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CONSTRUCTION OF GENERAL COLORED R MATRICES FOR THE YANG–BAXTER EQUATION AND q-BOSON REALIZATION OF QUANTUM ALGEBRA $\mathrm{sl}_q(2)$ WHEN q IS A ROOT OF UNITY*

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Through a general q-boson realization of quantum algebra $\mathrm{sl}_q(2)$ and its universal R matrix an operator R matrix with many parameters is obtained in terms of q-boson operators. Building finite-dimensional representations of q-boson algebra, we construct various colored R matrices associated with nongeneric representations of $\mathrm{sl}_q(2)$ with dimension-independent parameters. The "nonstandard" R matrices obtained by Lee-Couture and Murakami are their special examples. We also study the factorizable structure of some R matrices for the indecomposable representations used in its construction.

1. Introduction

According to Drinfeld, Jimbo and Reshetikhin,[1–3] solutions to the spectral parameter-free Yang–Baxter equation (YBE) can be constructed in terms of quantum universal enveloping algebras $U_q(L)$ (quantum algebras) of a simple Lie algebra L. Sometimes, those solutions are called R matrices. As an associative algebra over complex number field $\mathbb{C}$, $U_q(L)$ can be endowed with a quasi-triangular Hopf algebraic structure leading to the relations between YBE and $U_q(L)$.

Let $\{e_s\}$ be a basis for a certain Borel subalgebra $U_q(b^+)$ in $U_q(L)$ and $\{e^s\}$ the basis for its dual, $U_q(b^+)^0$. The quantum double theory defines the so-called universal R matrix

$$\mathcal{R} = \sum_s e_s \otimes e^s \in U_q(L) \otimes U_q(L) .$$

If $\rho^\lambda : U_q(L) \to \mathrm{End}\ (V^\lambda)$ is the representations of $U_q'(L)$ on space V^λ and

$$R_{12}^{\lambda_1\lambda_2} = \sum_s \rho^{\lambda_1}(e_s)\otimes \rho^{\lambda_2}(e^s) \otimes I^{\lambda_3} ,$$

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$$R_{13}^{\lambda_1\lambda_3} = \sum_s \rho^{\lambda_1}(e_s) \otimes I^{\lambda_2} \otimes \rho^{\lambda_3}(e^s) ,$$

$$R_{23}^{\lambda_2\lambda_3} = \sum_s I^{\lambda_1} \otimes \rho^{\lambda_2}(e_s) \otimes \rho^{\lambda_3}(e_s) , \qquad \in \text{End}\,(V^{\lambda_1} \otimes V^{\lambda_2} \otimes V^{\lambda_3}) ,$$

the quasi-triangular Hopf structure ensures that the R matrix

$$R^{\lambda\lambda'} = \rho^{\lambda} \otimes \rho^{\lambda'}(\mathcal{R}) = \sum_s \rho^{\lambda}(e_s) \otimes \rho^{\lambda'}(e^s) \tag{1.1}$$

satisfies the YBE

$$R_{12}^{\lambda_1\lambda_2} R_{13}^{\lambda_1\lambda_3} R_{23}^{\lambda_2\lambda_3} = R_{23}^{\lambda_2\lambda_3} R_{13}^{\lambda_1\lambda_3} R_{12}^{\lambda_1\lambda_2} . \tag{1.2}$$

So far a lot of R matrices have been derived according to the above-mentioned standard approach, and they are associated with generic irreducible representations of $U_q(L)$ when q is not a root of unity. These R matrices can be expressed in terms of q-CG coefficients of $U_q(L)$[3,4] and are called standard R matrices.

On the other hand, associated with representations of SU(2) having certain spins and the fundamental representations of other classical Lie algebras, a number of new R matrices for the YBE have been obtained[5-8] by using the extended Kauffman's diagram.[9,10] These R matrices are called nonstandard R matrices, because they possess quite different properties in comparison to the standard ones. Notice that some of them are also related to quantum superalgebras.[11] In order to understand the nonstandard R matrices, the present authors tried to construct them from the standard one by taking into account the concept of weight conservation.[12] More recently, we succeeded in constructing them in connection with nongeneric representations of quantum algebras at q, a root of unity;[14-16] the so-called q (deformed) boson realization theory was used.[17-19]

The purposes of this paper are to generalize the studies in Refs. 17–19 in a quite general framework and construct the completely new R matrices, such as the colored R matrices, by studying the representations of q-boson operators and the quantum algebra $\mathrm{sl}_q(2)$ in the nongeneric case where q is a root of unity. Notice that the representations of quantum algebras in this case are very useful in physics[20-23] and have been studied by many authors.[24-31] In Refs. 24 and 25, the new R matrices are considered for the generalized Potts model, but the nonstandard ones are not discussed. In this paper, we first build a general parameter-dependent q-boson realization of $\mathrm{sl}_q(2)$. It leads to a two-parameter-dependent q-boson realization of the universal R matrix for $\mathrm{sl}_q(2)$ and defines finite-dimensional representations with the dimension-independent parameters in the nongeneric case for $\mathrm{sl}_q(2)$. Associated with two such representations with different parameters, the general expression for new R matrices is obtained. If the two representations used here have the same dimension but the different parameters, the obtained R matrices are the colored R matrices, and the two parameters labeling the two representations are called colors of the R matrices. This is because their four-dimensional case directly results in the colored R matrix used by Murakami in constructing new Alexander polynomials.[32] When two colors become the same, the colored R matrices given by us not only lead to the nonstandard R matrices obtained before[5–8] in the lower-dimensional case, but also define higher-dimensional nonstandard R matrices as completely new results. Notice that some of the representations of $\mathrm{sl}_q(2)$ obtained in this paper can also be given from a generally mathematical point of view, but our q-boson realization method not only leads to these results very conveniently, but also emphasizes the finite-dimensional indecomposable (reducible, but not completely reducible) representations of $\mathrm{sl}_q(2)$. The latter cause the factorizable structures in the obtained colored R matrices.

It should be pointed out that the basic ideas and method in this paper can be almost directly applied to the case of quantum algebra $\mathrm{sl}_q(N)$ for $N \geq 3$.

2. The q-Boson Realizations of $\mathrm{sl}_q(2)$ with Many Parameters

The q-boson operators were introduced by different authors to realize quantum algebra $\mathrm{sl}_q(2)$ and $\mathrm{sl}_q(N)$. Up to now, they have been named by the terminology[33–38] as the so-called q-boson algebra $\mathcal{B}_q$.

Definition 2.1. The q-boson algebra is an associative algebra over $\mathbb{C}$ generated by q-boson operator a, $a^- \equiv a$ and $\hat{N}$ satisfying

$$a^+ a = [\hat{N}]\,, \qquad aa^+ = [\hat{N} + 1]\,, \qquad [\hat{N}, a^\pm] = {}^\pm a^\pm \tag{2.1}$$

where $q \in \mathbb{C}$ and

$$[f] = \frac{q^f - q^{-f}}{q - q^{-1}}$$

is defined for any operator or number f. Its n-multiple tensor algebra is called n-q-boson algebra and is denoted by $\mathcal{B}_q(n)$.

Definition 2.2. The image $B(U_q(L))$ of a quantum algebra $U_q(L)$ under an algebra homomorphism $B : U_q(L) \to \mathcal{B}_q(n)$ is called the q-boson realization of $U_q(L)$.

In fact, the mapping B embeds $U_q(L)$ into $\mathcal{B}_q(n)$ as a subalgebra. With the help of a direct calculation, we prove the following proposition.

Proposition 2.1. If functions $\alpha(\hat{N})$ and $\beta(\hat{N})$ of $\hat{N}$ satisfy

$$\alpha(\hat{N} - 1) \cdot \beta(\hat{N}) = [\lambda + 1 - \hat{N}]\,, \qquad \lambda \in \mathbb{C}\,, \tag{2.2}$$

then mapping $B : \mathrm{sl}_q(2) \to \mathcal{B}_q$,

$$\begin{aligned} \hat{J}_+ &\to J_+ = B(\hat{J}_+) = a^+ \cdot \alpha(\hat{N})\,, \\ \hat{J}_- &\to J_- = B(\hat{J}_-) = a \cdot \beta(\hat{N})\,, \\ \hat{J}_0 &\to J_0 = B(\hat{J}_0) = 2\hat{N} - \lambda\,, \end{aligned} \tag{2.3}$$

defines a q-boson realization of $\mathrm{sl}_q(2)$ with generators $\hat{J}_\pm$ and $\hat{J}_0$, satisfying

$$[\hat{J}_+, \hat{J}_-] = [\hat{J}_0] , \qquad [\hat{J}_0, \hat{J}_\pm] = \pm 2\hat{J}_\pm . \tag{2.4}$$

A solution,

$$\alpha(\hat{N}) = 1 , \qquad \beta(\hat{N}) = [\lambda + 1 - \hat{N}] ,$$

to Eq. (2.2) leads to a particular q-boson realization of $\mathrm{sl}_q(2)$,

$$J_+ = a^+ , \qquad J_- = a \cdot [\lambda + 1 - \hat{N}] , \qquad J_0 = 2\hat{N} - \lambda . \tag{2.5}$$

On the q Fock space F_q,

$$\{|n\rangle = a^{+n}|0\rangle \qquad |a|0\rangle = \hat{N}|0\rangle = 0 , \qquad n \in \mathbb{Z}^+ = \{0, 1, 2, \ldots\}\} ,$$

the q-boson realization (2.5) defines a Verma representation,

$$\begin{cases} J_+|n\rangle = |n+1\rangle , \\ J_-|n\rangle = [n][\lambda + 1 - n]|n-1\rangle , \\ J_0|n\rangle = (2n - \lambda)|n\rangle , \end{cases} \tag{2.6}$$

of $\mathrm{sl}_q(2)$, which has been obtained in Refs. 27 and 30.

As the generators of $\mathrm{sl}_q(2)$ can be expressed in terms of q-boson operators, it is natural to expect that a representation of $\mathrm{sl}_q(2)$ can be constructed from a representation of $\mathcal{B}_q(n)$. The following proposition provides us with a scheme for this purpose.

Proposition 2.2. If $\rho_F : \mathcal{B}_q(n) \longrightarrow \mathrm{End}(V)$ is a representation of $\mathcal{B}_q(n)$ on space V and B: $U_q(L) \longrightarrow \mathcal{B}_q(n)$ defines a q-boson realization of quantum algebra $U_q(L)$, then the following commutative diagram,

defines a representation $\rho : U_q(L) \longrightarrow \mathrm{End}(V)$.

In fact, we can explicitly write down $\rho = \rho_F \cdot B$ through

$$\rho(g) \cdot x = \rho_F(B(g)) \cdot x , \qquad \forall g \in U_q(L) , \qquad \forall x \in V . \tag{2.7}$$

Since the manipulation obtaining representations of $U_q(L)$ can be transformed into the discussion of representation for $\mathcal{B}_q(n)$, it is necessary to study the representation theory of $\mathcal{B}_q$.

3. Finite-Dimensional Representation of $\mathcal{B}_q$ and Its Realized Representation of $\mathrm{sl}_q(2)$ at q, a Root of Unity

To construct the new R matrices of $\mathrm{sl}_q(2)$ from its universal R matrix,

$$R = q^{J_0 \otimes J_{0/2}} \cdot \sum_{n=0}^{\infty} \left\{ (q^{J_{0/2}} \cdot J_+ \otimes q^{-J_{0/2}} \cdot J_-) q^{\frac{1}{2}n(n-1)} \cdot \frac{(1-q^{-2})^n}{[n]!} \right\} , \quad (3.1)$$

we must use the finite-dimensional representation (FDR) of $sl_q(2)$ and thus first study the FDR of q-boson algebra $\mathcal{B}_q$ in general. Notice that some FDR's of $\mathcal{B}_q$ have been obtained when q is a root of unity.[38] Now, we study the general aspects of construction for the FDR of $\mathcal{B}_q$.

Proposition 3.1. If $\mathcal{B}_q$ has an FDR, then q must be a root of unity.

Proof. Let V be the space of the FDR and dim $V < \infty$. Since $\mathbb{C}$ is algebraically closed, there exists $u \in V$ such that

$$\hat{N} u_0 = \lambda u_0 , \qquad \lambda \in \mathbb{C} .$$

Because u_0, $a^+ u_0$, $a^{+2} u_0, \ldots, a^{+m} u_0, \ldots$ are eigenvectors of N with different eigenvalues λ, $\lambda + 1$, $\lambda + 2, \ldots, \lambda + m, \ldots$, they are linearly independent. Owing to dim $V < \infty$, there must be $l \in \mathbb{Z}^+$ such that

$$(a^+)^l u_0 = 0 , \qquad (a^+)^{l-1} u_0 \neq 0 .$$

Define $f_0 = (a^+)^{l-1} u_0$. Similarly, there is $l' \in \mathbb{Z}^+$ such that

$$(a)^{l'} f_0 = 0 , \qquad (a)^{l'-1} f_0 \neq 0 .$$

Then we have

$$0 = a^+ a^{l'} f_0 = [\hat{N}] a^{l'-1} f_0 = [\lambda + l - l'](a^{l'-1} f_0) ,$$

$$0 = a \cdot a^{+l} u_0 = [\hat{N} + 1] a^{+l-1} u_0 = [\lambda + l](a^{+l-1} u_0) ,$$

i.e.

$$[\lambda + l - l'] = 0 = [\lambda + l] .$$

Thus $q^{2l'} = 1$ that is to say, q is a root of unity. The proposition is proved.

Define $v_0 = a^{l'-1} f_0$. The above proof also leads to the following proposition.

Proposition 3.2. In an FDR of $\mathcal{B}_q$, there exist a "vacuum state" v_0 such that

$$\hat{N} v_0 = 0 \ \mathrm{Mod}(p) , \qquad a \cdot v_0 = 0 .$$

According to Proposition 1, to study the problems related to the FDR of $\mathcal{B}_q$ we need only to focus on the nongeneric case where q is a root of unity. According to Proposition 2, we define

$$\{F(m) = a^{+m} v_0 | m \in \mathbb{Z}^+\}$$

as a space for representation space V of an FDR. Since dim $V < \infty$, there must be d $(< \infty) \in \mathbb{Z}^+$ succh that $F(d) = 0$. Then, the equation

$$0 = aa^+ F(d - 1) = [d]F(d - 1)$$

gives $[d] = 0$, i.e.

$$d = \alpha p \text{ (for } \alpha p = \text{odd) or } \frac{1}{2}p \text{ (for } \alpha p = \text{even)} , \qquad \alpha \in \mathbb{Z}^+ .$$

Without affecting construction of R matrices, we take k Mod(p) to be k simply for $k \in \mathbb{Z}^+$; we obtain an FDR of

$$\begin{cases} a^+ F(m) = \theta(d - 1 - m)F(m + 1) , \\ a \cdot F(m) = [m]F(m - 1) , \\ \hat{N} \cdot F(m) = m\, F(m) , \end{cases} \tag{3.2}$$

where $\theta(x) = 1$ for $x \geq 0$ and $\theta(x) = 0$ for $x < 0$. Denote V by Q_α as follows.

From this FDR of $\mathcal{B}_q$ and q-boson realization (2.3), we immediately construct an FDR of $\mathrm{sl}_q(2)$:

$$\begin{cases} J_+ F(m) = \theta(d - 1 - m)\alpha(m)F(m + 1) , \\ J_- F(m) = [m] \cdot \beta(m) \cdot F(m - 1) , \\ J_0 F(m) = (2m - \lambda)F(m) , \end{cases} \tag{3.3}$$

where $\alpha(m) = \alpha(m, \lambda)$ and $\beta(m) = \beta(m, \lambda)$ satisfy

$$\alpha(m - 1) \cdot \beta(m) = [\lambda + 1 - m] . \tag{3.4}$$

It can be proved that the FDR's (3.2) and (3.3) of $\mathcal{B}_q$ and $\mathrm{sl}_q(2)$ are indecomposable when $\alpha \geq 2$ and irreducible for $\alpha = 1$ (the proof is similar to that in Ref. 38).

4. General Formula of New R Matrices

In terms of the q-boson realization (2.3) of $\mathrm{sl}_q(2)$ the universal R matrix is rewritten as the q-boson operator form—q-boson realized R matrix:

$$R^{[\lambda_1,\lambda_2]} = q^{(2\hat{N}-\lambda_1)\otimes(2\hat{N}-\lambda_2)/2} \sum_{n=0}^{\infty} \frac{(1-q^{-2})^n}{[n]!} q^{\frac{1}{2}n(n-1)}$$

$$\times \{q^{\hat{N}-(\lambda_1/2)}a^+\alpha(\hat{N},\lambda_1) \otimes q^{-\hat{N}+(\lambda_2/2)}a\beta(\hat{N},\lambda_1)\}^n , \quad (4.1)$$

where $\alpha(\hat{N}, \lambda_i)$ and $\beta(\hat{N}, \lambda_i)$ satisfy

$$\alpha(\hat{N}-1, \lambda_i)\beta(\hat{N}, \lambda_i) = [\lambda_i + 1 - \hat{N}] .$$

In (4.1) we have introduced two parameters, λ_1 and λ_2, which are called the colors of the R matrix and whose meaning will be discussed.

In order to write out the R matrix of $R^{(\lambda_1,\lambda_2)}$ on certain representation spaces in a well-defined matrix form, we define a new basis,

$$\{\Psi_j(m) = F(j+m)| , \qquad m = j, j-1, \ldots, -j\} ,$$

for the space $Q_\alpha(=V)$ of the representation (3.3) of $\mathrm{sl}_q(2)$ or the representation (3.2) of $\mathcal{B}_q$, where

$$j = \frac{\alpha p - 1}{2} .$$

On this basis, the FDR $\rho^{j(\lambda)}$ of $\mathrm{sl}_q(2)$ is rewritten as

$$\begin{cases} J_+\Psi_j(m) = \theta(j-1-m)\alpha(j+m)\Psi_j(m+1) , \\ J_-\Psi_j(m) = [j+m]\beta(j+m)\Psi_j(m-1) , \\ J_0\Psi_j(m) = (2j+2m-\lambda)\Psi_j(m) , \end{cases} \quad (4.2)$$

and the FDR (3.2) of $\mathcal{B}_q$ with $j = (\alpha p - 1)/2$ is rewritten as

$$\begin{cases} a^+\Psi_j(m) = \theta(j-1-m)\Psi_j(m+1) , \\ a\Psi_j(m) = [j+m]\Psi_j(m-1) , \\ \hat{N}\Psi_j(m) = (j+m)\Psi_j(m+1) . \end{cases} \quad (4.3)$$

Using the universal R matrix (3.1) combined with the representation $\rho^{j(\lambda)}$ of $\mathrm{sl}_q(2)$, or the boson realized R matrix $R^{\lambda_1\lambda_2}$ combined with the representation ρ^j of $\mathcal{B}_q$, we obtain the general R matrix

$$R^{j_1(\lambda_1)j_2(\lambda_2)} = \rho^{j_1(\lambda_1)} \otimes \rho^{j_2(\lambda_2)}(\check{R})$$

$$\text{or} = (\rho^{j_1} \otimes \rho^{j_2})(R^{\lambda_1\lambda_2}) \tag{4.4}$$

with the matrix elements

$$(R^{j_1(\lambda)j_2(\mu)})^{m_1'm_2'}_{m_1m_2} = q^{2(j_1+m_1'-\lambda/2)(j_2+m_2'-\mu/2)} \Bigg\{ \delta^{m_1'}_{m_1}\delta^{m_2'}_{m_2}$$

$$+ \sum_{n=0}^{k} \frac{(1-q^{-2})^n}{[n]!} q^{(-1/2)n(n-1)+n(j_1-j_2+m_1'-m_2'-\lambda/2+\mu/2)} \tag{4.5}$$

$$\times \prod_{l=0}^{n} \alpha_{j_1,m_1+l-1}(\lambda)\beta_{j_2,m_2-l+1}(\mu)[j_2+m_2-l+1]\delta^{m_1'}_{m_1+n}\delta^{m_2'}_{m_2-n} \Bigg\},$$

where $k = \min(2j_1, 2j_2)$ and $\alpha_{j\cdot m}(\lambda) = \alpha(j+m, \lambda)$, $\beta_{j\cdot m}(\lambda) = \beta(j+m, \lambda)$.

Although q is only a cyclic parameter due to $q^p = 1$, the continuous parameters λ and μ are still contained in the obtained R matrices, and the continuous parameters $t_\lambda = q^{f(\lambda)}$ may be taken in those R matrices, which play the role of q in the standard R matrix. Introduction of these parameters λ and μ is the key to our studies.

Notice that there exist two different representations, $\rho^{j(\lambda)}$ and $\rho^{j(\mu)}$, of $sl_q(2)$ for $\lambda \neq \mu$ with the same dimension, $2j+1$. This feature appears only for the nongeneric case and can be regarded as a "purely quantum" phenomenon.

Definition 4.1. In the representation $\rho^{j(\lambda)}[(4.2)]$ of $sl_q(2)$ the continuous parameter λ which is independent of the dimension $2j+1$ of $\rho^{j(\lambda)}$ is called the color of the representation.

5. Colored R Matrices and Quantum Group Construction of the Obtained Nonstandard R Matrices

Now, we apply the general formula (4.5) to write some typical R matrices, including the nonstandard R matrices obtained before and the completely new ones.

(1) In the case with $q^2 = -1$, we have a 2×2 R matrix,

$$R^{1/2\,1/2} = R^{1/2(\lambda)1/2(\lambda)} = \begin{bmatrix} t & & & \\ & 1 & t-t^{-1} & \\ & & 1 & \\ & & & -t^{-1} \end{bmatrix}, \tag{5.1}$$

where $t = -q^{-\lambda} = -e^{(i\pi\lambda/4)}$ is a continuous parameter, which can also be taken to be real, and $j_1 = j_2 = 1/2$. In fact, $R^{1/2\,1/2}$ is just the unique 2×2 nonstandard R matrix given in Ref. 5.

(2) In the case with $q^3 = 1$ and $j_1 = j_2 = 1$, we have a 9×9 R matrix with the block diagonal structure

$$R^{11} = R^{1(\lambda)1(\lambda)} = \begin{bmatrix} A_1 & & & & \\ & A_2 & & & \\ & & A_3 & & \\ & & & A_2' & \\ & & & & A_1' \end{bmatrix}, \tag{5.2}$$

where

$$A_1 = t, \qquad A_1' = q^{-1}t^{-3},$$

$$A_2 = \begin{bmatrix} 1 & t - t^{-1} \\ 0 & 1 \end{bmatrix}, \qquad A_2' = \begin{bmatrix} qt^{-2} & q^{-1}t^{-3} - t^{-1} \\ 0 & qt^{-2} \end{bmatrix},$$

$$A_3 = \begin{bmatrix} t^{-1} & qt^{-1}Q_1 & (t - t^{-1})(1 - q^{-1}t^{-2}) \\ & q^{-1}t^{-1} & qt^{-1}Q_2 \\ & & t^{-1} \end{bmatrix},$$

$$Q_1 = (q - q^{-1})\alpha_{1,0}(\lambda)\beta_{1,0}(\lambda), \qquad Q_2 = -(q - q^{-1})\alpha_{1-1}(\lambda)\beta_{11}(\lambda),$$

and the continuous parameter in R^{11} is $t = q^{1-\lambda}$. The particular case of (5.2) with $Q_1 = Q_2$ is just the 9×9 nonstandard R matrix in Ref. 5.

In general, other nonstandard R matrices of $\mathrm{sl}_q(2)$ in Ref. 5 can also be obtained by using the general formula (4.5) and letting $j_1 = j_2 = 1, 3/2, 2, 5/2$ and defining $t = (-1)^{2j}q^{2j-1-\lambda}$.

Before we introduce the concept of the colored R matrix, we consider the simplest case, with $q^2 = -1$ and $j_1 = j_2 = 1/2 = j$. In this case, by associating with two different nongeneric representations of $\mathrm{sl}_q(2)$ with the same dimension $(2j + 1 = 2)$ and different colors (λ and μ), the 2×2 R matrix $R^{1/2(\lambda)1/2(\mu)}$ on the tensor space of $\rho^{1/2(\lambda)} \otimes \rho^{1/2(\mu)}$ is obtained from (4.5):

$$R^{1/2(\lambda)1/2(\mu)} = R^{1/2}(\lambda, \mu) = \begin{bmatrix} -t_\lambda t_\mu & & & \\ & t_\lambda t_\mu^{-1} & W(\lambda, \mu) & \\ & & t_\lambda^{-1} t_\mu & \\ & & & t_\lambda^{-1} t_\mu^{-1} \end{bmatrix} \tag{5.3}$$

where $t_\xi = q^{-(1/2)\xi}(\xi = \lambda, \mu)$ and

$$W(\lambda, \mu) = t_\lambda^2 t_\mu^{-2}(q - q^{-1})\alpha_{1/2,-1/2}(\lambda)\beta_{1/2,1/2}(\mu).$$

When $t = it_\lambda^{1/2}$, $\alpha_{1/2,-1/2}(\lambda) = t_\lambda^{-1}$ and $\beta_{1/2,1/2}(\mu) = t\mu$,

$$R^{1/2(\lambda)1/2(\mu)} = (\bar{t}_\lambda \bar{t}_\mu)^{1/2} \begin{bmatrix} \bar{t}_\lambda & & & \\ & \bar{t}_\lambda \bar{t}_\mu^{-1} & \bar{t}_\mu^{-1}(\bar{t}_\lambda^2 - 1) & \\ & & 1 & \\ & & & -\bar{t}_\mu^{-1} \end{bmatrix} \tag{5.4}$$

is just the R matrix used by Murakami[32] in constructing a new Alexander link polynomial where the parameters λ and μ (or t_λ and t_μ) are regarded as the colors of the link.

Definition 5.1. The R matrix $R^{j(\lambda)j(\mu)}$, defined on the product $\rho^{j(\lambda)} \otimes \rho^{j(\mu)}$ for two different representations, $\rho^{j(\lambda)}$ and $\rho^{j(\mu)}$ $(\lambda \neq \mu)$, with the same dimension $(2j + 1)$ is called the colored R matrix and (λ, μ) are called its colors.

In fact, the general formula also results in a high rank colored R matrix, such as the 9×9 R matrix $R^{11}(\lambda, \mu)$ for $q^3 = 1$:

$$R^{11}(\lambda, \mu) = R^{1(\lambda)1(\mu)} = \begin{bmatrix} B_1 & & & & \\ & B_2 & & & \\ & & B_3 & & \\ & & & B_2' & \\ & & & & B_1' \end{bmatrix},$$

$$B_1 = t_\lambda t_\mu , \qquad B_1' = q t_\lambda^{-1} t_\mu^{-1} , \tag{5.5}$$

$$B_2 = \begin{bmatrix} q^2 t_\mu & W_1(\lambda, \mu) \\ 0 & q^2 t_\lambda \end{bmatrix}, \qquad B_2' = \begin{bmatrix} q t_\lambda^{-1} & W_{-1}(\lambda, \mu) \\ 0 & q t_\mu^{-1} \end{bmatrix},$$

$$B_3 = \begin{bmatrix} q t_\lambda^{-1} t_\mu & X_1(\lambda, \mu) & W_0(\lambda, \mu) \\ & 1 & X_2(\lambda, \mu) \\ & & q t \lambda t_\mu^{-1} \end{bmatrix},$$

where $t_\lambda = q^{-\lambda}$, $t_\mu = q^{-\mu}$ and

$$W_1(\lambda, \mu) = -(1 - q) t_\lambda^{1/2} t_\mu^{1/2} \alpha_{10}(\lambda) \beta_{1,1}(\mu) ,$$

$$W_0(\lambda, \mu) = q(1 - q^2) \alpha_{1,-1}(\lambda) \alpha_{1,0}(\lambda) \beta_{1,0}(\mu) \beta_{1,1}(\mu) ,$$

$$W_{-1}(\lambda, \mu) = (1 - q) q^2 t_\lambda^{-1/2} t_\mu^{-1/2} \alpha_{1,-1}(\lambda) \beta_{10}(\mu) ,$$

$$X_1(\lambda, \mu) = (1 - q) t_\lambda^{-1/2} t_\mu^{1/2} \alpha_{1,0}(\lambda) \beta_{1,0}(\mu) ,$$

$$X_2(\lambda, \mu) = -(1 - q) t_\lambda^{-1/2} t_\mu^{1/2} \alpha_{1,-1}(\lambda) \beta_{1,1}(\mu) .$$

Notice that the colors λ and μ in the colored R matrix $R^{j(\lambda)j'(\mu)} = R^{jj'}(\lambda, \mu)$ appear as if they were the dynamic spectral parameter because of the Yang–Baxter equation

$$(\check{R}^{j_1j_2}(\lambda, \mu) \otimes I)(I \otimes \check{R}^{j_3j_2}(\gamma, \lambda))(\check{R}^{j_3j_1}(\gamma, \lambda) \otimes I)$$

$$= (I \otimes \check{R}^{j_3j_1}(\gamma, \lambda))(\check{R}^{j_3j_2}(\gamma, \lambda) \otimes I)(I \otimes \check{R}^{j_1j_2}(\lambda, \mu)) , \quad (5.6)$$

which is formally satisfied by $R^{jj'}(\lambda, \mu)$.

6. The Factorization Structure and Other Properties of New *R* Matrices

In constructing the new R matrices, we use the nongeneric condition $q^p = 1$ so that some zeros, $[\alpha p] = 0$, appear in the denominator of the universal R matrix R. Now, we must find out for which kind of representation the universal R matrix still makes sense in the nongeneric case. With the $\mathrm{sl}_q(2)$ case as an example, we first decompose the universal R matrix (3.1) into two parts,

$$R = R_0 + R_p ,$$

where

$$R_0 = q^{J_0 \times J_0/2} \sum_{n \neq kp} \frac{(1 - q^{-2})^n}{[n]!} (q^{(1/2)J_0}J_+ \otimes q^{-(1/2)J_0}J_-)^n \cdot q^{-(1/2)n(n-1/2)} , \quad (6.1)$$

$$R_p = q^{J_0 \times J_0/2} \sum_{k=0}^{\infty} (1 - q^{-2})^{kp} q^{-(1/2)kp(kp-1)} q^{(1/2)kpJ_0} J_+^{kp} \otimes q^{-(1/2)kpJ_0}(L_-)^k ,$$

$\hat{L}_- = (J_-)^p/[p]!$ being the Lusztig operator. Then, we conclude that if $\hat{L}_-$(or $\hat{L}_+$) is well defined for a representation ρ, that universal R matrix makes sense. For the representation (3.3), $\hat{L}_-$ is obviously well defined and we can use the universal matrix R freely. In fact, the practical calculations obtaining new R matrices in the last subsection also show the validity of R.

Since some nongeneric representations we used are indecomposable, we need to consider the factorization structure in terms of them. In fact, If ρ_σ ($\sigma = a, b$) are indecomposable representations of $U_q(L)$ on the space V_σ, $V_\sigma^{J_\sigma}$ are the invariant subspaces in V_σ and $\rho_\sigma^{[J_\sigma]}$ are the corresponding subrepresentation on $V_\sigma^{J_\sigma}$, i.e.

$$\rho_\sigma = \begin{bmatrix} \rho_\sigma^{[J_\sigma]} & A' \\ 0 & B' \end{bmatrix} ,$$

where A' and B' are certain matrices, the universal R matrix $R = \Sigma\, e_\theta \otimes e^\theta \in U_q(L) \otimes U_q(L)$ will define an R matrix,

$$R^{ab} = \rho_a \otimes \rho_b(R) = \sum_\theta \rho_a(e_\theta) \otimes \rho_b(e^\theta) ,$$

with the factorization structure

$$R^{ab} = \begin{bmatrix} R^{J_a J_b} & A \\ 0 & B \end{bmatrix},$$

where

$$R^{J_a J_b} = \rho_a^{[J_a]} \otimes \rho_b^{[J_b]}(R) \in \mathrm{End}(V_a^{J_a} \otimes V_b^{J_b})$$

is the R matrix associated with $\rho_a{}^{[J_a]}$ and $\rho_b{}^{[J_b]}$. Let us take an example to illustrate the above conclusion.

Where $q^2 = -1$ and $\alpha = 2$, we obtain on $Q_2 \otimes Q_2$ an R matrix $R^{3/2(\lambda)3/2(\mu)} = R^{3/2}(\lambda, \mu)$ with $j_1 = j_2 = 3/2$, which possesses the factorization structure

$$R^{3/2(\lambda)3/2(\mu)} = \left[\begin{array}{c|cc|c} & 0 & X_{-1/2}(\lambda, \mu) & \\ & 0 & 0 & \\ R^{1/21/2}(\lambda, \mu) & \vdots & \vdots & 0 \\ & 0 & 0 & \\ \hline 0 & & M & \end{array}\right], \tag{6.2}$$

where $R^{1/21/2}(\lambda, \mu) = R^{1/2(\lambda)1/2(\mu)}$ is the 4×4 colored R matrix given in (5.3) and

$$M = \text{Block diag}(M_1, M_2, M_3, M_4),$$

$$M_1 = \begin{bmatrix} t_\lambda^3 t_\mu^{-1} & W_{-1/2}(\lambda, \mu) \\ & t_\lambda^{-1} t_\mu^3 \end{bmatrix},$$

$$M_2 = \begin{bmatrix} t_\lambda^5 t_\mu^{-1} & X_0^{(1)}(\lambda, \mu) & X_0^{(2)}(\lambda, \mu) & W_0(\lambda, \mu) \\ & t_\lambda^3 t_\mu & 0 & X_0^{(3)}(\lambda, \mu) \\ & & t_\lambda t_\mu^3 & X_0^{(4)}(\lambda, \mu) \\ & & & t_\lambda^{-1} t_\mu^5 \end{bmatrix},$$

$$M_3 = \begin{bmatrix} -t_\lambda^5 t_\mu & W_{1/2}(\lambda, \mu) & X_{1/2}(\lambda, \mu) \\ & -t_\lambda t_\mu^5 & 0 \\ & & t_\lambda^3 t_\mu^3 \end{bmatrix},$$

$$M_4 = \begin{bmatrix} t_\lambda^5 t_\mu^3 & W_{3/2}(\lambda, \mu) & \\ & t_\lambda^3 t_\mu^5 & -t_\lambda^5 t_\mu^5 \end{bmatrix},$$

where we have defined

$$W_{-1/2}(\lambda, \mu) = 4q^{-1}t_\lambda t_\mu \alpha_{3/2,-3/2}(\lambda)\alpha_{3/2,-1/2}(\lambda)\beta_{3/2,1/2}(\mu)\beta_{3/2,-1/2}(\mu) ,$$

$$W_{1/2}(\lambda, \mu) = -4t_\lambda^3 t_\mu^3 \alpha_{3/2,-1/2}(\lambda)\alpha_{3/2,1/2}(\lambda)\beta_{3/2,3/2}(\mu)\beta_{3/2,1/2}(\mu) ,$$

$$W_{3/2}(\lambda, \mu) = -4t_\lambda^4 t_\mu^4 \alpha_{3/2,1/2}(\lambda)\beta_{3/2,3/2}(\mu) ,$$

$$X_{-1/2}(\lambda, \mu) = -2t_\mu^2 \alpha_{3/2,-1/2}(\lambda)\beta_{3/2,-1/2}(\mu) ,$$

$$X_0^{(1)}(\lambda, \mu) = -2q^{-1}t_\lambda^4 \alpha_{3/2,-3/2}(\lambda)\beta_{3/2,3/2}(\mu) ,$$

$$X_0^{(2)}(\lambda, \mu) = -4qt_\lambda^3 t_\mu \alpha_{3/2,-3/2}(\lambda)\alpha_{3/2,-1/2}(\lambda)\beta_{3/2,3/2}(\mu)\beta_{3/2,1/2}(\mu) ,$$

$$X_0^{(3)}(\lambda, \mu) = 4qt_\lambda t_\mu^3 \alpha_{3/2,-1/2}(\lambda)\alpha_{3/2,1/2}(\lambda)\beta_{3/2,1/2}(\mu)\beta_{3/2,-1/2}(\mu) ,$$

$$X_0^{(4)}(\lambda, \mu) = 2q^{-1}t_\mu^4 \alpha_{3/2,1/2}(\lambda)\beta_{3/2,-1/2}(\mu) ,$$

$$X_{1/2}(\lambda, \mu) = -2t_\lambda^4 t_\mu^2 \alpha_{3/2,-1/2}(\lambda)\beta_{3/2,3/2}(\mu) .$$

Notice that if the representation ρ used in constructing the R matrix is completely reducible, i.e. $\rho = \rho^J \oplus \rho^L$, then the obtained R matrix $R_\rho = \rho \otimes \rho(R)$ possesses the completely factorizable structure

$$R_\rho = \begin{bmatrix} R^{JJ} & & & \\ & R^{LJ} & & \\ & & R^{JL} & \\ & & & R^{LL} \end{bmatrix} \tag{6.3}$$

where $R^{fg} = \rho^f \otimes \rho^g(R)$ $(f, g = L, J)$ are the R matrices defined by the subrepresentations ρ_J and ρ_L, which still satisfy the YBE.

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References

1. V. G. Drinfeld, "Quantum groups," in *Proc. ICM* (Berkeley, 1986).
2. M. Jimbo, *Lett. Math. Phys.* **10**, 63 (1985); **11**, 247 (1986).
3. N. Yu. Reshetikhin, LOMI preprints E-4 and E-11 (1987).
4. A. N. Kirillov and N. Yu. Reshetikhin, LOMI preprint (1988).
5. H. C. Lee, *Nato Workshop on Physics and Geometry* (Lake Tahoe, 1989); H. C. Lee, M. Couture and N. C. Schmeing, Chalk River preprint CRNL-TP-1125R, 1988.

6. M. L. Ge, G. C. Liu and K. Xue, *J. Phys.* **A4**, 2679 (1991).
7. M. L. Ge and K. Xue, *Phys. Lett.* **A146**, 245 (1990).
8. M. L. Ge and K. Xue, *J. Math. Phys.* **32**, 1301 (1991).
9. L. H. Kauffman, *Ann. Math. Studies* **115**, 1 (1988).
10. M. L. Ge, K. Xue and Y. S. Wu, *Int. J. Mod. Phys.* **A6**, 3735 (1990).
11. T. Deguchi, *J. Phys. Soc. Japan* **68**, 3441 (1989).
12. M. L. Ge, C. P. Sun, L. Y. Wang and K. Xue, *J. Phys.* **A23**, L645 (1990).
13. M. L. Ge, X. F. Liu, C. P. Sun and K. Xue, *J. Phys.* **A24**, 4955 (1991).
14. C. P. Sun, K. Xue, X. F. Liu and M. L. Ge, *J. Phys.* **A24**, L545 (1991).
15. C. P. Sun, X. F. Liu and M. L. Ge, *J. Math. Phys.* **32**, 2409 (1991).
16. M. L. Ge, X. F. Liu and C. P. Sun, *Phys. Lett.* **A155**, 137 (1991).
17. L. C. Biedenharn, *J. Phys.* **A22**, L873 (1989).
18. C. P. Sun and H. C. Fu, *J. Phys.* **A22**, 1983 (1989).
19. A. J. MacFarlane, *J. Phys.* **A22**, 4551 (1989).
20. L. Alverez-Gaume, G. Gomez and G. Sirra, *Nucl. Phys.* **B330**, 347 (1990).
21. V. Pasquire and H. Saleur, *Nucl. Phys.* **B330**, 523 (1990).
22. N. Yu. Reshetikhin and F. Smirnov, Harvard preprint (1990).
23. F. Smirnov and L. A. Takhtajian, Colorado preprint (1990).
24. E. Date, M. Jimbo, K. Miki and T. Miwa, *Phys. Lett.* **A148**, 45 (1990).
25. E. Date, M. Jimbo, K. Miki and T. Miwa, *Commun. Math. Phys.* **137**, 133 (1991)
26. G. Lusztig, *Adv. Math.* **70**, 237 (1988).
27. M. Rosso, *Commun. Math. Phys.* **117**, 581 (1988); **124**, 307 (1989).
28. G. Lusztig, *Contemp. Math.* **82**, 59 (1989).
29. P. Roche and D. Arnaudon, *Lett. Math. Phys.* **17**, 295 (1989).
30. C. P. Sun, J. F. Lu and M. L. Ge, *J. Phys. Lett.* **A24**, L1199 (1990).
31. C. De Concini and V. G. Kac, "*Representations of Quantum Groups*.
32. J. Murakami, Osaka preprint (1990).
33. L. C. Biedenharn, "Overview of quantum groups," Duke preprint (1990).
34. C. P. Sun and M. L. Ge, *J. Math. Phys.* **32**, 595 (1991).
35. T. Hayashi, *Commun. Math. Phys.* **127**, 129 (1990).
36. Y. J. Ng, *J. Phys.* **A23**, 1023 (1990).
37. M. Chaichan, P. Kulish and J. Lukierski, *Phys. Lett.* **B234**, 401 (1990).
38. C. P. Sun and M. L. Ge, *J. Phys.* **A24**, 3265 (1991).