra, the algebra of operators x and $\mathrm{d}/\mathrm{d}x$, and the super HW algebra, respectively, the above theory specifies the boson realization of the universal enveloping algebra (UEA) A of a Lie algebra [22, 23], the differential realization of the UEA A of a Lie algebra [24], and the UEA A of a Lie superalgebra [25].

The focus of this paper is placed on the q-boson realization of quantum algebras and superalgebras. In this sense, S is the many-state q-boson algebra $B_{\mathrm{q}}(N)$ generated by

$$g_i = \overbrace{1 \otimes 1 \otimes \ldots \otimes 1}^{i-1} \otimes g \otimes 1 \otimes \ldots \otimes 1\,,$$

where $g = a^{\pm}$ and $Q^{\pm}$; A is a quantum algebra or quantum superalgebra. The representation of $B_{\mathrm{q}}(N)$ used in constructing representation of A is defined by

$$\rho(g_i) = \underset{1,}{1} \otimes \underset{2,}{1} \ldots \otimes \underset{i-1,}{1} \otimes \underset{i,}{\tau(g)} \otimes \underset{i+1,}{1} \otimes \underset{\cdots,}{\ldots} \otimes \underset{N}{1}\,, \qquad \tau(g) \in \mathrm{End}\,V(\xi_i, \lambda_i)$$

where τ is the cyclic representation (2.3). The product space

$$V = V(\xi_1, \lambda_1) \otimes V(\xi_2, \lambda_2) \otimes \ldots \otimes V(\xi_i, \lambda_i) \otimes \ldots \otimes V(\xi_N, \lambda_N)$$

is thus the carrier space of ρ. Usually, the q-boson realization of A with generators $\{x_i\}$ is formally expressed as

$$\hat{\chi}_2 = \varphi(\chi_2) = \sum_{m_i,\, n_i;\ \pm r_i \in \boldsymbol{Z}^+} C_{\{m_i,\, n_i,\, r_i\}} \prod_{i=1}^{N} a_i^{+m_i} a_i^{n_i} Q_i^{r_i}\,, \tag{4.2}$$

where $Q^r = (Q^+)^r$ for $r \geqslant 0$ and $= (Q^-)^{-r}$ for $r < 0$ and $C_{\{m_i,\, n_i,\, r_i\}}$ are the coefficients to be determined. Correspondingly, the q-boson realization of the representation of A is defined by

$$\bar{\rho}(\chi_i) = \sum_{m_i,\, n_i;\ \pm r_i \in \boldsymbol{Z}^+} C_{\{m_i,\, n_i,\, r_i\}} \prod_{i=1}^{N} \rho(a_i^+)^{m_i} \rho(a_i)^{n_i} \rho(Q_i)^{r_i}\,. \tag{4.3}$$

As shown in the following discussion, the representations constructed through eqs. (4.3) are cyclic for the well-chosen q-boson realizations φ. The key to our construction is that the cyclic representation (2.3) is used.

5. – Cyclic representations of quantum superalgebras $U_q osp(1,2)$ and $U_q osp(1,4)$.

Following the definition of $U_q osp(1,2)$ in ref. [26], we first write down the algebraic relations of generators $V_\pm$ and $K^\pm(=q^{\pm H})$ for $Y_q osp(1,2)$

$$\begin{cases} V_+V_- + V_-V_+ = -\dfrac{1}{4}[2H], \\ K^+V_\pm = q^{\pm 1}V_\pm K^+, \qquad K^-V_\pm = q^{\pm 1}V_\pm K^-. \end{cases} \tag{5.1}$$

Its q-boson realization was given in ref. [27] as follows:

$$\begin{cases} \varphi\colon\; U_q\, osp(1,2) \to B_q \\ V_\pm \mapsto \hat{V}_\pm = \varphi(\hat{V}_\pm) = i\sqrt{\left[\dfrac{1}{2}\right]}a^\pm, \\ H \mapsto \hat{H} = \varphi(\hat{H}) = \dfrac{1}{2}\left(\hat{N} + \dfrac{1}{2}\right). \end{cases} \tag{5.2}$$

Then, we obtain an αp-dimensional representation $\hat{\pi}(\xi(\alpha), \lambda)$,

$$\begin{cases} \hat{V}_+\,|n(\lambda)\rangle = \dfrac{i}{2}\sqrt{\left[\dfrac{1}{2}\right]}\,|n+1(\lambda)\rangle, & 0 \leqslant n \leqslant \alpha p - 1, \\ \hat{V}_+\,|\alpha p - 1(\lambda)\rangle = \dfrac{i}{2}\sqrt{\left[\dfrac{1}{2}\right]}\,\xi(\alpha)\,|0(\lambda)\rangle, & \\ \hat{V}_-\,|n(\lambda)\rangle = \dfrac{i}{2}\sqrt{\left[\dfrac{1}{2}\right]}\,[\lambda+n]\,|n-1(\lambda)\rangle, & 1 \leqslant n \leqslant \alpha p - 1, \\ \hat{V}_-\,|0(\lambda)\rangle = \dfrac{i}{2}\sqrt{\left[\dfrac{1}{2}\right]}\,\xi(\alpha)^{-1}[\lambda]\,|\alpha p - 1(\lambda)\rangle, & \\ \hat{K}^\pm\,|n(\lambda)\rangle = q^{\pm}\left(n+\lambda+\dfrac{1}{2}\right)|n(\lambda)\rangle. & \end{cases} \tag{5.3}$$

of $U_q osp(1,2)$. Obviously, it is a cyclic representation because

$$(\hat{V}_+)^{\alpha p} = \left(\frac{i}{2}\sqrt{\left[\frac{1}{2}\right]}\right)^{\alpha p}\xi(\alpha)I, \qquad (\hat{V}_-)^{\alpha p} = \left(\frac{i}{2}\sqrt{\left[\frac{1}{2}\right]}\right)\xi(\alpha)^{-1}[\lambda][\lambda+1]\dots[\lambda+\alpha p-1].$$

When $\alpha \geqslant 2$, it is reducible and $\{f(m)\,|\,0 \leqslant m \leqslant p-1\}$ is an invariant subspace carrying an irreducible cyclic representation equivalent to $\hat{\pi}(\xi(1), \lambda)$ defined by (5.3).

Applying the same method, we obtain the p^2-dimensional cyclic representation of $U_q osp(1,4)$ [28, 29]:

$$
(5.4)\quad \begin{cases}
e_1 F(m_1, m_2) = [m_2 + \lambda_2] F(m_1+1, m_2-1), & 0 \leqslant m_1 \leqslant p-2,\ 1 < m_2 \leqslant p-1, \\
e_1 F(p-1, m_2) = [m_2 + \lambda_2] \xi_1 F(0, m_2-1), & 1 \leqslant m_2 \leqslant p-1, \\
e_1 F(m_1, 0) = \xi_2^{-1} [\lambda_2] F(m_1+1, p-1), & 0 \leqslant m_1 \leqslant p-2, \\
e_1 F(p-1, 0) = \xi_2^{-1} [\lambda_2] \xi_1 F(0, p-1); & \\
f_1 F(m_1, m_2) = [m_1 + \lambda_1] F(m_1-1, m_2+1), & 1 \leqslant m_1 \leqslant p-1,\ 0 \leqslant m_2 \leqslant p-2, \\
f_1 F(m_1, p-1) = [m_1 + \lambda_1] \xi_2 F(m_1-1), & 1 \leqslant m_1 \leqslant p-1, \\
f_1 F(0, m_2) = \xi_1^{-1} [\lambda_1] F(p-1, m_2+1), & 0 \leqslant m_2 \leqslant p-2, \\
f_2 F(0, p-1) = \xi_1^{-1} [\lambda_1] \xi_2 F(p-1, 0); & \\
e_2 F(m_1, m_2) = \sqrt{\left[\frac{1}{2}\right]} F(m_1, m_2+1), & 0 \leqslant m_2 \leqslant p-1, \\
e_2 F(m_1, p-1) = \sqrt{\left[\frac{1}{2}\right]} \xi_2 F(m_1, 0), & \\
f_2 F(m_1, m_2) = \sqrt{\left[\frac{1}{2}\right]} [m_2 + \lambda_2] F(m_1, m_2-1), & 1 \leqslant m_2 \leqslant p-1, \\
f_2 F(m_1, 0) = \sqrt{\left[\frac{1}{2}\right]} \xi_2^{-1} [\lambda_2] F(m_1, p-1), & \\
K_1^{\pm} F(m_1, m_2) = q^{\pm(m_1 - m_2)} F(m_1, m_2), & \\
K_2^{\pm} F(m_1, m_2) = q^{\pm(1/2 + m_2)} F(m_1, m_2); &
\end{cases}
$$

from its q-boson realization [28, 29]:

$$
(5.5)\qquad \begin{cases}
\hat{e}_1 = a_1^+ a_2, & \hat{f}_1 = a_2^+ a_1, \\
\hat{e}_2 = \sqrt{\left[\frac{1}{2}\right]} a_2^+, & \hat{f}_2 = \sqrt{\left[\frac{1}{2}\right]} a_2, \\
K_1^{\pm} = Q_1^{\pm} Q_2^{\mp}, & K_2^{\pm} = q^{\pm 1/2} Q_2^{\pm},
\end{cases}
$$

on the space $V = V(\xi_1, \lambda_1) \otimes V(\xi_2, \lambda_2)$:

$$
\{F(m_1, m_2) = f_{m_1} \otimes f_{m_2} \mid f_{m_1} \in V(\xi_1, \lambda_1),\ f_{m_2} \in V(\xi_2, \lambda_2)\}.
$$

Notice that the above results (5.3) and (5.4) are obtained for this first time to our best knowledge and a similar method can be used to construct cyclic representations of other quantum superalgebras through their explicit realizations, some of which are given in ref. [29].

6. – Cyclic representations of $(C_2)_q$.

In this section, we discuss the q-boson realization of cyclic representations of quantum algebra $(C_2)_q$ with generators E_1, E_2, F_1, F_2, H_1 and H_2 satisfying

$$(6.1)\quad \begin{cases} [H_1, E_1] = E_1\,, & [H_1, E_2] = -2E_2\,, \quad [H_1, H_2] = 0\,,\\ [H_2, E_1] = -E_1\,, & [H_2, E_2] = E_2\,,\\ [H_1, F_1] = -F_1\,, & [H_1, F_2] = 2E_2\,,\\ [H_2, F_1] = E_1\,, & [H_2, F_2] = -F_2\,,\\ [E_1, F_2] = 0\,, & [E_2, F_1] = 0\,,\\ [E_1, F_1] = [H_1]\,, & [E_2, F_2] = [H_2]\,,\\ G_1^3 G_2 - [3]_{q^2} G_1^2 G_2 G_1 + [3]_{q^2} G_1 G_2 G_1^2 - G_2 G_1^3 = 0\,, & \left([f]_t = \dfrac{t^f - t^{-f}}{t - t^{-1}}\right),\\ G_2^2 G_1 - (q + q^{-1}) G_2 G_1 G_2 + G_1 G_2^2 = 0\,, & G = E, F\,. \end{cases}$$

Using its q-boson realization [19],

$$(6.2)\quad \begin{cases} \hat{E}_1 = a_1^+ a_2\,, \quad \hat{F} = a_2^+ a_1\,, \quad \hat{H}_1 = \hat{N}_1 - \hat{N}_2\,, \quad \hat{K}_1^\pm = Q_1^\pm Q_2^\mp\,,\\ \hat{E}_2 = \dfrac{1}{[2]} a_2^{+2}\,, \quad \hat{F}_2 = -\dfrac{1}{[2]} a_2^2\,, \quad \hat{H}_2 = N_2 + \dfrac{1}{2}\,, \quad \hat{K}_2^\pm = Q_2^\pm q^{\pm 1/2}\,, \end{cases}$$

we obtain its cyclic representation

$$(6.3a)\quad \begin{cases} \hat{E}_1 |m, n) = [\mu + n]\, |m+1, n-1)\,, & m \neq p-1\,,\ n \neq 0\,,\\ \hat{E}_1 |p-1, n) = \xi[\mu + n]\, |0, n-1)\,, & n \neq 0\,,\\ \hat{E}_1 |m, 0) = \eta^{-1}[\mu]\, |m+1, p-1)\,, & m \neq p-1\,,\\ \hat{E}_1 |p-1, 0) = \xi\eta^{-1}[\mu]\, |0, p-1)\,, & \\ \hat{F}_1 |m, n) = [\lambda + m]\, |m-1, n+1)\,, & m \neq 0\,,\ n \neq p-1\,,\\ \hat{F}_1 |m, p-1) = \eta^{-1}[\lambda + m]\, |m-1, 0)\,, & m \neq 0\,,\\ \hat{F}_1 |0, n) = \xi^{-1}[\lambda]\, |p-1, n+1)\,, & \\ \hat{F}_1 |0, p-1) = \eta\xi^{-1}[\lambda]\, |p-1, 0)\,, & \\ \hat{K}_1^\pm |m, n) = q^{\pm(m-n+\lambda-\mu)} |m, n)\,; & \end{cases}$$

$$
(6.3b)\quad \begin{cases}
\hat{E}_2\,|m, n) = \dfrac{1}{[2]}\,|m, n+2)\,, & 0 \leqslant n \leqslant p-3\,,\\[2mm]
\hat{E}_2\,|m, p-1) = \dfrac{1}{[2]}\xi\,|m, 1)\,, & \\[2mm]
\hat{E}_2\,|m, p-2) = \dfrac{1}{[2]}\xi\,|m, 0)\,, & \\[2mm]
\hat{F}_2\,|m, n) = -\dfrac{1}{[2]}[\mu+n][\mu+n-1]\,|m, n-2)\,, & 2 \leqslant n \leqslant p-1\,,\\[2mm]
\hat{F}_2\,|m, 0) = -\dfrac{1}{[2]}[\mu][\mu-1]\,|m, p-2)\,, & \\[2mm]
\hat{F}_2\,|m, 1) = -\dfrac{1}{[2]}[\mu+1][\mu]\,|m, p-1)\,, & \\[2mm]
\hat{K}_2^{\pm}\,|m, n) = q^{\pm(n+1/2+\mu)}\,|m, n) &
\end{cases}
$$

on the space $V = V(\xi_1 = \xi, \lambda_1 = \lambda) \otimes V(\xi_2 = \eta, \lambda_2 = \mu)$:

$$\{|m, n) = f_m \otimes f_n \mid f_m \in V(\xi, \lambda),\ f_n \in V(\eta, \mu)\}\,.$$

Now, let us consider the reduction of the representation (6.3). When p is even, $(-1)^{m+n}$ is invariant under the action of the representation and so there exist two invariant subspaces

$$W^{\pm} : \{|m, n) \mid (-1)^{m+n} = \pm 1\}\,.$$

They carry two irreducible cyclic FDRs, respectively, for λ and $\mu \neq 0$. However, when p is odd, E_2 and F_2 mix W^+ and W^- with each other because they respectively change $|m, 0)$ into $|m, 2)$, $|m, 2)$ into $|m, 4) \ldots |m, p-1)$ into $|m, 1)$ and $|m, 1)$ into $|m, 3)$, …, $|m, p-4)$ into $|m, p-2)$, $|m, p-2)$ into $|m, 0)$, *i.e.* E_2 and F_2 mix $|m, 0)$, $|m, 2)$, …, $|m, p-1)$ and $|m, 1)$, $|m, 3)$, …, $|m, p-2)$ due to odd p. In this case, the representation (6.3) is irreducible.

Finally, we discuss the constraint of the representation (6.3) of $(C_2)_q$ on its subalgebra $sl_q(2)$ generated by E_1, F_1 and H_1. Obviously, there are p-invariant subspaces $W(k)\,(0 \leqslant k \leqslant p-1)$:

$$\{f_k(m) = |m, k-m),\ 0 \leqslant m \leqslant k;\ f_k(m) = |m, k+p-m),\ k+1 \leqslant m \leqslant p-1\}$$

of $sl_q(2)$ and

$$V = \sum_{k=0}^{p-1}\!\oplus\, W(k)\,.$$

Then, a p-dimensional cyclic representation of $sl_q(2)$ is obtained on $W(k)$ as follows:

$$
(6.4)\qquad \begin{cases}
E_1 f_k(m) = [m + k - m] f_k(m)\,, & m \neq k\,, p-1\,, \\
E_1 f_k(k) = \eta^{-1}[\mu] f_k(k+1)\,, & \\
E_1 f_{p-1}(p-1) = \xi\eta^{-1}[\mu] f_{p-1}(0)\,, & \\
E_1 f_k(p-1) = \xi[\mu + k] f_k(0)\,, & \\
F_1 f_k(m) = [\lambda + m] f_k(m-1)\,, & m \neq 0,\ \ k;\ \ k \neq p-1\,, \\
F_1 f_k(0) = \xi^{-1}[\lambda] f_k(p-1)\,, & \\
F_1 f_k(k+1) = \eta^{-1}[\lambda + k - 1] f_k(k)\,, & \\
F_1 f_{p-1}(0) = \eta\xi^{-1}[\lambda] f_{p-1}(p-1)\,, & \\
k_1^{\pm} f_k(m) = q^{\pm(2m-k+\lambda-\mu)} f_k(m)\,, & \\
k_2^{\pm} f_k(m) = q^{\pm(k-m+\mu+1/2)} f_k(m)\,. &
\end{cases}
$$

In the above representation, since

$$E_1 f_k(p-1) = \alpha f_k(0)\,, \qquad F_1 f_k(0) = \beta f_k(p-1)$$

and $\alpha = \xi[\mu + k] \neq 0$ and $\beta = \xi^{-1}[\lambda] \neq 0$, a similar discussion shows that it is equivalent to the standard cyclic representation of $sl_q(2)$ given by De Concini and Kac in ref. [12], in which only two parameters are contained. In the representation (6.4), the extra parameters are introduced by the similarity transformation of the basis.

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