New Realization of the Loop Algebras and Their Indecomposable Modules.

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(ricevuto l'8 Febbraio 1989)

Summary. — A new realization of the loop algebra $\hat{G}$ (untwisted affine Kac-Moody algebra) is given on the enveloping field $\bar{\Omega}$ of the Bose algebra $\mathscr{H}$. By making use of this new realization nontrivial infinite-dimensional indecomposable representations and finite-dimensional representations of $\hat{G}$ are constructed on $\bar{\Omega}$ and its quotient spaces. Finally, as an explicit example, the loop algebra $\widehat{SU}(2)$ associated with Lie algebra $SU(2)$ is discussed in detail.

PACS 03.65.Fd – Algebraic methods.

1. – Introduction.

Indecomposable modules of physically relevant Lie algebras have been suggested [1-3] for the description of unstable particles. Gruber and his cooperators have studied indecomposable representations of some physically relevant Lie algebras on their universal enveloping algebras [4-6]. By making use of the method of Bose realization used to study the indecomposable representations of Lie algebra [7] and Lie superalgebras [8], we discussed Virasoro and Kac-Moody algebras [9], which have appeared as a new kind of symmetry algebra in many areas of physics [10].

The Bose algebra $\mathscr{H}$, also called the Heisenberg-Weyl algebra, is defined by

$$(1)\quad \mathscr{H}\colon \{a_i^+, a_i, E \mid i=1,2,\dots,N\}, \qquad [a_i, a_j^+] = \delta_{ij}E, \qquad [a_i^+, E] = [a_i, E] = 0\,.$$

In ref. [9], because the representatives of element E in the indecomposable representations of the Bose algebra are unit matrices, the indecomposable representations of loop algebras constructed in [9] are more trivial. However, it is significant in mathematical physics to consider the representation of $\mathscr{H}$ in which the representative of E is not a unit matrix [11].

In this paper we give firstly a new realization of loop algebra $\hat{G}$ on the enveloping field $\bar{\Omega}$ of $\mathscr{H}$ in sect. 2. Then we construct a class of nontrivial infinite-dimensional indecomposable representations of loop algebra $\hat{G}$ in which the representative of E is not unit matrix on $\bar{\Omega}$ and its quotient spaces in sect. 3. In sect. 4, a class of finite-dimensional (irreducible) representations is induced on quotient spaces of $\bar{\Omega}$. Finally, in sect. 5, as an explicit example, the loop algebra $\widehat{SU}(2)$ associated with Lie algebra $SU(2)$ is discussed in detail.

The symbols $\boldsymbol{N}$, $\boldsymbol{N}^+$, $\boldsymbol{Z}$ denote the set of nonnegative integers, the set of positive integers and the set of all integers, respectively. The symbol $\boldsymbol{C}$ denotes complex number field.

According to PBW theorem, the basis of universal enveloping algebra $\mathfrak{A}(\mathscr{H})$ of the Bose algebra $\mathscr{H}$ can be chosen as

$$\left\{ f(m_i, n_i, n) \equiv \left[\prod_{i=1}^{N} a_i^{+m_i} a_i^{n_i} \right] E \middle| m_i, n_i, n \in \boldsymbol{N} \right\}. \tag{2}$$

Because $\mathfrak{A}(\mathscr{H})$ is an Ore ring, one can introduce the enveloping field Ω of $\mathscr{H}$ (also called Heisenberg field) which incorporates in a natural way, quotients of polynomials of the generators of $(\mathscr{H})$ [1]. $\mathfrak{A}(\mathscr{H})$ is a subring of $\mathscr{H}$. As an algebra, Ω has a subalgebra $\bar{\Omega}$ with the basis

$$\left\{ F(m_i, n_i, n) \equiv \left[\prod_{i=1}^{N} a_i^{+m_i} a_i^{n_i} \right] E^n \middle| m_i, n_i, n \in \boldsymbol{Z} \right\}. \tag{3}$$

It can be regarded as the extension of the basis of $\mathfrak{A}(\mathscr{H})$ to $m_i, n_i, n \in \boldsymbol{Z}$.

2. – New realization of loop algebra G.

Let T be a finite-dimensional faithful representation of Lie algebra G with generators $\{X_\alpha | \alpha = 1, 2, \ldots, N\}$ that satisfy the Lie product

$$[X_\alpha, X_\beta] = \sum_{\nu=1}^{N} C_{\alpha\beta}^{\nu} X_\nu, \tag{4}$$

where complex numbers $C_{\alpha\beta}^{\nu}$ are structure constants. There exists a subalgebra of $\bar{\Omega}$ generated by

$$\left\{ X_\alpha^m \middle| X_\alpha^m \equiv \sum_{j,k=1}^{N} T(X_\alpha)_{jk} a_j^+ a_k E^{m-1},\ m \in \boldsymbol{Z} \right\}. \tag{5}$$

It is easy to prove that

$$[X_\alpha^m, X_\beta^n] = \sum C_{\alpha\beta}^\nu X_\nu^{m+n} \tag{6}$$

which is just the commutation relations of the loop algebra (also called untwisted affine Kac-Moody algebra, or Kac-Moody algebra without central term). Therefore, (5) generates a loop algebra $\hat{G}$ associated with the Lie algebra G. The expression X_α^m is a new Bose realization of loop algebra $\bar{\Omega}$ which is different from that in ref. [9].

For example, for the Pauli's representation of Lie algebra $SU(2)$, we can obtain the Bose realization of the loop algebra $\widehat{SU}(2)$ associated with $SU(2)$

$$\begin{cases} \sigma_+^m = a_1^+ a_2 E^{m-1}, \qquad \sigma_-^m = a_2^+ a_1 E^{m-1}, \\ \sigma_3^m = [a_1^+ a_1 - a_2^+ a_2] E^{m-1} \qquad\qquad (m \in \boldsymbol{Z}), \end{cases} \tag{7}$$

where $\sigma_1, \sigma_2, \sigma_3$ are Pauli's matrices and $\sigma_\pm = \frac{1}{2}(\sigma_1 \pm i\sigma_2)$.

3. – Indecomposable representations of $\hat{G}$.

The enveloping field $\bar{\Omega}$ is regarded as a left-module of $\mathscr{H}$, then indecomposable representation of $\mathscr{H}$ is obtained:

$$\begin{cases} \rho(a_k^+) F(m_i, n_i, n) = F(m_i + \delta_{ik}, n_i, n), \\ \rho(a_k) F(m_i, n_i, n) = F(m_i, n_i + \delta_{ik}, n) + m_k F(m_i - \delta_{ik} + \delta_{ij}, n_i, n+1), \\ \rho(E) F(m_i, n_i, n) = F(m_i, n_i, n+1). \end{cases} \tag{8}$$

Let J be a left ideal of $\bar{\Omega}$ and $\bar{\Omega}/J$ the quotient module, then $\rho(E)$ is not a unit matrix on $\bar{\Omega}/J$ if $(E - \lambda 1) \notin J$.

Making use of the following expression:

$$\Gamma(X_\alpha^m) = \sum_{jk} T(X_\alpha)_{jk} \rho(a_j^+) \rho(a_k) \rho(E)^{m-1}, \tag{9}$$

the indecomposable representation of $\hat{G}$ on $\bar{\Omega}$ is obtained as

$$\Gamma(X_\alpha^m) F(m_i, n_i, n) = \sum_{jk} T(X_\alpha)_{jk} [F(m_i + \delta_{ij}, n_i + \delta_{ik}, m+n-1) + \\ + m_k F(m_i - \delta_{ik} + \delta_{ij}, n_i, m+n)]. \tag{10}$$

The relations $\{a_i - \mu_i 1 | i = 1, \ldots, N, \mu_i \in \boldsymbol{C}\}$ generate a left ideal I of $\bar{\Omega}$. For the quotient space $V=\tilde{\Omega}/I$, a basis can be chosen as

$$V: \{D(m_i,n) \equiv F(m_i, 0, n) \operatorname{Mod} I | m_i, n \in \boldsymbol{Z}\}.$$

The representation (10) induces on V the representation

$$\Gamma(X_\alpha^m) D(m_i, n) = \sum_{jk} T(X_\alpha)_{jk} [\mu_k D(m_i + \delta_{ij}, m+n-1) + m_k D(m_i - \delta_{ik} + \delta_{ij}, m+n)]. \tag{11}$$

1) The case with $\mu_i \neq 0$. It is observed that the value $\left(\sum_{i=1}^N m_i\right)$ cannot decrease in (11), the integer $M \in \boldsymbol{Z}$ defines a $\hat{G}$-invariant subspace V_M of V with basis

$$V_M: \left\{D(m_i, n) | \sum_{i=1}^N m_i \geq M, m_i, n \in \boldsymbol{Z}\right\}.$$

Since there does not exist an invariant complementary subspace for any invariant subspace V_M, the representation on V is indecomposable. It is easy to see that there is an invariant subspace chain

$$\ldots \supset V_{-2} \supset V_{-1} \supset V_0 \supset V_1 \supset V_2 \supset \ldots.$$

The representation on each quotient space $V(M,K) = V_M/V_{M+K}$ with basis $(K \in \boldsymbol{N}^+)$

$$V(MK): \left\{H(m_i,n) \equiv D(m_i, n) \operatorname{Mod} V_{M+K} | M \leq \sum_{i=1}^N m_i \leq M+K-1\right\}$$

is obtained from (11) as

$$\Gamma(X_\alpha^m) H(m_i, n) = \sum_{jk=1}^N T(X_\alpha)_{jk} [\mu_k H(m_i + \delta_{ij}, m+n-1) + m_k H(m_i - \delta_{ik} + \delta_{ij}, m+n)]. \tag{12}$$

The representation (12) is the infinite-dimensional indecomposable representation when $K \geq 2$. In the case of $K=1$, the representation on $V(M,1)$ becomes

$$\Gamma(X_\alpha^m) H(m_i, n) = \sum_{jk=1}^N T(X_\alpha)_{jk} m_k H(m_i - \delta_{ik} + \delta_{ij}, m+n), \tag{13}$$

which is the infinite-dimensional irreducible representation.

2) The case with $\mu_1=\mu_2=\ldots=\mu_N=0$. In the case of $\mu_1=\mu_2=\ldots=\mu_N=0$, the representation (11) becomes

$$\dot{\Gamma}(X_\alpha^m)\,D(m_i,n)=\sum_{jk=1}^{N}T(X_\alpha)_{jk}\,m_k\,D(m_i-\delta_{ik}+\delta_{ij},m+n)\,. \tag{14}$$

It is noted that the value $\left(\sum_{i=1}^{N}m_i\right)$ cannot change in (14), the integer $R\in\boldsymbol{Z}$ defines a $\hat{G}$-invariant subspace $V^{[R]}$ of V with basis

$$V^{[R]}:\left\{D(m_i,n)|\sum_{i=1}^{N}m_i=R,\ m_i,n\in\boldsymbol{Z}\right\}$$

and V can be decomposed as

$$V=\sum_{R\in\boldsymbol{Z}}^{\oplus}V^{[R]}.$$

Thus, the representation (14) on V is complete reducible. The representation subduced on each $V^{[R]}$ is the infinite-dimensional irreducible representation.

4. – Finite-dimensional representations.

The relation $\{E-\lambda 1,1|\lambda\neq 0,\lambda\in\boldsymbol{C}\}$ generates a left ideal J of V. A basis for the quotient space $W=V/J$ can be chosen as

$$W:\{P(m_i)\equiv D(m_i,0)\,\mathrm{Mod}\,J|m_i\in\boldsymbol{Z}\}\,.$$

The representation (11) induces on W a new representation

$$\Gamma(X_\alpha^m)\,P(m_i)=\sum_{jk}T(X_\alpha)_{jk}\,[\mu_k\,\lambda^{m-1}P(m_i+\delta_{ij})+m_k\,\lambda^mP(m_i-\delta_{ik}+\delta_{ij})]\,. \tag{15}$$

When $\mu_1=\mu_2=\ldots=\mu_N=0$, (15) is complete reducible, otherwise, decomposable. From (15), it can be seen that the subspace W^+ spanned by

$$W^+:\{P(m_i)|m_i\in\boldsymbol{N},i=1,\ldots,N\} \tag{16}$$

is a $\hat{G}$-invariant subspace. The representation subduced on W^+ is (15) with the condition $m_1,m_2,\ldots,m_N\in\boldsymbol{N}$.

a) The case with $\mu_i\neq 0$. In the case of $\mu_i\neq 0$, the representation on W^+ is indecomposable. It is noted that $\sum_{i=1}^{N}m_i$ in $P(m_i)$ cannot decrease under the action of the representation Γ of $\hat{G}$, therefore, for $M\in\boldsymbol{N}$, there exists a $\hat{G}$-invariant subspace W_M^+ of W^+ with basis

$$W_M^+ : \left\{ P(m_i) | \sum_{i=1}^{N} m_i \geq M; m_i \in \boldsymbol{N} \right\} \tag{17}$$

and a $\hat{G}$-invariant subspace chain

$$W^+ = W_0^+ \supset W_1^+ \supset W_2^+ \supset \ldots \supset W_M^+ \supset W_{M+1}^+ \supset \ldots .$$

For $K \in \boldsymbol{N}^+$, we can construct quotient spaces $W^+[M, K]$ with basis

$$W^+(M, K) : \left\{ Q(m_i) \equiv P(m_i) \operatorname{Mod} W_{M+K}^+ | M \leq \sum_{i=1}^{N} m_i \leq M + K - 1; m_i \in \boldsymbol{N} \right\}.$$

It is easy to prove that the dimension of $W^+[M, K]$ is

$$\dim W^+[M, K] = \sum_{t=M}^{M+K-1} \frac{(N+t-1)!}{(N-1)!\, t!} . \tag{18}$$

The representation on $W^+[M, K]$ can be obtained from (15):

$$\Gamma(X_\alpha^m) Q(m_i) = \sum_{jk} T(X_\alpha)_{jk} [\mu_k \lambda^{m-1} Q(m_i + \delta_{ij}) + m_k \lambda^m Q(m_i - \delta_{ik} + \delta_{ij})] . \tag{19}$$

In the case of $K \geq 2$, this representation is indecomposable. When $K = 1$, the representation on $W^+[M, 1]$ is

$$P(X_\alpha^m) Q(m_i) = \sum_{jk} T(X_\alpha)_{jk} m_k \lambda^m Q(m_i - \delta_{ik} + \delta_{ij}) \qquad \left(\sum_{i=1}^{N} m_i = M \right) \tag{20}$$

that is an irreducible representation with dimension

$$\dim W^+(M, 1) = \frac{(M+N-1)!}{M!\,(N-1)!} . \tag{21}$$

b) The case with $\mu_1 = \mu_2 = \ldots = \mu_N = 0$. In the case of $\mu_1 = \mu_2 = \ldots = \mu_N = 0$, the representation on W^+ is complete reducible. In fact, W^+ can be decomposed as

$$W^+ = \sum_{R \in \boldsymbol{N}}^{\oplus} W^{+[R]} , \tag{22}$$

where $W^{+[R]}$ is spanned by

$$W^{+[R]} : \left\{ P(m_i) | \sum_{i=1}^{N} m_i = R; R, m_i \in \boldsymbol{N} \right\}. \tag{23}$$

The representation subduced on every $W^{+[R]}$ can be obtained as

$$(24)\qquad \Gamma(X_\alpha^m)P(m_i)=\sum_{jk}T(X_\alpha)_{jk}m_kP(m_i-\delta_{ik}+\delta_{ij})\qquad\left(\sum_{i=1}^{N}m_i=R;\ R,m_i\in\boldsymbol{N}\right),$$

that is an irreducible representation with dimension

$$(25)\qquad \dim W^{+[R]}=\frac{(N+R-1)!}{(N-1)!R!}\,.$$

5. – Representation of loop algebra $\widehat{SU}(2)$.

According to Bose realization (7) of $\widehat{SU}(2)$ and expression (9), the indecomposable representation of $\widehat{SU}(2)$ on the enveloping field $\bar{\Omega}$ of 2-states Bose algebra $\mathscr{H}$

$$(26)\qquad \bar{\Omega}\colon\{F(m_1,m_2,n_1,n_2,n)\equiv a_1^{+m_1}a_2^{+m_2}a_1^{n_1}a_2^{n_2}E^n\,|\,m_1,m_2,n_1,n_2,n\in\boldsymbol{Z}\}$$

can be obtained:

$$(27)\qquad \begin{cases}\Gamma(\sigma_+^m)F(m_1,m_2,n_1,n_2,n)=F(m_1+1,m_2,n_1,n_2+1,n+m-1)+\\ \qquad\qquad +m_2F(m_1+1,m_2-1,n_1,n_2,m+n)\,,\\ \Gamma(\sigma_-^m)F(m_1,m_2,n_1,n_2,n)=F(m_1,m_2+1,n_1+1,n_2,n+m-1)+\\ \qquad\qquad +m_1F(m_1-1,m_2+1,n_1,n_2,m+n)\,,\\ \Gamma(\sigma_3^m)F(m_1,m_2,n_1,n_2,n)=F(m_1+1,m_2,n_1+1,n_2,m+n-1)-\\ -F(m_1,m_2+1,n_1,n_2+1,m+n-1)+(m_1-m_2)F(m_1,m_2,n_1,n_2,m+n)\,.\end{cases}$$

Firstly, let us discuss the infinite-dimensional representation. The representation on $V=\bar{\Omega}/I$ is obtained from (27):

$$(28)\qquad \begin{cases}\Gamma(\sigma_+^m)D(m_1,m_2,n)=\mu_2D(m_1+1,m_2,n+m-1)+\\ \qquad\qquad +m_2D(m_1+1,m_2-1,m+n)\,,\\ \Gamma(\sigma_-^m)D(m_1,m_2,n)=\mu_1D(m_1,m_2+1,n+m-1)+\\ \qquad\qquad +m_1D(m_1-1,m_2+1,m+n)\,,\\ \Gamma(\sigma_3^m)D(m_1,m_2,n)=\mu_1D(m_1+1,m_2,m+n-1)-\\ \qquad -\mu_2D(m_1,m_2+1,m+n-1)+(m_1-m_2)D(m_1,m_2,m+n)\,.\end{cases}$$

In the case of $\mu_1\neq 0$ or $\mu_2\neq 0$, the representation (28) is indecomposable. It is observed that the value (m_1+m_2) cannot decrease in (28), the integer $M\in \mathbf{Z}$ defines an invariant subspace V_M of V with basis

(29) $$V_M\colon \{D(m_1,m_2,n)|m_1+m_2\geqslant M\,;m_1,m_2\in \mathbf{Z}\}\,.$$

There exists an invariant subspace chain

$$\ldots V_{-2}\supset V_{-1}\supset V_0\supset V_1\supset V_2\supset\ldots .$$

For $K\in \mathbf{N}^+$, we can define a quotient space $V(M,K)$ with basis

(30) $$V[M,K]\colon \{H(m_1,m_2,n)\equiv \equiv D(m_1,m_2,n)\ \mathrm{Mod}\, V_{M+K}\,|M\leqslant m_1+m_2\leqslant M+K-1\}\,.$$

The representation on $V(M,K)$ is obtained from (28):

(31) $$\begin{cases} \Gamma(\sigma_+^m)H(m_1,m_2,n)=\mu_2H(m_1+1,m_2,n+m-1)+ \\ \qquad\qquad +m_2H(m_1+1,m_2-1,m+n)\,, \\ \Gamma(\sigma_-^m)H(m_1,m_2,n)=\mu_1H(m_1,m_2+1,n+m-1)+ \\ \qquad\qquad +m_1H(m_1-1,m_2+1,m+n)\,, \\ \Gamma(\sigma_3^m)H(m_1,m_2,n)=\mu_1H(m_1+1,m_2,n+m-1)- \\ \qquad -\mu_2H(m_1,m_2+1,m+n-1)+(m_1-m_2)H(m_1,m_2,m+n) \\ \qquad\qquad (M\leqslant m_1+m_2\leqslant M+K-1)\,. \end{cases}$$

If we define an "angular momentum basis" for $V(M,K)$

(32) $$|(n),j,s\rangle=\frac{H(j+s,j-s,n)}{\sqrt{(j+s)!\,(j-s)!}}\,,$$

where $j=M/2,\ (M+1)/2,\ (M+2)/2,\ \ldots,\ (M+K-1)/2,\ s\in \mathbf{Z}$, the representation (31) becomes

(33) $$\begin{cases} \Gamma(\sigma_+^m)|(n),j,s\rangle=(j+s+1)^{1/2}\mu_2|(m+n-1),j+\tfrac12,s+\tfrac12\rangle+ \\ \qquad\qquad +[j(j+1)-s(s+1)]^{1/2}|(m+n),j,s+1\rangle\,, \\ \Gamma(\sigma_-^m)|(n),j,s\rangle=(j-s+1)^{1/2}\mu_1|(m+n-1),j+\tfrac12,s-\tfrac12\rangle+ \\ \qquad\qquad +[j(j+1)-s(s-1)]^{1/2}|(m+n),j,s-1\rangle\,, \\ \Gamma(\sigma_3^m)|(n),j,s\rangle=\mu_1(j+s+1)^{1/2}|(m+n-1),j+\tfrac12,s+\tfrac12\rangle+ \\ \qquad +\mu_2(j-s+1)^{1/2}|(m+n-1),j+\tfrac12,s-\tfrac12\rangle+2s|(m+n),j,s\rangle\,, \end{cases}$$

that is an infinite-dimensional indecomposable representation in the case of $K \geqslant 2$. When $K=1$, the representation (32) becomes

$$(34)\quad \begin{cases} \Gamma(\sigma_+^m)|(n),j,s\rangle = [j(j+1)-s(s+1)]^{1/2}|(m+n),j,s+1\rangle\,, \\ \Gamma(\sigma_-^m)|(n),j,s\rangle = [j(j+1)-s(s-1)]^{1/2}|(m+n),j,s-1\rangle\,, \\ \Gamma(\sigma_3^m)|(n),j,s\rangle = 2s\,|(m+n),j,s\rangle \qquad \left[j=\dfrac{M}{2}\,,\ s\in \boldsymbol{Z}\right], \end{cases}$$

that is an infinite-dimensional irreducible representation.

In the case of $\mu_1=\mu_2=0$, the representation (28) becomes

$$(35)\quad \begin{cases} \Gamma(\sigma_+^m)\,D(m_1,m_2,n) = m_2 D(m_1+1,m_2-1,m+n)\,, \\ \Gamma(\sigma_-^m)\,D(m_1,m_2,n) = m_1 D(m_1-1,m_2+1,m+n)\,, \\ \Gamma(\sigma_3^m)\,D(m_1,m_2,n) = (m_1-m_2)\,D(m_1,m_2,m+n)\,. \end{cases}$$

It is observed that the value m_1+m_2 does not change in (35), the integer $R\in\boldsymbol{Z}$ defines an invariant subspace $V^{[R]}$

$$(36)\quad V^{[R]}\colon \{D(m_1,m_2,n)|m_1+m_2=R;\ m_1,m_2\in\boldsymbol{Z}\}\,.$$

The quotient space $V=\bar{\Omega}/I$ can be decomposed as

$$(37)\quad V=\sum_{R\in\boldsymbol{Z}}^{\oplus} V^{[R]}\,.$$

Therefore, the representation on V is complete reducible. On every $V^{[R]}$, the representation is (35) with the condition $m_1+m_2=R$. By defining "angular momentum basis" for V

$$(38)\quad |(n),j,s\rangle = \frac{D(j+s,j-s,n)}{\sqrt{(j+s)!\,(j-s)!}}\,,$$

where $j=R/2=0, 1/2, 1, 3/2, \ldots$, $s\in\boldsymbol{Z}$, we can obtain the representation on $V^{[R]}$

$$(39)\quad \begin{cases} \Gamma(\sigma_+^m)|(n),j,s\rangle = [j(j+1)-s(s+1)]^{1/2}|(m+n),j,s+1\rangle\,, \\ \Gamma(\sigma_-^m)|(n),j,s\rangle = [j(j+1)-s(s-1)]^{1/2}|(m+n),j,s-1\rangle\,, \\ \Gamma(\sigma_3^m)|(n),j,s\rangle = 2s\,|(n),j,s\rangle \qquad \left(j=\dfrac{R}{2}\,,\ s\in\boldsymbol{Z}\right), \end{cases}$$

that is an infinite-dimensional irreducible representation.

Now, let us discuss finite-dimensional representations.

The representation induced on W can be obtained from (28):

$$\text{(40)}\quad \begin{cases} \Gamma(\sigma_+^m)P(m_1,m_2)=\mu_2\lambda^{m-1}P(m_1+1,m_2)+m_2\lambda^m P(m_1+1,m_2-1)\,, \\ \Gamma(\sigma_-^m)P(m_1,m_2)=\mu_1\lambda^{m-1}P(m_1,m_2+1)+m_1\lambda^m P(m_1-1,m_2+1)\,, \\ \Gamma(\sigma_3^m)P(m_1,m_2)=\mu_1\lambda^{m-1}P(m_1+1,m_2)-\mu_2\lambda^{m-1}P(m_1,m_2+1)+ \\ \qquad +(m_1-m_2)\lambda^m P(m_1,m_2) \quad (\lambda\neq 0, m_1, m_2\in \boldsymbol{Z})\,. \end{cases}$$

The representation subduced on W^+ is (40) with the condition $m_1, m_2 \in \boldsymbol{N}$. In the case of $\mu_1\neq 0$ or $\mu_2\neq 0$, the representation on W^+ is indecomposable. The representation on $W^+(M,K)$ is

$$\text{(41)}\quad \begin{cases} \Gamma(\sigma_+^m)Q(m_1,m_2)=\mu_2\lambda^{m-1}Q(m_1+1,m_2)+m_2\lambda^m Q(m_1+1,m_2-1)\,, \\ \Gamma(\sigma_-^m)Q(m_1,m_2)=\mu_1\lambda^{m-1}Q(m_1,m_2+1)+m_1\lambda^m Q(m_1-1,m_2+1)\,, \\ \Gamma(\sigma_3^m)Q(m_1,m_2)=\mu_1\lambda^{m-1}Q(m_1+1,m_2)-\mu_2\lambda^{m-1}Q(m_1,m_2+1)+ \\ \qquad +(m_1-m_2)\lambda^m Q(m_1,m_2) \quad (K\geqslant 2,\ \mu_1\neq 0 \text{ or } \mu_2\neq 0)\,, \end{cases}$$

with dimension

$$\dim W^+(MK)=\frac{1}{2}K[2M+K-1]\,.$$

The representation on $W^+(M,1)$ is

$$\text{(42)}\quad \begin{cases} \Gamma(\sigma_+^m)Q(m_1,m_2)=m_2\lambda^m Q(m_1+1,m_2-1)\,, \\ \Gamma(\sigma_-^m)Q(m_1,m_2)=m_1\lambda^m Q(m_1-1,m_2+1)\,, \\ \Gamma(\sigma_3^m)Q(m_1,m_2)=(m_1-m_2)\lambda^m Q(m_1,m_2) \quad (m_1+m_2=M, \mu_1\neq 0 \text{ or } \mu_2\neq 0)\,. \end{cases}$$

If we define an "angular momentum basis" for $W^+(M,1)$,

$$\text{(43)}\qquad |j,s\rangle=\frac{Q(j+s,j-s)}{\sqrt{(j+s)!\,(j-s)!}},$$

where $j=M/2=0,1/2,1,3/2,\ldots$, $s=j,j-1,j-2,\ldots,-j$, the representation (42) becomes

$$\text{(44)}\qquad \begin{cases} \Gamma(\sigma_\pm^m)|j,s\rangle = \lambda^m\sqrt{(j\mp s)(j\pm s+1)}\,|j,s\pm 1\rangle\,, \\ \Gamma(\sigma_3^m)|j,s\rangle = \lambda^m 2s\,|j,s\rangle\,, \end{cases}$$

that is an irreducible representation with dimension $M+1=2j+1$.

In the case of $\mu_1 = \mu_2 = 0$, the representation on W^+ becomes

$$(45)\qquad \begin{cases} \Gamma(\sigma_+^m)P(m_1, m_2) = m_2\lambda^m P(m_1+1, m_2-1)\,, \\ \Gamma(\sigma_-^m)P(m_1, m_2) = m_1\lambda^m P(m_1-1, m_2+1)\,, \\ \Gamma(\sigma_3^m)P(m_1, m_2) = (m_1 - m_2)\lambda^m P(m_1, m_2)\,, \qquad (\lambda \neq 0, m_1, m_2 \in \boldsymbol{N})\,, \end{cases}$$

that is a complete reducible representation. Obviously, W^+ can be decomposed as

$$W^+ = \sum_{R \in \boldsymbol{N}}^{\oplus} W^{+[R]}\,,$$

where $W^{+[R]}$ is spanned by

$$(46)\qquad W^{+[R]}\colon \{P(m_1, m_2) | m_1 + m_2 = R;\ m_1, m_2 \in \boldsymbol{N}; R \in \boldsymbol{N}\}\,.$$

The representation subduced on $W^{+[R]}$ is (45) with the condition $m_1 + m_2 = R$. If we define an "angular momentum basis" for $W^{+[R]}$

$$(47)\qquad |j, s\rangle = \frac{P(j+s, j-s)}{\sqrt{(j+s)!\,(j-s)!}}\,,$$

where $j = R/2 = 0, 1/2, 1, 3/2, \ldots$, $s = j, j-1, j-2, \ldots, -j$, then we can obtain the representation on $W^{+[R]}$

$$(48)\qquad \begin{cases} \Gamma(\sigma_\pm^m)|j, s\rangle = \lambda^m \sqrt{(j \mp s)(j \pm s + 1)}\,|j, s \pm 1\rangle\,, \\ \Gamma(\sigma_3^m)|j, s\rangle = \lambda^m 2s\,|j, s\rangle\,, \end{cases}$$

that is an irreducible representation with dimension $R + 1 = 2j + 1$.

According to (44) (or (48)), we can construct irreducible representations with any dimension $d \in \boldsymbol{N}$. For example, when $j = 1$, we can the obtain the irreducible representation of $\widehat{SU}(2)$ with dimension 3 (where $\lambda \neq 0$):

$$\Gamma(\sigma_+^m) = \lambda^m \begin{bmatrix} 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{bmatrix}, \quad \Gamma(\sigma_-^m) = \lambda^m \begin{bmatrix} 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{bmatrix}, \quad \Gamma(\sigma_3^m) = 2\lambda^m \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

* * *

The authors would like to thank Prof. Zhao-Yan Wu for his useful discussions.

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● RIASSUNTO (*)

Si dà una nuova realizzazione dell'algebra ad ansa $\hat{G}$ (algebra di Kac-Moody affine non intrecciata) sul campo inviluppante $\bar{\Omega}$ dell'algebra di Bose $\mathscr{H}$. Usando questa nuova realizzazione si elaborano rappresentazioni non scomponibili a dimensioni infinite non triviali e rappresentazioni a dimensioni finite di $\hat{G}$ su $\bar{\Omega}$ e i suoi spazi quozienti. Infine si discute in dettaglio, come esempio esplicito, l'algebra ad ansa $\widehat{SU}(2)$ associata all'algebra di Lie $SU(2)$.

(*) *Traduzione a cura della Redazione.*

Новая реализация алгебры петель и неприводимые модули.

Резюме (*). — Предлагается новая реализация алгебры петель $\hat{G}$ (раскрученная аффинная алгебра Как-Муди) на огибающем поле $\bar{\Omega}$ алгебры Бозе $\varkappa$. Используя эту новую реализацию, конструируются нетривиальные бесконечномерные неприводимые представления и конечномерные представления $\hat{G}$ на $\bar{\Omega}$ и их частные пространства. В заключение, подробно обсуждается пример алгебры петель $\widehat{SU}(2)$, связанной с алгеброй Ли $SU(2)$.

(*) *Переведено редакцией.*