Letters in Mathematical Physics **23**: 19–27, 1991.

The Cyclic Representations of the Quantum Superalgebra $U_q\,\mathrm{osp}(2, 1)$ with q a Root of Unity*

CHANG-PU SUN[1,2], HONG-CHEN FU[2], and MO-LIN GE[1]
[1]*Theoretical Physics Division, Nankai Institute of Mathematics, Tianjin 300071, P.R. China*
[2]*Physics Department, Northeast Normal University, Changchun 130024 P.R. China*

(Received: 15 April 1991; revised version 17 June 1991)

Abstract. By generalizing De Concini and Kac's cyclic representation theory of quantum groups at roots of unity, the cyclic representations of the quantum superalgebra $U_q\,\mathrm{osp}(2, 1)$ are constructed in three classes: irreducible representations with single multiplicities, irreducible representations with the multiplicities larger than one, and indecomposable representations.

AMS subject classifications (1991). 22E99, 81Q99.

1. Introduction

The quantum group, quantum universal enveloping algebra (quantum algebra) and their representation theories are deeply rooted in integrable nonlinear physics models associated with the Yang–Baxter equation (YBE) [1–4]. Recently, much attention has been given to the representations of quantum algebras in the nongeneric case where q is a root of unity [5–9]. Especially, the cyclic representations of quantum algebras have been built by De Concini and Kac [10] within a general framework and the explicit constructions for $U_q\,\mathrm{sl}(n + 1)$ and $U_q\,A_2^{(2)}$ have been given and associated with the Potts model by Date *et al.* [11, 12]. We have also obtained the concrete cyclic representations of $\mathrm{sl}_q(2)$ [13] by generalizing the q-deformed boson realization method [14–17].

In this Letter, taking the principal ideas of De Concini and Kac's work into account, we construct and study the cyclic representations of the quantum superalgebra [18–20] with $U_q\,\mathrm{osp}(2, 1)$ as an explicit example. We obtain the following cyclic representations of $A = U_q\,\mathrm{osp}(2, 1)$:

(1) A class of irreducible cyclic representations with single multiplicities;
(2) some irreducible representations with multiplicities larger than one, which are labeled by two integers and are also cyclic.
(3) some indecomposable representations with two labels, one of which is cyclic.

* This work is supported in part by the National Science Foundation in China.

If we use the lattice points to represent the basis vectors, we will find the basis for the above representations possess certain 'topologies'.

2. The Algebra $A = U_q osp(2, 1)$ and its Regular Representation

The quantum superlagebra $A = U_q osp(2, 1)$ is an associative algebra over the complex number field $\mathbb{C}$ generated by the elements V_+, V_- and $Q^{\pm} = q^{\pm H}$ satisfying the algebraic relations

$$V_+ V_- + V_- V_+ = -\tfrac{1}{4}[2H], \quad [H, V_\pm] = \pm\tfrac{1}{2}V, \tag{2.1}$$

where $[f] = (q^f - q^{-f})/(q - q^{-1})$ for any operator or number f. By introducing the appropriate coproduct, antipode, and co-unit, A can be defined as a Hopf algebra and the corresponding universal R-matrix for the YBE can be obtained from the quantum double [17, 19]. Let

$$e_\pm = 2V_\pm, \qquad h = 2H, \qquad K^\pm = q^{\pm h}. \tag{2.2}$$

Then, it follows from the induction and Equations (2.1) that

$$\begin{aligned} &K^+ e_\pm = q^\pm e_\pm K^+, \qquad K^- e_\pm = q^\mp e_\pm K^-, \\ &e_+ e_-^{m+1} = (-1)^{m+1} e_-^{m+1} e_+ + (-1)^{m+1} e_-^m \cdot C_m(K^\pm), \end{aligned} \tag{2.3}$$

where

$$\begin{aligned} &C_{2m}(K^\pm) = ([m+1] - [m]) \frac{K^+ q^{-m} - K^- q^m}{q - q^{-1}}, \\ &C_{2m+1}(K^\pm) = [m+1] \left(\frac{K^+ q^{-m} - K^- q^m}{q - q^{-1}} - \frac{K^+ q^{-m-1} - K^- q^{m+1}}{q - q^{-1}} \right). \end{aligned} \tag{2.4}$$

From Equations (2.3) and (2.4), it is easy to observe that the elements $e_\pm^{2p}$ and $(K^+)^p$ belong to the center Z_p of the algebra A. According to the Schur lemma, the central element $e_\pm^{2p}$ and $(K^\pm)^p$ are a constant multiple of the unit matrix in an irreducible representation with finite dimension. This fact is the key to the construction of cyclic representations of A.

Regarding the algebra A itself as an A-left module, we write down the regular representation of A

$$\begin{aligned} &e_+ X(2m, n, s) \\ &\quad = X(2m, n+1, s) + [m](q - q^{-1})^{-1} \times \\ &\qquad \times \{q^{n-m}(q - 1)X(2m-1, n, s+1) + \\ &\qquad + q^{-n+m}(1 - q^{-1})X(2m-1, n, s+1)\}, \\ &e_+ X(2m+1, n, s) \\ &\quad = -X(2m+1, n+1, s) - ([m+1] - [m])(q - q^{-1})^{-1} \times \\ &\qquad \times \{q^{n-m}X(2m, n, s+1) - q^{-n+m}X(2m, n, s-1)\}, \end{aligned} \tag{2.5}$$

$$e_- X(m, n, s) = X(m+1, n, s),$$

$$K^{\pm} X(m, n, s) = q^{\pm m \pm n} X(m, n, s \pm 1).$$

from Equation (2.3) on the basis

$$\{X(m, n, s) = e_-^m e_+^n K^s \mid m, n \in \mathbb{Z}^+ = \{0, 1, 2, \dots\};\ s \in \{0, \pm 1, \pm 2, \dots,\}\}$$

Let $I(K^{\pm})$ be a left ideal generated by $K^{\pm} - q^{\pm\lambda}$ ($\lambda \in \mathbb{C}$). Representation (2.5) defines a quotient representation

$$\begin{aligned}
&e_+ X(2m, n) = X(2m, n+1) + [m]\{[n - m + 1 + \lambda] - \\
&\quad - [n - m + \lambda]\} X(2m - 1, n),\\
&e_+ X(2m+1, n) = -X(2m+1, n+1) - ([m+1] - [m])[n - m + \lambda] X(2m, n),\\
&e_- X(m, n) = X(m+1, n), \qquad\qquad (2.6)\\
&K^{\pm} X(m, n) = q^{\mp m \pm n \pm \lambda} X(m, n),
\end{aligned}$$

on the quotient space $V(\lambda) = A/I(K^{\pm})$ with the basis

$$\{X(m, n) = X(m, n, 0) \text{ Mod } I(K^{\pm}) \mid m, n \in \mathbb{Z}^+\}.$$

This representation is the starting point of all the discussions in this Letter.

3. Cylinder Type Representations

Since e_-^{2p} is a central element of A, the left ideal $J(\alpha)$ generated by $e_-^{2p} - \alpha$ ($\alpha \in \mathbb{C}$) is a two-sided ideal. For the quotient space $W_\lambda(\alpha) = V(\lambda)/J(\alpha)$ spanned by

$$\{\bar{X}(m, n) \equiv \alpha^{-l(m)} X(m, n) \text{ Mod } J(\alpha) \mid X(m, n) \in V(\lambda), m = 2p \cdot l(m) + \bar{m}, 0 \leqslant \bar{m} \leqslant 2p - 1;\ l(m) \in \mathbb{Z}^+\},$$

we have a cyclic condition

$$\bar{X}(m + kp, n) = \bar{X}(m, n), \quad m, n \in \mathbb{Z}^+;\ k \in 2\mathbb{Z}^+. \tag{3.1}$$

Choosing a basis $\{\bar{X}(m, n) \mid m = 0, 1, 2, \dots, 2p - 1;\ n \in \mathbb{Z}^+\}$ for $W_\lambda(\alpha)$, we obtain an infinite-dimensional representation of A induced by Equation (3.6) as follows

$$\begin{aligned}
&e_+ \bar{X}(2m, n) = \bar{X}(2m, n+1) + [m]\{[n - m + 1 + \lambda] - \\
&\quad - [n - m + \lambda]\} \bar{X}(2m - 1, n),\\
&e_- \bar{X}(2m+1, n) = -\bar{X}(2m+1, n+1) - \{[m+1] - \\
&\quad - [m]\}[n - m + \lambda] \bar{X}(2m, n),\\
&e_- \bar{X}(m, n) = \bar{X}(m+1, n), \quad m \neq 2p - 1, \qquad\qquad (3.2)\\
&e_- \bar{X}(2p - 1, n) = \alpha \bar{X}(0, n),\\
&K^{\pm} \bar{X}(m, n) = q^{\mp m \pm n \pm \lambda} \bar{X}(m, n).
\end{aligned}$$

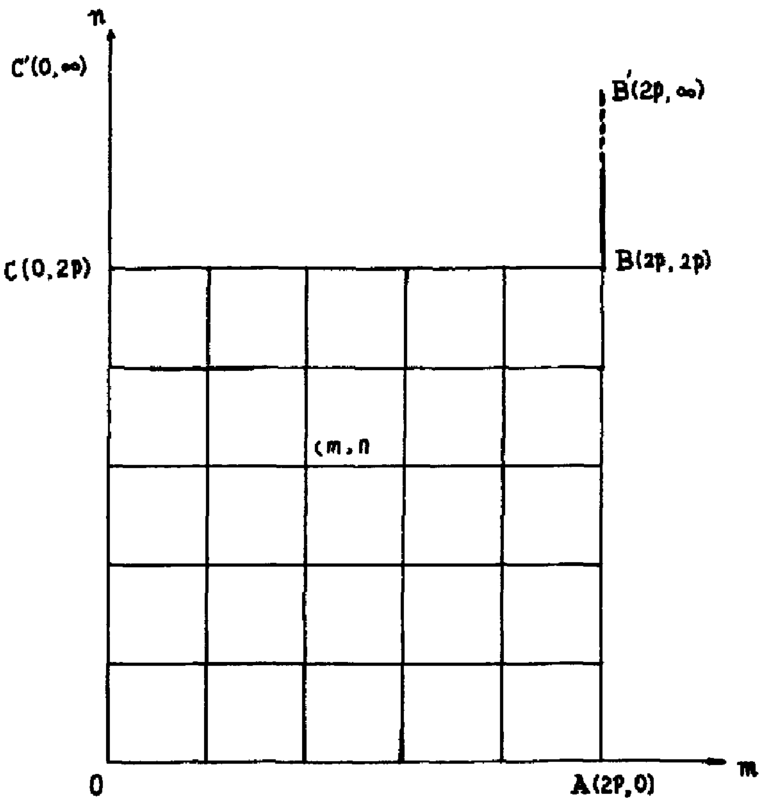

Fig. 1. The diagram of the lattice for the representations.

Now, we formally describe the basis for representation (3.2) using a simple topological terminology. As illustrated in Figure 1, we let the basis vectors $\bar{X}(m, n)$ be represented by the lattice points (m, n) on rectangle domain $OAB'C'$ on the plane R^2. Due to the cyclic condition (3.1), we identify all the points $(2p, y)$ $(0 \leqslant y(\in R) \leqslant \infty)$ on AB' with the corresponding points $(0, y)$ on OC' so that the two-dimensional domain $OABC$ forms a cylinder. The actions of the representation on the basis $\bar{X}(m, n)$ can be described by the displacement of the points (m, n) on the cylinder in this sense.

In order to obtain finite-dimensional representations, which are useful in the construction of solutions for the YBE, we consider an invariant subspace chain

$$W_\lambda(\alpha) = W_\lambda(\alpha)^0 \supset W_\lambda(\alpha)^1 \supset W_\lambda(\alpha)^2 \cdots \supset W_\lambda(\alpha)^N \supset \cdots,$$

where the invariant subspace

$$W_\lambda(\alpha)^N : \{\bar{X}(m, n + N) \mid n \in \mathbb{Z}^+, 0 \leqslant m \leqslant 2p - 1\}$$

for a given $N \in \mathbb{Z}^+$ results from the fact that the label n never decreases under the action of representation (3.2). On a quotient space $W_\lambda(\alpha)^N_M = W_\lambda(\alpha)^N / W_\lambda(\alpha)^{N+M}$ ($M \in \mathbb{Z}^+, M \neq 0$):

$$\{\tilde{X}(m, n) = \bar{X}(m, n) \text{ Mod } W_\lambda(\alpha)^{N+M} \mid 0 \leqslant m \leqslant 2p - 1, N \leqslant n \leqslant N + M - 1\}$$

with the dimension $2pM$, representation (3.2) induces a new representation

$$\begin{aligned}
&e_+ \tilde{X}(2m, n) = \tilde{X}(2m, n+1) + [m]\{n - m + 1 + \lambda] - [n - m + \lambda]\} \times \\
&\quad \times X(2m-1, n), \quad N \leqslant n \leqslant N + M - 2, \\
&e_+ \tilde{X}(2m, M + N - 1) = [m]([N' - m + 1 + \lambda] - [N' - m + \lambda]) \times \\
&\quad \times \tilde{X}(2m - 1, M + N - 1), \quad N' = N + M - 2, \\
&e_+ \tilde{X}(2m+1, n) = -\tilde{X}(2m + 1, n + 1) - \{[m+1] - [m]\}[n - m + \lambda] \times \\
&\quad \times \tilde{X}(2m, n), \quad N \leqslant n \leqslant M + N - 2, \qquad (3.3) \\
&e_+ \tilde{X}(2m + 1, M + N - 1) = -\{[m + 1] - [m]\}[N + M - 1 - m + \lambda] \times \\
&\quad \times X(2m, M + N - 1), \\
&e_- \tilde{X}(m, n) = \tilde{X}(m + 1, n), \quad 0 \leqslant m \leqslant 2p - 2, \quad e_- \tilde{X}(2p - 1, n) = \alpha \tilde{X}(0, n) \\
&K^\pm \tilde{X}(m, n) = q^{\mp m \pm n \pm \lambda} \tilde{X}(m, n).
\end{aligned}$$

When $M = 1$, representation (3.3) is irreducible; when $M \geqslant 2$, there exist invariant subspaces $W_\lambda(\alpha, t)^N_M$:

$$\{\tilde{X}(m, n) \in W_\lambda(\alpha)^N_M \mid N + t \leqslant n \leqslant M + N - 1\}, \quad 1 \leqslant t \leqslant N - 1,$$

namely, representation (3.3) is reducible. However, it can be easily proved that there is no complementary invariant subspace to $W_\lambda(\alpha, t)^N_M$ and so representation (3.3) is not completely reducible. Thus, Equation (3.3) defines an indecomposable representation of A when $M \geqslant 2$. For example, when $p = 3$ and $M = 1$, we explicitly write down a six-dimensional representation

$$\begin{aligned}
e_+ &= -[\lambda + N]E_{12} + ([N + \lambda] - [\lambda + N - 1])E_{23} + 2[\lambda + N - 1]E_{34} + \\
&\quad + ([\lambda + N - 2] - [\lambda + N - 1])E_{45} - [\lambda + N - 2]E_{56}, \\
e_- &= E_{21} + E_{32} + E_{43} + E_{54} + \alpha E_{16}, \qquad (3.4) \\
K^\pm &= q^{\pm\lambda \pm N} E_{11} + q^{\pm\lambda + N \mp 1} E_{22} + q^{\pm\lambda \pm N \mp 2} E_{33} + q^{\pm\lambda \pm N} E_{44} + \\
&\quad + q^{\pm\lambda \pm N \mp 1} E_{55} + q^{\pm\lambda \pm N \mp 2} E_{66},
\end{aligned}$$

where E_{ij} is a 6×6 matrix with elements $(E_{ij})_{kl} = \delta_{ik}\delta_{jl}$. Using this concrete result, we easily check $e_-^{2p} = \alpha$ and other algebraic relations for A.

It is worth noticing that the representation given by (3.3) for $M = 1$ is irreducible, but it possesses weights with multiplicities larger than one. This fact shows a completely 'quantum' picture caused by the q-deformation with $q^p = 1$ because this situation does not appear for the classical Lie algebras or Lie superalgebras with rank one. We also point out that a similar discussion associated with another left ideal generated by $e_{\mp}^{2p} - \beta$ ($\beta \in \mathbb{C}$) can be proceeded in a parallel way.

4. The Torus-Type Representations

Now, we can discuss a representation of A on a basis with another 'topology – torus. Let $L(\xi_\pm)$ be a left ideal generated by central elements $e_\pm^{2p} - \xi_\pm$ ($\xi_\pm \in \mathbb{C}$). The quotient space $Q(\xi_\pm) = V(\lambda)/L(\xi_\pm)$ is spanned by

$$\{\hat{X}(m,n) = \xi_+^{-l(m)}\xi_-^{l(n)}X(m,n)\mathrm{Mod}\, L(\xi_\pm) \mid m = 2l(m)p + \bar{m},\, n = 2l(n)p + \bar{n},\\ l(m), l(n) \in \mathbb{Z}^+, 0 \leqslant \bar{m}, \bar{n} \leqslant 2p-1\},$$

where

$$\hat{X}(m+kp, n+jp) = \hat{X}(m,n), \quad m,n \in \mathbb{Z}^+, k,j \in 2\mathbb{Z}^+.$$

are the cyclic conditions. These conditions determine the basis for $Q(\xi_\pm)$ as

$$\{\hat{X}(m,n) \mid 0 \leqslant m, n \leqslant 2p-1\}.$$

Then, the representation induced by Equation (2.6) on $Q(\xi_\pm)$ reads

$$\begin{aligned}
e_+\hat{X}(2m,n) &= \hat{X}(2m,n+1) + [m]\{[n-m+1+\lambda] - \\
&\quad - [n-m+\lambda]\}\hat{X}(2m-1,n),\\
e_+\hat{X}(2m+1,n) &= -\hat{X}(2m+1,n+1) - \{[m+1] - \\
&\quad - [m]\}[n-m+\lambda]\hat{X}(2m,n), \quad n \neq 2p-1,\\
e_+\hat{X}(2m,2p-1) &= \xi_+\hat{X}(2m,0) + [m]\{n-m+1+\lambda] - \\
&\quad - [n-m+\lambda]\}\hat{X}(2m-1,2p-1),\\
e_+\hat{X}(2m+1,2p-1) &= -\xi_+\hat{X}(2m+1,0) - \{[m+1] - \\
&\quad - [m]\}[n-m+\lambda]\hat{X}(2m,2p-1),\\
e_-\hat{X}(m,n) &= \hat{X}(m+1,n), \qquad e_-\hat{X}(2p-1,n) = \xi_-\hat{X}(0,n),\\
K^\pm\hat{X}(m,n) &= q^{\mp m \pm N \pm \lambda}\hat{X}(m,n).
\end{aligned} \tag{4.1}$$

This is a $4p^2$-dimensional irreducible representation without highest and lowest weights.

Now, we discuss the properties of representation (4.1). It is observed from (4.1) that all the basis vectors $\hat{X}(n, n+k)$ ($0 \leqslant n \leqslant -k+2p-1$) correspond to a same weight $(n+k) - n + \lambda = \lambda + k$, namely, the multiplicity of the weight $\lambda + k$ is larger than one. Like representation (3.3), such a finite-dimensional representation (4.1) only appears in the completely 'quantum' cases resulting from the q-deformation for $q^p = 1$. Moreover, the cyclic conditions

$$\hat{X}(2p,n) = \hat{X}(0,n) \quad , \hat{X}(m,2p) = \hat{X}(m,0) \tag{4.2}$$

enable us to represent the basis vector $\hat{X}(m,n)$ by a lattice point (m,n) on a torus $S^1 \times S^1\{(x,y) \mid (x+2p, y+2p) = (x,y)\}$, which is formed by identifying the points $(x,2p)$ with $(x,0)$ and $(2p,y)$ with $(0,y)$, respectively, for the square domain $OABCD$ in Figure 1. In this sense, we say the representation (4.1) possesses the 'topology' of a torus.

5. Circle-Type Representations

In this section, we try to construct the representations of A, whose basis is labeled only by one index and is represented by the lattice points on a circle. To this end, we return to Equation (2.6) and consider the left ideal $H(\alpha)$ generated by $e_+ - \alpha e_-^{2p-1}$ ($\alpha \in \mathbb{C}$). On the quotient space $U = V(\lambda)/H(\alpha)$:

$$\{X(m) = X(m,0) \text{ Mod } H(\alpha) \mid m \in \mathbb{Z}^+\}$$

the representation (2.6) induces an infinite-dimensional representation

$$\begin{aligned} &e_+ X(2m) = \alpha X(2m+2p-1) + [m]([\lambda+1-m] - [\lambda-m])X(2m-1), \\ &e_+ X(2m+1) = -\alpha X(2m+2p) - ([m+1] - [m])]\lambda - m]X(2m), \\ &e_+ X(m) = X(m+1), \\ &K^{\pm} X(m) = q^{\pm\lambda\mp m} X(m). \end{aligned} \tag{5.1}$$

By direct calculation, we easily check that representation (5.1) indeed satisfies the basic relation (2.1).

Now, we consider the finite-dimensional representation. Let S_p be a left ideal generated by $e_-^{2p} - \beta$ ($\beta \in \mathbb{C}$). Then

$$\{\bar{X}(m) \equiv X(m) \text{ Mod } S_\beta \mid 0 \leqslant m \leqslant 2p-1\}$$

the basis for the quotient space $F = U/S$. A $2p$-dimensional representation on F immediately follows from Equation (5.1):

$$\begin{aligned} &e_+ \bar{X}(2m) = ([m]([\lambda+1-m] - [\lambda-m]) + \alpha\beta)\bar{X}(2m-1), \quad m \neq 0, \\ &e_+ \bar{X}(2m+1) = -([\lambda-m]([m+1] - [m]) + \alpha\beta)\bar{X}(2m), \\ &e_- \bar{X}(m) = \bar{X}(m+1), \quad 0 \leqslant m \leqslant 2p-2, \\ &e_- \bar{X}(2p-1) = \beta\bar{X}(0), \\ &e_+ \bar{X}(0) = \alpha\bar{X}(2p-1). \end{aligned} \tag{5.2}$$

When $\alpha, \beta \neq 0$, this representation has neither the lowest nor the highest weights; when $\alpha = 0$ ($\beta = 0$), it only has a highest weight (lowest weight). We point out that the lowest weight representation (5.2) for $\beta = 0$ can also be obtained on the cyclic Verma module $\mathscr{F}$:

$$\{F(m) = e_+^m|\lambda\rangle \mid e_-|\lambda\rangle = 0, K^{\pm}|\lambda\rangle = q^{\mp\lambda}, 0 \leqslant m \leqslant 2p-1\}$$

with the lowest weights $-\lambda$. As an example, we write down a six-dimensional representation ($p = 3$):

$$e_+ = -([\lambda] + \alpha\beta)E_{12} + ([\lambda] - [\lambda - 1] + \alpha\beta)E_{23} + (2[\lambda - 1] - \alpha\beta)E_{34} + \\ + ([\lambda - 2] - [\lambda - 1] + \alpha\beta)E_{45} - ([\lambda - 2] + \alpha\beta)E_{56} + \alpha E_{61},$$

$$e_- = E_{21} + E_{32} + E_{43} + E_{54} + E_{65} + \beta E_{16},$$

$$K_\pm = q^{\pm\lambda}(E_{11} + E_{44}) + q^{\pm(\lambda-1)}(E_{22} + E_{55}) + q^{\pm(\lambda-2)}(E_{33} + E_{66}).$$

6. Short Discussion

Up to now, we have obtained various representations of the quantum superalgebra $A = U_q\,\mathrm{osp}(2, 1)$ where the last one (5.2) can be understood as the super analogue of the cyclic representation of $\mathrm{sl}_q(2)$ (see the remark 4.2-(b) in ref. [9]). We hope the main ideas in this Letter can be applied to other quantum superalgebras. Since recent studies, show that the new representations of quantum algebra $\mathrm{sl}_q(2)$ resulted in some new solutions for the YBE [21–23], we can obtain new solutions of the YBE from the representations of the quantum superalgebra in this Letter. We have done this, in fact, for the parameters $\alpha, \beta = 0$ and detailed results are being prepared for publication. For the general cases with arbitrary parameters α, β and $\xi_\pm$, we may possibly generalize the scheme built by Jimbo *et al.* [10, 11].

Among all the representations presented in this Letter, (5.1) and (5.2) are obviously inequivalent because they have completely different properties in their dimensions. Representations (3.2) and (4.1) with different 'topologies' are also mutually unequivalent. In fact, there are only the equivalent representations defined by (3.3) for different N.

Acknowledgement

Many thanks to Prof. Z. Y. Wu and Drs X. F. Liu and K. Xue for the helpful discussions.

References

1. Drinfeld, V. G., *Proc. ICM*, Berkeley, 1986, p. 798.
2. Jimbo, L., *Lett. Math. Phys.* **10**, 63 (1985); **11**, 247 (1986).
3. Reshetikhin, N. Yu, LOMI preprints E-4 and E-11 (1987).
4. Rosso, M., *Comm. Math. Phys.* **117**, 581 (1988); **124**, 307 (1989).
5. Lusztig G., *Contemp. Math.* **82**, 59 (1989).
6. Roche, P. and Arnaudon, D., *Lett. Math. Phys.* **17**, 295 (1989).
7. Sun, C. P., Lu, J. F., and Ge, M. L., *J. Phys.* **A23**, L1199 (1990).
8. Pasquier, V. and Saleur, H., *Nuclear Phys.* **B330** 523 (1990).
9. ALverez-Gaume, L., Gomez, G., and Sirra, G., *Nuclear Phys.* **B330**, 523 (1990).
10. Concini, C. and Kac, V. G., Representations of quantum groups at roots of 1, preprint (1990).

11. Date, E., Jimbo, M., Mike, K., and Miwa, T., RIMS-703 preprint.
12. Date, E., Jimbo, M., Mike, K., and Miwa, T., RIMS-706 preprint.
13. Sun, C. P. and Ge, M. L., The cyclic boson operators and new representations of quantum algebra, preprint (1991).
14. Biedenharn, L. C., *J. Phys.* **A22**, L873 (1989).
15. Sun, C. P. and Fu, H. C., *J. Phys.* **A22**, L983 (1989).
16. Macfarlane, A. J., *J. Phys.* **A22**, 4551 (1989).
17. Chaichian, M., Kulish, P., and Lukierski, J., *Phys. Lett.* **B237**, 401 (1990).
18. Kulish, P. and Reshetikhin, N. Yu., *Lett. Math. Phys.* **19**, 143 (1989).
19. Saleur, H., *Nuclear Phys.* **B366**, 363 (1990).
20. Floreanini, R., Spiridonov, V. P., and Vinet, L., *Phys. Lett.* **B242**, 383 (1990).
21. Sun, C. P., Xue, K., Liu, X. F., and Ge, M. L., *J. Phys.* **A24**, L545 (1991).
22. Ge. M. L., Liu, X. F., and Sun, C. P., Submitted to *Lett. Math. Phys.* (1991).
23. Ge, M. L., Liu, X. F., and Sun, C. P., *Phys. Lett.* **A155**, 137 (1991).