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ALGEBRAIC CONSTRUCTION OF 't HOOFT'S QUANTUM EQUIVALENCE CLASSES

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Most recently 't Hooft has postulated (G. 't Hooft, *Class. Quantum Grav.* **16**, 3263 (1999)) that quantum states at the "atomic scale" can be understood as equivalence classes of primordial states governed by a dissipative deterministic theory underlying quantum theory at the "Planck scale". Defining invariant subspaces clearly for primordial states according to a given evolution, we mathematically reformulate 't Hooft's theory as a quotient space construction with the time-reversible evolution operator induced naturally. With this observation and some analysis, 't Hooft's theory is generalized beyond his case where the evolution at the "Planck scale" is a power of a one-time-step evolution or the time is discrete.

1. Introduction

To probe the physical differences in locality and causality between the so-called Planck scale physics such as quantum gravity and the usual quantum field theories in some flat background space–time, Gerard 't Hooft postulated[1,2] that there should be a dissipative deterministic theory underlying the usual quantum theory. In his theory, the generic quantum mechanics is no longer the crucial starting point. Rather, a deterministic theory with dissipation of information at the Planck scale is needed to derive quantum mechanics at the atomic scale. Quantum state used to make probabilistic prediction about physical phenomenon is then shown to be a derived concept.

In 't Hooft's opinion, at the atomic scale quantum states are equivalence classes of primordial states at the Planck scale. If we only care the temporal evolution of equivalence classes, the information within each equivalence class can be ignored. Then from a non-time-reversible evolution, which characterizes a deterministic process with dissipation at the Planck scale, we can obtain a time-reversible evolution of the properly defined equivalence classes for primordial states. Taking the equivalence classes to be quantum states we are then able to introduce a unitary evolution

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law at the atomic scale. Apparently, here the central problem is how to classify the Planck scale states with respect to a deterministic evolution. 't Hooft's solution to this problem is as follows. He argued that two Planck scale states are equivalent at the atomic scale if, after some finite time interval, they evolve into the same state. This leads to a natural definition of equivalence classes: Two states are in the same equivalence class if and only if they evolve into the same state after some finite time interval. Quantum states are identified with these equivalence classes.

To see 't Hooft's idea clearly, we will make use of algebraic terminologies such as quotient space and induced representation of operators. We will first properly define an invariant subspace of primordial states related to the equivalence classes defined by 't Hooft. Then we can identify the space of quantum states, which is spanned by the equivalence classes according to 't Hooft, with the quotient space and naturally reformulate the time-reversible evolution at the atomic scale by the mechanism of induced representation of the dissipative deterministic evolution operator on the quotient space. Finally, we extend 't Hooft's theory to cases where the evolution of primordial states is not necessarily a power of one-time-step evolution at the Planck scale ('t Hooft has implicitly assumed the one-time-step evolution law in the case of discrete time variable).

2. Quotient Representation of Quantum States

In 't Hooft's theory,[1] primordial states at the Planck scale need not form a linear space. Generally they can be denoted by a set

$$\Sigma = \{\phi_i | i \in I\}\,,$$

where I stands for an index set. The underlying deterministic evolution is a transformation U (usually depending on time) of Σ to itself. It can be represented by a matrix with the entry 0 or 1 if I is a countable set. The determinism requires that there be at most one nonzero entry in each column. Otherwise, the system will be forced to evolve into an uncertain state, namely, a superposition of several elements that is not in Σ. As U is an evolution operator, we write it as $U = U(t_f, t_i)$ by convention. Physically, it represents the evolution in the time interval $[t_i, t_f]$. Certainly the evolution should satisfy the so-called semigroup condition

$$\begin{gathered} U(t_f, t_m)U(t_m, t_i) = U(t_f, t_i)\,, \\ U(t,t) = 1\,. \end{gathered} \tag{1}$$

In general, U is singular, namely, it has no inverse. Such singular operator describes deterministic process with dissipation. As a matter of fact, under such an evolution some states will disappear and some states will evolve into the same state, or in other words, some states with a different past may have the same deterministic fate. 't Hooft thinks that, if two states evolve in such a way that their futures are identical, they should represent the same state at the atomic scale. In this view, he divides the elements of Σ into equivalence classes $\bar{\phi}_j$, ϕ_{i_1} and ϕ_{i_2} ($i_1, i_2 \in I$) being in the same equivalence class if they are evolved into the same state after finite time interval. Denote by $\Xi = \{\bar{\phi}_j | j \in J\}$ the set of the equivalence classes, where J is another index set. Then 't Hooft postulates that the space of quantum states is spanned by $\{\bar{\phi}_j | j \in J\}$ and claims that the reduced evolution on the space of quantum states is reversible.

Now let us analyze 't Hooft's theory from mathematical point of view as follows. Let V be the vector space spanned by $\{\phi_i | i \in I\}$. Then $U(t_f, t_i)$ can be extended to a linear transformation of V. We will call V the space of primordial states in spite of the fact that generally it contains elements (such as $\phi_i + \phi_j$) which are not primordial states originally defined by 't Hooft. Let V_1 denote the subspace of V consisting of the vectors annihilated by $U(0, t)$ at some t, namely, a vector v belongs to V_1 if and only if there exists some $U(t, 0)$ such that $U(t, 0)v = 0$. Now it is easy to observe that *the space of quantum states is none other than the quotient space*

$$V/V_1 = \{|\phi\rangle \triangleq \phi + V_1 | \phi \in V\} .$$

It is also easy to notice that 't Hooft's construction implies the assumption that the evolution operator $U(t_2, t_1)$ only depends on the difference of t_2 and t_1, i.e. we can write

$$U(t_2, t_1) = U(t_2 - t_1) . \tag{2}$$

Since the time is discrete in the Planck scale, for the minimum time interval Δ, we can label

$$U(t_2, t_1) = U(m_2\Delta, m_1\Delta) \triangleq \mathbf{U}(m_2, m_1)$$

with integers m_2 and m_1. With this notation, Eq. (2) means that all of the one-time-step evolutions are identical, namely,

$$\mathbf{U}(m_1 + 1, m_1) = \mathbf{U}(m_2 + 1, m_2)$$

or

$$\mathbf{U}(1, 0) = \mathbf{U}(2, 1) = \cdots = \mathbf{U}(m + 1, m) .$$

Obviously, this equation is equivalent to the condition Eq. (2) in the case of discrete time.

Indeed, if this is the case, a nonsingular evolution law of the quantum states naturally follows from $U(t_2, t_1)$. Otherwise, generally the evolution operator at the Planck scale cannot be reduced to the space of quantum states at the atomic space. Mathematically, this is because, for a linear transformation in End(V) to have an induced action on the quotient space V/V_1, V_1 should be invariant with respect to it. Let $\bar{v} \equiv |\nu\rangle$ denote the equivalence class containing v. We notice that V_1 is invariant under $U(t_2, t_1)$ in this case. Thus $U(t_2, t_1)$ induces a natural action on the quotient space V/V_1. We denote the induced operator by $\overline{U(t_2, t_1)}$, then we have

$$\overline{U(t_2, t_1)}\bar{v} = \overline{U(t_2, t_1)v} . \tag{3}$$

It is easily seen that $\overline{U(t_2,t_1)}$ is nonsingular, i.e. zero is not its eigenvalue. In fact, if $\overline{U(t_2,t_1)}\bar{v} = \bar{0}$, then $U(t_2,t_1)v \in V_1$. Thus there exists some t such that $U(t,0)U(t_2,t_1)v = 0$. It then follows that

$$\begin{aligned} U(t,0)U(t_2,t_1)v &= U(t_2+t,t_2)U(t_2,t_1)v \\ &= U(t_2+t,t_1)v = U(t_2-t_1+t,0)v = 0\,. \end{aligned} \tag{4}$$

By definition this means $v \in V_1$, i.e. $\bar{v} = \bar{0}$. This proves the nonsingularity of $\overline{U(t_2,t_1)}$. Now the unitary of $\overline{U(t_2,t_1)}$ remains to be established. We will handle this problem in a special case below.

For more general case, if the condition $U(t_2,t_1) = U(t_2-t_1)$ is not satisfied, to guarantee the nonsingularity of $\overline{U(t_2,t_1)}$, the definition of the invariant subspace V_1 needs to be modified. It seems that we should define V_1 in the following way:

> A vector v belongs to V_1 if and only if there exist finitely many t_i $(i = 1,2,\ldots,r)$ such that $U(t_1,t_2)U(t_3,t_4)\cdots U(t_{r-1},t_r)v = 0$.

Unfortunately, in this definition the physical meaning of V_1 is not clear. Let us return to the case with the condition $U(t_2,t_1) = U(t_2-t_1)$. Following 't Hooft, we consider a system with *discrete time coordinates.* Actually, it stands for the same one-step evolution process

$$U(n+1,n) = U(1,0)\,, \qquad n \in Z^+ \tag{5}$$

in time unit $\Delta = 1$. We can assume that the time t takes values in Z^+, the set of non-negative integers. Then, we have

$$U(t) = U(t,0) = U(1,0)^t$$

for the *discrete time* $t \in Z^+$. For such a periodic system the invariant subspace V_1 is

$$V_1 = \{v \in V | \exists\ t \in Z^+ \text{ s.t. } U(t)v = 0\}\,. \tag{6}$$

In Refs. 1 and 2 't Hooft presented a simple example to illustrate his theory. Fit into the above mathematical framework, the example goes as follows: V is four-dimensional:

$$V = \text{span}\{v_1, v_2, v_3, v_4\} \tag{7}$$

and

$$U(1,0) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

with respect to the basis $\{v_1, v_2, v_3, v_4\}$. It is easily seen that

$$V_1 = \mathrm{span}\{v_1 - v_4\}\,,$$

$$\frac{V}{V_1} = \mathrm{span}\{\bar{v}_1, \bar{v}_2, \bar{v}_3\}$$

and the induced evolution operator is

$$\overline{U(1,0)} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

with respect to the basis $\{\bar{v}_1, \bar{v}_2, \bar{v}_3\}$. Clearly, $\overline{U(1,0)}$ is unitary relative to a properly defined inner product. This is not at all accidental. In fact, *if the space of primordial states is finite-dimensional, a dissipative deterministic evolution at the Planck scale always induces a unitary evolution at the atomic scale if only we choose an inner product on the space of quantum states adequately. We will present the proof elsewhere.*

3. Generalize Dynamics

We now turn to consider general evolution process which is not a power of the one-step-evolution, such as scattering process, with time variable approaching infinity. Assume such a process is described by an evolution operator $U(0, +\infty) \triangleq W$ at the Planck scale.

As above, let V be the space of primordial states. Suppose V is finite-dimensional. Inspired by 't Hooft's theory, we postulate that the space of quantum states at the atomic scale is the quotient space V/V_1 with V_1 defined as follows:

$$V_1 = \{v \in V | \exists\ n \in Z^+ \text{ s.t. } W^n v = 0\}\,.$$

Suppose that the characteristic polynomial of W is

$$p_w(\lambda) = \prod_{i=0}^{r} (\lambda - \lambda_i)^{m_i}\,, \tag{8}$$

where $\lambda_0 = 0$ and $\lambda_j \neq 0$ for $j \neq 0$. Obviously, V_1 is just the kernel of W^{m_0}, namely,

$$V_1 = \mathrm{Ker}\, W^{m_0} = \{v \in V | W^{m_0} v = 0\} \tag{9}$$

and the characteristic polynomial of the induced operator $\overline{W}$ is

$$p_{\bar{w}}(\lambda) = \prod_{i=1}^{r} (\lambda - \lambda_i)^{m_i}\,. \tag{10}$$

Therefore, $\overline{W} \in \mathrm{End}(V/V_1)$ is nonsingular. Let us go on to deal with the unitarity problem of $\overline{W}$.

$\overline{W}$ is called unitarizable if it is diagonalizable and all of its eigenvalues are of modulus 1. By definition, if $\overline{W}$ is unitarizable, there exists a basis $\{\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_m\}$ of

V/V_1 such that $\overline{W}\bar{v}_j = e^{i\theta_j}\bar{v}_j$ $(j = 1, 2, \ldots, m)$ where θ_j is a real number. Therefore, if we define the "canonical" inner product (,) on V/V_1 satisfying $(\bar{v}_i, \bar{v}_j) = \delta_{ij}$, then $\overline{W}$ is unitary with respect to it. We have shown that if an operator is unitarizable it can be made unitary by properly defining an inner product on the space that it acts on, as the term suggests. The converse statement is trivially true, as one easily sees. As for the unitarizability condition for $\overline{W}$, it is not difficult to show that $\overline{W}$ is unitarizable if and only if the minimal polynomial of W is of the form $p(\lambda) = \lambda^n \prod_{j=1}^{m}(\lambda - e^{i\theta_j})$ where θ_j $(j = 1, 2, \ldots, m)$ are different nonzero real numbers.

If $\overline{W}$ is not unitarizable we can construct a unitary operator from $\overline{W}$ by the polar decomposition of $\overline{W}$. Explicitly, we define

$$\overline{U}_{\overline{w}} = \overline{W}(\overline{W}^{+}\overline{W})^{-1/2}\,. \tag{11}$$

It is then elementary to show the unitarity of $\overline{U}_{\overline{w}}$. Certainly, $\overline{U}_{\overline{w}}$ depends on the inner product on V/V_1. But it is always unitary with respect to the chosen inner product. It is also clear that when $\overline{W}$ is unitarizable $\overline{U}_{\overline{w}}$ coincides with $\overline{W}$ if we choose the "canonical" inner product on V/V_1. In general, there does not exist a canonical way to construct a unitary operator from $\overline{W}$. This corresponds to the fact that there does not exist a canonical way to introduce an inner product on V/V_1.

We proceed to present a method of obtaining the matrix representation of $\overline{W}$. Denote by $(\operatorname{Ker} W^{m_0})^{\perp}$ the orthogonal complement to the subspace $\operatorname{Ker} W^{m_0}$ in V with respect to the inner product on V. Then we have the decomposition

$$V = \operatorname{Ker} W^{m_0} \oplus (\operatorname{Ker} W^{m_0})^{\perp}\,.$$

Obviously, the operator $W^{+m_0}W^{m_0}$ is hermitian and $(\operatorname{Ker} W^{m_0})^{\perp}$ is a $W^{+m_0}W^{m_0}$-invariant subspace. Thus the restriction of $W^{+m_0}W^{m_0}$ to $(\operatorname{Ker} W^{m_0})^{\perp}$ is also hermitian. Hence there are eigenvectors $v_1, v_2, \ldots, v_d$ of $W^{+m_0}W^{m_0}$ such that they constitute a basis of $(\operatorname{Ker} W^{m_0})^{\perp}$. Choose a basis $\{v_{d+1}, v_{d+2}, \ldots, v_N\}$ of $\operatorname{Ker} W^{m_0}$. Then $\{v_i | i = 1, 2, \ldots, N\}$ is a basis of V and $\{\bar{v}_i \triangleq \bar{v}_i + V_1 | i = 1, 2, \ldots, d\}$ is a basis of V/V_1. Suppose that P is the projection operator upon $(\operatorname{Ker} W^{m_0})^{\perp}$. Then the operator PWP has the following matrix representation

$$PWP = \begin{pmatrix} M & 0 \\ 0 & 0 \end{pmatrix} \tag{12}$$

with respect to the basis $\{v_i | i = 1, 2, \ldots, N\}$, where M is the matrix representation of $\overline{W}$ with respect to the basis $\{\bar{v}_i | i = 1, 2, \ldots, d\}$.

To illustrate the above arguments, let us take

$$W = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

as an example. Its characteristic polynomial is $(\lambda - 1)^2\lambda^2$. Thus the invariant subspace $V_1 = \operatorname{Ker} W^2$. By simple calculation we have

$$V_1^{\perp} = \operatorname{span}\{v_1, v_2\}\,, \qquad V_1 = \operatorname{span}\{v_3, v_4\}\,,$$

where

$$v_1 = (0,0,0,1)^{\mathrm{T}}\,, \qquad v_2 = (1,1,1,0)^{\mathrm{T}}\,,$$

$$v_3 = (1,0,-1,0)^{\mathrm{T}}\,, \qquad v_4 = (0,1,-1,0)^{\mathrm{T}}\,.$$

Then we obtain the matrix representation $M = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ of $\overline{W}$ relative to the basis $\{\bar{v}_1, \bar{v}_2\}$.

In summary, in the finite-dimensional case, if we can find an adequate classification of the so-called primordial states at the Planck scale such that the information concerning the time irreversibility could be reasonably ignored, then at the atomic scale we can manage to obtain a unitary evolution, describing a quantum mechanical process. When we pass to the infinite-dimensional case, the situation becomes subtle and hard to manage. In particular, it should be difficult to identify the subspace V_1. Nevertheless, the central idea of classifying primordial states and identifying quantum states with equivalence classes is applicable without difficulty. 't Hooft has shown us two elegant examples: The classical motion with limit cycles and the massless neutrinos moving as a plane in space–time.[1,2] We wish to try our hands on discrete infinite-dimensional case in the next section.

4. Remarks

To conclude this letter we should give some remarks on our mathematical reformulation and the physical generalization for 't Hooft's equivalence class theory.

Firstly, a correct quantum theory requires a Hilbert space with properly defined inner product to define probability. But it is not at all clear how to endow the space of equivalence classes with such an inner product even though there may be a natural inner product on the space of primordial states. Thus to establish the unitarity of the induced evolution is really a problem if one does not know in advance what the physical system at the atomic scale seems to be. So a gap remains to be bridged between the so-called Planck scale physics and the atomic scale physics even if 't Hooft's theory proves to be correct.

Secondly, we also notice that mathematically there is something in common between the 't Hooft's equivalence class idea and the representation theory of the Heisenberg–Weyl (HW) algebra.[4] In this theory the Fock space of bosons follows from the quotient space construction for the HW algebra[4] generated by $[a, a^{+}] = 1$. The primordial description of the HW algebra is its master representation M with the Poincaré–Birkhoff–Witt (PBW) basis, but it is not unitary for $M(a) \neq M(a^{+})^{\dagger}$. However, from the master representation M, a unitary representation u which satisfies $u(a) = u(a^{+})^{\dagger}$ can be constructed on the quotient space for a certain HW-invariant subspace. Based on this observation, we can construct the Fock space as a span of the equivalence class of certain primordial states obeying non-reversible evolution.

Finally, it is a challenge to understand quantum decoherence or wave function collapse in quantum measurement[5] from the underlying deterministic theory at a deeper level. However, like the hidden variable theory, which has been rejected by experiments till now, quantum measurement problems such as quantum-classical correspondence, quantum dissipation and quantum entanglement[5,6] must be faced if we are to take 't Hooft's theory seriously.

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