Letters in Mathematical Physics **23**: 169–178, 1991.

New Boson Representations of the $\mathrm{sl}_q(2)$ with Multiplicity Two and New Solutions to the Yang–Baxter Equation at $q^p = 1$*

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(Received: 4 February 1991; revised version: 18 June 1991)

Abstract. When q is a root of unity, the representations of the quantum universal enveloping algebra $\mathrm{sl}_q(2)$ with multiplicity two are constructed from the q-deformed boson realization with an arbitrary parameter which is in a very general form and is first presented in this Letter. The new solutions to the Yang–Baxter equation are obtained from these representations through the universal R-matrix.

AMS subject classifications (1991). 81R10, 82B23, 17B35.

1. Introduction

Developments in nonlinear physics such as exactly solvable models in statistical mechanics and in low-dimensional quantum field theory, have made it clear that the Yang–Baxter equation is a key to the complete integrability of many physical systems, and much attention has been paid to its solutions in recent years [1–3]. For this reason, the representation theory of quantum universal enveloping algebra (also called quantum algebra for short) has aroused the interest of physicists and mathematicians because of the work of Drinfeld and other authors, which allows one to construct R-matrices (solutions to the Yang–Baxter equation without a parameter) from the representations of a quantum algebra through the so-called universal R-matrix.

As one knows, there are many ways of obtaining the representations of a quantum algebra [4–8]. One such method is the q-deformed boson realization method which is very simple but at the same time powerful [9–12]. This method was developed independently by different authors and has proved to be very useful. Recently, we have used different, q-deformed boson realizations of the quantum algebras $\mathrm{sl}_q(n)$ and $C_q(n)$ to study their representations, especially in the nongeneric case ($q^p = 1$), and have obtained some interesting results [12–14]. This Letter is a development of our previous work. We will first construct a class of λ-dependent boson realizations of $\mathrm{sl}_q(2)$ related to its regular representation, where λ is an arbitrary parameter and then, in the nongeneric case, present the representations of

* This work is supported in part by the National Foundation of Natural Science of China.

$sl_q(2)$ with multiplicity two, which are completely new so far as we know. As an important result and an example of a new type of R-matrices, in Section 5 we will explicitly give an R-matrix with a continuous parameter λ and a cyclic parameter q, which has a different block diagonal structure from that of the corresponding standard R-matrix and, hence, is different from those discussed in reference [15].

2. $sl_q(2)$ Representations and Related Boson Realizations

First, let us briefly describe two representations of the quantum algebra $sl_q(2)$. It is well known that $sl_q(2)$ is generated by the elements J_+, J_-, and J_3 which satisfy the commutation relations

$$[J_\pm, J_3] = \pm 2J_3, \qquad [J_+, J_-] = [J_3] = (q^{J_3} - q^{-J_3})/(q - q^{-1}). \tag{2.1}$$

Using these relations, one can get a representation of $sl_q(2)$ on the vector space

$$\{X(m, n) = J_+^m J_-^n \,|\lambda\rangle \mid J_3 \,|\lambda_3\rangle = \lambda \,|\lambda\rangle, m, n \in \mathbb{Z}^+ = \{0, 1, 2, \ldots\}\}.$$

Explicitly, one has

$$\begin{aligned} &J_+ X(m, n) = X(m+1, n), \\ &J_- X(m, n) = X(m, n+1) + [m][2n - m + 1 - \lambda]X(m-1, n), \\ &J_3 X(m, n) = (2(m-n) + \lambda)X(m, n). \end{aligned} \tag{2.2}$$

Actually, this is the quotient representation on $\Omega(\lambda) = sl_q(2)/I$,

$$\{X(m, n) = X(m, n, 0) \,\mathrm{Mod}\, I \mid m, n \in \mathbb{Z}^+\}$$

induced by the regular representation of $sl_q(2)$ on its own linear space

$$\{X(m, n, \gamma) = J_+^m J_-^n J_3^\gamma \mid m, n, \gamma \in \mathbb{Z}^+\},$$

where I is the left ideal generated by $(J_3 - \lambda)$. If another condition $J_+ \,|\lambda\rangle = 0$ is imposed on $|\lambda\rangle$, then one gets the representation

$$\begin{aligned} &J_+ X(n) = [n][\lambda - n + 1]X(n-1), \\ &J_- X(n) = X(n+1), \\ &J_3 X(n) = (\lambda - 2n)X(n), \end{aligned} \tag{2.3}$$

on the so-called quantum Verma module, where $X(n) = J_-^n \,|\lambda\rangle$.

Now, we turn to the general construction of the q-deformed boson realization of a quantum algebra from any representation of it. The q-deformed boson realization of a quantum algebra G_q is defined as the image of a homomorphic mapping from G_q to $\mathscr{B}_q(n)$, where $\mathscr{B}_q(n)$ is the so-called q-deformed boson algebra generated by a_i^+, $a_i^- \equiv a_i$ and N_i $(i = 1, 2, \ldots, n)$ which satisfy

$$\begin{aligned} &a_i^+ a_i = [N_i], \qquad a_i a_i^+ = [N_i + 1], \\ &[N_i, a_i^\pm] = \pm a_i^\pm, \qquad [a_i, a_j] = [a_i^+, a_j^+] = 0, \\ &[a_i^+, a_j] = [a_i, a_j^+] = 0, \quad i \neq j. \end{aligned} \tag{2.4}$$

Suppose V is the representation space carrying a representation ρ of the quantum algebra G_q, and each basis vector f of V is marked with $m_1, m_2, \ldots, m_n \in \mathbb{Z}^+$, i.e. one can write $f = X(m_1, m_2, \ldots, m_n)$, then the linear mapping

$$M = X(m_1, m_2, \ldots, m_n) \to a_1^{+m_1} a_2^{+m_2}, \ldots, a_n^{+m_n} |0\rangle$$

is obviously a one-to-one mapping between V and the Fock space $\mathscr{F}_q(n)$:

$$\{a_1^{+m_1} a_2^{+m_2}, \ldots, a_n^{+m_n} |0\rangle \mid a_i |0\rangle = N_i |0\rangle = 0,$$
$$i = 1, 2, \ldots, n; m_1, m_2, \ldots, m_n \in \mathbb{Z}^+\}.$$

If there exists a mapping B from G_q to $\mathscr{B}_q(n)$ such that the commutivity of the diagram

$$\begin{array}{ccc} V & \xrightarrow{M} & \mathscr{F}_q(n) \\ {\scriptstyle \rho(g)}\downarrow & & \downarrow{\scriptstyle B(g)} \\ V & \xrightarrow{M} & \mathscr{F}_q(n) \end{array}$$

is guaranteed for any $g \in G_q$, one can easily check $B(G_q)$ is a boson realization of G_q, which we call the boson realization associated with the representation ρ. We would like to point out that with the realization $B(G_q)$, we cannot obtain any new representations different from ρ on the Fock space $\mathscr{F}_q(n)$. But if we choose other representation spaces, we can expect to obtain some new results. Here, we will not consider this problem further because it is not the purpose of this Letter.

Having introduced the general principle, we now consider the quantum algebra $\mathrm{sl}_q(2)$ to show how this principle works. According to the principle, one can write down the boson realization

$$\begin{aligned} J_+ &= a_1^+, \\ J_- &= a_2^+ + a_1[2N_2 - N_1 + 1 - \lambda], \\ J_3 &= 2(N_1 - N_2) + \lambda, \end{aligned} \tag{2.5}$$

associated with the representation (2.2) after some simple calculation. Similarly, from the representation (2.3), one obtains the realization

$$\begin{aligned} J_+ &= a_1[\lambda - N_1 + 1], \\ J_- &= a_1^+, \\ J_3 &= (\lambda - 2N_1). \end{aligned} \tag{2.6}$$

In the next section, we will generalize the above realizations to a more general form, from which we can construct many new representations essentially different from (2.2) and (2.3) in the nongeneric case.

3. A More General Boson Realization of $\mathrm{sl}_q(2)$

The realizations (2.5) and (2.6) lead us to seek a more general realization of $\mathrm{sl}_q(2)$

$$\begin{aligned} J_+ &= a_1^+\alpha_1(N_1, N_2) + a_2\alpha_2(N_1, N_2), \\ J_- &= a_1\alpha_3(N_1, N_2) + a_2^+\alpha_4(N_1, N_2), \\ J_3 &= \alpha_5(N_1, N_2), \end{aligned} \tag{3.1}$$

where α_i $(i = 1, 2, \ldots, 5)$ are the operator-valued functions of N_1 and N_2 to be determined. Using the algebraic relations $[N_i, a_i^{\pm}] = \pm a_i^{\pm}$ $(i = 1, 2)$, one can easily prove

$$\begin{aligned} a_1^+ f(N_1, N_2) &= f(N_1 - 1, N_2)a_1^+, \qquad a_1 f(N_1, N_2) = f(N_1 + 1, N_2)a_1, \\ a_2^+ f(N_1, N_2) &= f(N_1, N_2 - 1)a_2^+, \qquad a_2 f(N_1, N_2) = f(N_1, N_2 + 1)a_2, \end{aligned} \tag{3.2}$$

for any function f defined by the Taylor series in N_1 and N_2. With (3.2) in mind, from the commutation relation (2.1) one obtains the constraint relations

$$\begin{aligned} &\alpha_5(N_1 + 1, N_2) - \alpha_5(N_1, N_2) = \alpha_5(N_1, N_2 - 1) - \alpha_5(N_1, N_2) = 2, \\ &\alpha_5(N_1 - 1, N_2) - \alpha_5(N_1, N_2) = \alpha_5(N_1, N_2 + 1) - \alpha_5(N_1, N_2) = -2, \end{aligned} \tag{3.3}$$

$$\begin{aligned} &\alpha_1(N_1, N_2 + 1)\alpha_4(N_1, N_2) = \alpha_1(N_1, N_2)\alpha_4(N_1 + 1, N_2), \\ &\alpha_2(N_1 - 1, N_2)\alpha_3(N_1, N_2) = \alpha_2(N_1, N_2)\alpha_3(N_1, N_2 - 1) \end{aligned} \tag{3.4}$$

and

$$\begin{aligned} &[N_1]\alpha_1(N_1 - 1, N_2)\alpha_3(N_1, N_2) - [N_1 + 1]\alpha_1(N_1, N_2)\alpha_3(N_1 + 1, N_2) + \\ &\quad + [N_2 + 1]\alpha_2(N_1, N_2 + 1)\alpha_4(N_1, N_2) - [N_2]\alpha_2(N_1, N_2)\alpha_4(N_1, N_2 - 1) \\ &= [\alpha_5(N_1, N_2)]. \end{aligned} \tag{3.5}$$

We do not intend to deal with these equations in the most general way. Instead, we hope to find some solutions general enough to meet our need.

The equations in (3.3) strongly suggest that we choose $\alpha_5(N_1, N_2) = 2(N_1 - N_2) + \lambda$, where λ is an arbitrary parameter. As for the other four functions to be determined, we need to consider the following three different cases

(1) one of them is zero,
(2) two of them are zeros,
(3) none of them is zero.

Careful calculation shows that in the first two cases, if

$$\begin{aligned} &\alpha_1(N_1 - 1, N_2)\alpha_3(N_1, N_2) = [2N_2 - N_1 + 1 - \lambda] \\ &\quad (\text{when } \alpha_2 = 0 \text{ or } \alpha_4 = 0 \text{ or } \alpha_2 = \alpha_4 = 0) \end{aligned} \tag{3.6}$$

or

$$\begin{aligned} &\alpha_2(N_1, N_2)\alpha_4(N_1, N_2 - 1) = [2N_1 - N_2 + 1 + \lambda] \\ &\quad (\text{when } \alpha_1 = 0 \text{ or } \alpha_3 = 0 \text{ or } \alpha_1 = \alpha_3 = 0), \end{aligned} \tag{3.7}$$

then (3.5) will be satisfied, and that in the third case one can choose

$$\begin{aligned}\alpha_1(N_1, N_2)\alpha_3(N_1+1, N_2) &= -q^{N_1-2N_2+\lambda}/(q-q^{-1}),\\ \alpha_2(N_1, N_2+1)\alpha_4(N_1, N_2) &= -q^{N_2-2N_1-\lambda}/(q-q^{-1}).\end{aligned} \tag{3.8}$$

It is now natural to ask whether there exist explicit solutions to Equations (3.4)–(3.8) besides those provided by (2.5) and (2.6). To answer this question, we give the following three simple realizations

$$J_+ = a_2, \quad J_- = a_2^+[2N_1 - N_2 + \lambda] + a_1, \quad J_3 = 2(N_1 - N_2) + \lambda, \tag{3.9}$$

$$J_+ = a_2, \quad J_- = a_2^+[2N_1 - N_2 + \lambda], \quad J_3 = 2(N_1 - N_2) + \lambda, \tag{3.10}$$

$$\begin{aligned}J_+ &= a_1^+ q^{1/2N_1 - N_2 + (2\lambda-1)/4} + a_2 q^{-N_1 + 1/2N_2 - (2\lambda+1)/4}\\ J_- &= -\frac{1}{q-q^{-1}}\left(a_1 q^{1/2N_1 - N_2 + (2\lambda-1)/4} + a_2^+ q^{-N_1 + 1/2N_2 - (2\lambda+1)/4}\right),\\ J_3 &= 2(N_1 - N_2) + \lambda.\end{aligned} \tag{3.11}$$

Obviously, they correspond to the above three cases, respectively.

4. λ-Dependent Representations of $sl_q(2)$

We consider the representations on the Fock space $\mathscr{F}_q(2)$

$$\{f(m_1, m_2) = a_1^{+m_1} a_1^{+m_2}|0\rangle \mid m_1, m_2 \in \mathbb{Z}^+;\ a_i|0\rangle = N_i|0\rangle = 0, i = 1, 2\}.$$

From the general realization (3.1), we have

$$\begin{aligned}J_+f(m_1, m_2) &= \alpha_1(m_1, m_2)f(m_1+1, m_2) + [m_2]\alpha_2(m_1, m_2)f(m_1, m_2-1),\\ J_-f(m_1, m_2) &= [m_1]\alpha_3(m_1, m_2)f(m_1-1, m_2) + \alpha_4(m_1, m_2)f(m_1, m_2+1),\\ J_3f(m_1, m_2) &= \alpha_4(m_1, m_2)f(m_1, m_2),\end{aligned} \tag{4.1}$$

where, of course, $\alpha_i = (m_1, m_2)$ $(i = 1, 2, \ldots, 5)$ should satisfy (3.3)–(3.5) with N_1 and N_2 replaced by m_1 and m_2. In the following, for simplicity, we suppose that if $\alpha_i(N_1, N_2) \neq 0$, then $\alpha_i(m_1, m_2) \neq 0$ for any $m_1, m_2 \in \mathbb{Z}^+$.

In the generic case, (4.1) is an infinite-dimensional irreducible representation and we cannot get a finite-dimensional representation from it. But if q is a root of unity, this conclusion is no longer true. In fact, when $q^p = 1$, we have the following three typical cases.

(1) $\alpha_i(N_1, N_2) \neq 0$ $(i = 1, 2, \ldots, 5)$. There are two kinds of $sl_q(2)$-invariant subspaces

$$W_i(\beta_i)\colon \{f(m_1, m_2) \mid m_i \geqslant \beta_i p, \beta_i \in \mathbb{Z}^+\}, \quad i = 1, 2.$$

so, on the quotient space

$$Q(\beta_1, \beta_2) = \mathscr{F}_q(2) \Big/ \sum_{i=1}^{2} W_i(\beta_i)\colon$$

$$\left\{f(m_1, m_2) \text{ Mod } \sum_{i=1}^{2} W_i(\beta_i) \mid 0 \leqslant m_i \leqslant \beta_i p - 1, i = 1, 2\right\},$$

(4.1) induces a $\beta_1\beta_2 p^2$-dimensional representation.

(2) $\alpha_1(N_1, N_2) = 0$ and $\alpha_i(N_1, N_2) \neq 0$ ($i = 1, 2, 3, 4, 5$). The subspaces

$$V_1\colon \{f(m_1, m_2) \mid m_1 \leqslant n, n \in \mathbb{Z}^+\}$$

and

$$V_2\colon \{f(m_1, m_2) \mid m_2 \geqslant \gamma p, \gamma \in \mathbb{Z}^+\}$$

are invariant under $\mathrm{sl}_q(2)$ and on the quotient space $Q(n, \gamma) = V_1/V_2$:

$$\{f(m_1, m_2) \text{ Mod } V_2 \mid 0 \leqslant m_1 \leqslant n, 0 \leqslant m_2 \leqslant \gamma p - 1\}$$

one gets an $(n+1)\gamma p$-dimensional representation induced by (4.1).

(3) $\alpha_1(N_1, N_2) = \alpha_3(N_1, N_2) = 0$ and $\alpha_i(N_1, N_2) \neq 0$ ($i = 2, 4, 5$). For a given $m_1 \in \mathbb{Z}^+$, there is an $\mathrm{sl}_q(2)$-invariant subspace

$$S = \{f(m_1, m_2) \mid m_2 \geqslant \delta p, \delta \in \mathbb{Z}^+\}$$

and on the corresponding quotient space $Q(m_1, \delta)$:

$$\{f(m_1, m_2) \text{ Mod } S \mid 0 \leqslant m_2 \leqslant \delta p - 1\}$$

we get a δp-dimensional representation.

For explicit examples, let us discuss the realizations (3.10) and (3.11). According to the above results, it follows from (3.10) that

$$\begin{aligned}
&J_+ F(m) = [m]F(m-1),\\
&J_- F(m) = [\lambda - m]F(m+1), \qquad J_- F(\delta p - 1) = 0,\\
&J_3 F(m) = (\lambda - 2m)F(m),
\end{aligned} \tag{4.2}$$

where $F(m) = f(0, m)$ Mod $S(m = 0, 1, \ldots, \delta p - 1)$. Likewise, from (3.11), one has the representation

$$\begin{aligned}
J_+ \tilde{F}(m_1, m_2) &= q^{(1/2)m_1 - m_2 + ((2\lambda - 1)/4)}\theta(m_1 - \beta_1 p + 1)\tilde{F}(m_1 + 1, m_2) + \\
&\quad + [m_2]q^{-m_1 + (1/2)m_2 - ((2\lambda + 1)/4)}\tilde{F}(m_1, m_2 - 1),\\
J_- \tilde{F}(m_1, m_2) &= -\frac{1}{q - q^{-1}}\, q^{(1/2)m_1 - m_2 + ((2\lambda - 1)/4)}[m_1]\tilde{F}(m_1 - 1, m_2) - \\
&\quad - \frac{1}{q - q^{-1}}\, q^{-m_1 + (1/2)m_2 - ((2\lambda + 1)/4)}\theta(m_2 - \beta_2 p + 1)\tilde{F}(m_1, m_2 + 1),\\
J_3 \tilde{F}(m_1, m_2) &= (2(m_1 - m_2) + \lambda)\tilde{F}(m_1, m_2),
\end{aligned} \tag{4.3}$$

where

$$\tilde{F}(m_1, m_2) = f(m_1, m_2) \text{ Mod } \sum_{i=1}^{2} W_i(\beta_i) \quad (m_i = 0, 1, 2, \ldots, \beta_i p - 1, i = 1, 2)$$

and

$$\theta(x) = \begin{cases} 1, & x < 0, \\ 0, & x \geqslant 0. \end{cases}$$

One can easily prove that both (4.3) and (4.2) are indecomposable representations.

5. λ-Dependent R-matrices

As is well known, given a representation of a quantum algebra, one can obtain an R-matrix by substituting it into the universal R-matrix associated with the quantum algebra. In this section, using the λ-dependent representations of $\mathrm{sl}_q(2)$, we will construct new R-matrices in this way, i.e. through the universal R-matrix

$$\mathcal{R} = q^{(1/2)J_3 \otimes J_3} \sum_{n=0}^{\infty} \frac{(1-q^{-2})^n}{[n]!} (q^{(1/2)J_3} J_- \otimes q^{-(1/2)J_3} J_-)^n q^{(1/2)n(n-1)}. \tag{5.1}$$

Let us discuss two examples.

For the first example, we consider the representation (4.2). For convenience we rewrite it as

$$\begin{aligned} &J_+ \psi_J(M) = [J+M]\psi_J(M-1), \\ &J_- \psi_J(M) = [\lambda_-(J+M)]\psi_J(M+1), \qquad J_-\psi_J(J) = 0, \\ &J_3 \psi_J(M) = (\lambda - 2(J+M))\psi_J(M), \end{aligned} \tag{5.2}$$

where we have defined

$$\psi_J(M) = F(m+J) \quad (M = -J, -J+1, \dots, J) \quad \text{and} \quad J = \frac{\delta p - 1}{2}.$$

Substituting the matrix form of (5.2) into (5.1), we obtain

$$\begin{aligned} (R^{JJ})^{M_1', M_2'}_{M_1, M_2} = {}& q^{(1/2)(\lambda - 2J - 2M_1')(\lambda - 2J - 2M_2')} \delta^{M_1'}_{M_1} \delta^{M_2'}_{M_2} + q^{(1/2)(\lambda - 2J - 2M_1')(\lambda - 2J - 2M_2')} \times \\ & \times \sum_{n=1}^{\infty} \frac{(1-q^{-2})^n}{[n]!} q^{-(1/2)n(n-1)} q^{n(M_2' - M_1')} \times \\ & \times \prod_{l=1}^{n} [J + M_1' + l][\lambda - J - M_2' + l]\, \delta^{M_1'}_{M_1 - n} \delta^{M_2'}_{M_2 + n}. \end{aligned} \tag{5.3}$$

Setting $J = 1/2$ and $p = 2$, we have

$$R^{(1/2)(1/2)} = q^{(1/2)\lambda^2 - \lambda} \times \begin{bmatrix} t & & & \\ & 1 & & \\ & t - t^{-1} & 1 & \\ & & & -t^{-1} \end{bmatrix}, \quad q^2 = -1, \tag{5.4}$$

where $t = -q^{-\lambda}$. It is worth pointing out that the R-matrix given by (5.4) has been obtained elsewhere by means of the extended Kauffman's diagram technique, but here it appears as a natural result of the quantum algebra theory.

For the second example, a more complicated one, let us study the representation (4.3). We will restrict our attention to the special case that $p=2$ ($q^p=-1$) and $\beta_1=\beta_2=1$. Then, it turns out to be a four-dimensional indecomposable representation of $\mathrm{sl}_q(2)$ on the space spanned by the basis vectors $\tilde{F}(0,0)$, $\tilde{F}(1,1)$, $\tilde{F}(1,0)$ and $\tilde{F}(0,1)$. It should be emphasized that in this case, $\tilde{F}(0,0)$ and $\tilde{F}(1,1)$ have the same weight λ. This fact, as will be seen, gives rise to a new block diagonal structure of the R-matrix. If we order the basis as

$$\tilde{F}(0,1)<\tilde{F}(0,0)<\tilde{F}(1,1)<\tilde{F}(1,0)$$

the matrix form of the representation is

$$J_+=\begin{bmatrix} 0 & 0 & 0 & 0\\ q^{1/4-\lambda/2} & 0 & 0 & 0\\ q^{+5/4+\lambda/2} & 0 & 0 & 0\\ 0 & q^{-1/4+\lambda/2} & q^{3/4-\lambda/2} & 0 \end{bmatrix},$$

$$J_-=-\frac{1}{q-q^{-1}}\begin{bmatrix} 0 & q^{-1/4-\lambda/2} & q^{-3/4+\lambda/2} & 0\\ 0 & 0 & 0 & q^{1/4+\lambda/2}\\ 0 & 0 & 0 & q^{-5/4-\lambda/2}\\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$J_3=\begin{bmatrix} -2+\lambda & & & \\ & \lambda & & 0\\ 0 & & \lambda & \\ & & & 2+\lambda \end{bmatrix}.$$

Substituting these matrices into (5.1) and properly arranging the obtained matrix elements, we can write the R-matrix as

$$R=\text{block diag}(A_1,A_2,A_3,A_2',A_1'), \tag{5.5}$$

where

$$A_1=-q^{(1/2)\lambda^2}t^2,\qquad A_1'=-q^{(1/2)\lambda^2}t^{-2},$$

$$A_2=-q^{(1/2)\lambda^2}\begin{bmatrix} -t & 0 & q^{-(1/2)t} & t^2\\ & -t & -1 & -q^{1/2}t\\ & & -t & 0\\ & & & -t \end{bmatrix},$$

$$A_2'=-q^{(1/2)\lambda^2}\begin{bmatrix} -t^{-1} & 0 & -q^{(1/2)}t^{-1} & -1\\ & -t^{-1} & t^{-2} & q^{-(1/2)}t^{-1}\\ & & -t^{-1} & 0\\ & & & -t^{-1} \end{bmatrix},$$

$$A_3 = -q^{(1/2)\lambda^2}\begin{bmatrix} 1 & -t^{-1} & -q^{-(1/2)} & -q^{(1/2)} & -t & 0 \\ & -1 & & & & -t \\ & & -1 & & & q^{(1/2)} \\ & & & -1 & & q^{-(1/2)} \\ & & & & -1 & -t^{-1} \\ & & & & & 1 \end{bmatrix}, \quad t = q^{\lambda}.$$

For comparison, we point out that if we use the four-dimensional angular momentum representation of $\mathrm{sl}_q(2)$ instead of the representation (4.3), the R-matrix will be in the form

$$R = \text{block diag}(A_1, A_2, A_3, A_4, A'_3, A'_2, A'_1), \tag{5.6}$$

where A_i $(i = 1, 2, 3, 4)$ and A_i $(i = 1, 2, 3)$ are $i \times i$ matrices. We notice that the block diagonal structures of (5.5) and (5.6) are different. We would like to reiterate that this is due to the fact that in the representation we have used, the multiplicity of the weight λ is two.

6. Conclusion

We have seen that with λ-dependent representations, we can obtain new R-matrices. In all our discussion, the introduction of the arbitrary parameter λ and the nongeneric condition $(q^p = 1)$ play a crucial role. The method discussed in this Letter is expected to be applied to other quantum algebras. We have also found that not only the standard R-matrices, which can be expressed through generic q-deformed C-G coefficients according to the Reshetikhin approach, but also other types of R-matrices can be dealt with from a quantum algebra point of view. The universal R-matrix is indeed universal.

Finally, we would like to point out in the nongeneric case $(q^{2p} = 1)$ in order that the universal R-matrix (5.1) makes sense, at least one of $J^p_\pm$ should be zero because $[p]! = 0$. In the above two examples where $q^4 = 1$, it is apparent that $J^2_\pm = 0$. More generally, one can prove that if the q-deformed boson realization of $\mathrm{sl}_q(2)$ is given in the form of (3.1) with $\alpha_1(N_1, N_2)$ or $\alpha_4(N_1, N_2) = 0$, then at least one of $J^p_\pm$ is zero when $q^{2p} = 1$. This fact enables us to obtain more R-matrices in the way reported in this Letter.

Acknowledgement

The authors are indebted to Dr K. Xue for helpful discussions.

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