LETTER

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Non-thermal radiation of black holes off canonical typicality

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Abstract – We study the Hawking radiation of black holes by considering the canonical typicality. For the universe consisting of black holes and their outer part, we directly obtain a non-thermal radiation spectrum of an arbitrary black hole from its entropy, which only depends on a few external qualities (known as hairs), such as mass, charge, and angular momentum. Our result shows that the spectrum of the non-thermal radiation is independent of the detailed quantum tunneling dynamics across the black hole horizon. We prove that the black hole information paradox is naturally resolved by taking into account the correlation between the black hole and its radiation in our approach.

Introduction. – The study of black hole physics have elicited many important results, such as area law [1] and Hawking radiation [2,3], which have tremendous impacts on many related researches in different areas of physics [4–7]. One of them is the study of the black hole information paradox. The thermal radiation of a black hole directly leads to an information loss [8,9], which contradicts with the properties of unitarity in quantum mechanics.

In 2000, Parikh and Wilczek [10] considered the problem of black hole information loss from a consistent perspective of quantum tunneling, and a non-thermal radiation spectrum was discovered. Such spectrum allows the correlation between subsequently emitted particles. Zhang *et al.* [11] shows that by taking the correlation of black hole radiation into account, the black hole information is conserved. Thus they declared that the black hole information problem is explained.

Remarkably, all the radiation spectra studied case by case through the quantum tunneling method [12] have a simple form and perfectly satisfy the requirement of information conservation. This observation hints at an even deep origin of the non-thermal nature of the Hawing radiation without referring specially to the geometry of black hole's horizon [13], as well as the exact quantum tunneling dynamics. To reveal such origin of non-thermal spectra, we need an even general derivation of radiation spectra.

In this letter, we prove that the non-thermal spectrum of Hawking radiation can be derived with the general principle of canonical typicality [14–16] without referring to the dynamics of the particle tunneling. The radiation spectrum is directly obtained by making use of the black hole's entropy and maintaining the non-canonical part, which matches exactly with the non-thermal feature of the radiation. Besides black holes, in specific finite systems [17,18], we also observed their non-canonical statistic behavior, that is, their distribution is not a perfect thermal equilibrium distribution. This implies that the non-thermal property of black hole radiation is the inevitable result of the finite system statistics that goes slightly off canonical typicality. With our general formalism, we derive the radiation spectra of several black holes, which are well consistent with the previous results achieved from the perspective of quantum tunneling [10,19]. Further, the information carried by Hawking radiation is discussed in our framework, and then we clarify that the so-called black hole information loss paradox is due to the ignorance of correlation between the black hole and its radiation. It is worth mentioning that by introducing icezones [20–22] to replace firewalls [23,24], Stojkovic *et al.* have argued that once the correlation

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between the radiation fields is taken into consideration, the black hole information paradox will no longer exists. Such correlation, in our approach, is just the primary cause that results in the black hole radiation off thermal distribution.

Radiation spectrum off canonical typicality. – We first consider the universe U, which consists of the system of interest B (*e.g.*, black hole) and the environment O (see fig. 1). The whole universe is assumed to be in an arbitrary pure universe state

$$|\Psi\rangle = \sum_b \sum_o \frac{C(b,o)}{\sqrt{\Omega_{\mathrm{U}}}} |b\rangle \otimes |o\rangle , \qquad (1)$$

where $\Omega_{\mathrm{U}} = \Omega_{\mathrm{U}}(E_{\mathrm{U}})$ is the total number of micro-states for the universe with energy E_{U}, and $C(b,o)$ is the coefficient of state $|b\rangle \otimes |o\rangle$. And $|b\rangle$ and $|o\rangle$ are the eigenstates of B and O, respectively. Without losing generality, we assume the orthogonal conditions $\langle b_i|b_j\rangle = \delta_{ij}$, and $\langle o_k|o_l\rangle = \delta_{kl}$. The normalization of state $|\Psi\rangle$, $\sum_b \sum_o |C(b,o)|^2 / \Omega_{\mathrm{U}} = 1$, directly implies the average value of $|C(b,o)|^2$, $\overline{|C(b,o)|^2} = 1$. Let $|\Psi_b\rangle = \sum_o C(b,o)|o\rangle$. Then the universe state becomes a maximum entanglement-like state, *i.e.*, $|\Psi\rangle = \sum_b |b\rangle \otimes |\Psi_b\rangle / \sqrt{\Omega_{\mathrm{U}}}$. As a result, ignoring the environment O, the reduced density matrix of B is obtained as

$$\rho_{\mathrm{B}} = \mathrm{Tr}_{\mathrm{O}}(|\Psi\rangle\langle\Psi|) = \frac{1}{\Omega_{\mathrm{U}}} \sum_{b_i,b_j} \left\langle \Psi_{b_i}|\Psi_{b_j}\right\rangle |b_i\rangle\langle b_j| , \qquad (2)$$

where Tr_{O} means tracing over the variables of O, and $\left\langle \Psi_{b_i}|\Psi_{b_j}\right\rangle = \delta_{ij}\langle\Psi_b|\Psi_b\rangle$ have been proved in ref. [15]. For the environment O supported in a high-dimension Hilbert space, according to the central limit theorem, the average over a large enough subset O of U is the same as that over U, so that

$$\langle\Psi_b|\Psi_b\rangle = \sum_o |C(b,o)|^2 = \Omega_{\mathrm{O}}(E_{\mathrm{U}} - E_b) . \qquad (3)$$

Here, $\Omega_{\mathrm{O}}(E_{\mathrm{O}})$ is the number of O's micro-states with energy $E_{\mathrm{O}} = E_{\mathrm{U}} - E_b$, and E_b is the eigenenergy of $|b\rangle$. Thus, the reduced density matrix of B is simplified as

$$\rho_{\mathrm{B}} = \sum_b \frac{\Omega_{\mathrm{O}}(E_{\mathrm{U}} - E_b)}{\Omega_{\mathrm{U}}} |b\rangle\langle b| . \qquad (4)$$

When the total energy of system B is taken as a certain value, namely, $E_b = E$, the number of B's micro-states, denoted as $\Omega_{\mathrm{B}}(E) \equiv \Omega_{\mathrm{U}}/\Omega_{\mathrm{O}}(E_{\mathrm{U}} - E)$, is fully contributed by the degrees of freedom that degenerated in the macro energy state of B with eigenenergy E. When B is specific as a black hole, E corresponds to the total mass M of the black hole. The reduced density matrix of B is thus written as $\rho_{\mathrm{B}} = \sum_b |b\rangle\langle b| / \Omega_{\mathrm{B}}(E)$, which implies that B obeys the micro-canonical distribution. In this case, the entropy of B, $S_{\mathrm{B}} = \ln \Omega_{\mathrm{B}}(E)$, has been proved to be proportional to the area of B's boundary in some specific model [6,7]. And this is the so-called entanglement entropy area theorem, which results in a possible explanation for the origin of the black hole's entropy.

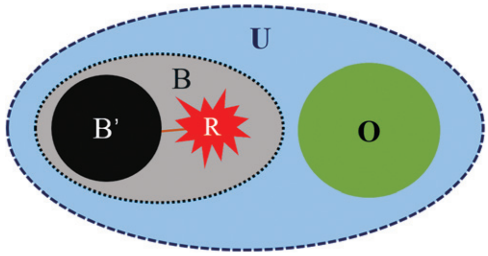

Fig. 1: (Colour online) The relation between universe U, the system of interest B, and the environment O. U consists of B and O, and B is further divided into subsystems R and B′.

Now we look at B's subsystem R, as shown in fig. 1, the rest of B is denoted as $\mathrm{B}' = \mathrm{B} - \mathrm{R}$. The reduced density matrix of B can be rewritten as $\sum_{r,b'} |r,b'\rangle\langle r,b'| / \Omega_{\mathrm{B}}(E)$, where $|r\rangle$ and $|b'\rangle$ are the eigenstates of R and B′ with eigenenergies E_r and $E_{b'}$, respectively. And $E_r + E_{b'} = E$ is the constraint condition given by energy conservation. The reduced density matrix of R

$$\rho_{\mathrm{R}} = \mathrm{Tr}_{\mathrm{B}'}(\rho_{\mathrm{B}}) = \sum_r \frac{\Omega_{\mathrm{B}'}(E - E_r)}{\Omega_{\mathrm{B}}(E)} |r\rangle\langle r| \qquad (5)$$

is obtained by tracing over B'. Here, $\Omega_{\mathrm{B}'}(E - E_r)$ is the number of micro-states of B′ with energy $E - E_r$, and it can be rewritten as $\Omega_{\mathrm{B}'}(E - E_r) = \exp[S_{\mathrm{B}'}(E - E_r)]$, where $S_{\mathrm{B}'}(E - E_r)$ is the entropy of B′. Therefore, we can further write eq. (5) as

$$\rho_{\mathrm{R}} = \mathrm{Tr}_{\mathrm{B}'}(\rho_{\mathrm{B}}) = \sum_r \mathrm{e}^{-\Delta S_{\mathrm{BB}'}(E_r,E)} |r\rangle\langle r| , \qquad (6)$$

where $\Delta S_{\mathrm{BB}'}(E_r, E) \equiv S_{\mathrm{B}}(E) - S_{\mathrm{B}'}(E - E_r)$ is the difference in entropy between B and B′. We can clearly see from eq. (6) that only when $\Delta S_{\mathrm{BB}'}$ depends on E_r linearly, the spectrum of R is perfectly thermal. What should be mentioned here is that we do not expand $\Delta S_{\mathrm{BB}'}$ only up to the first order of E_r, as done in most studies in the thermodynamic limit. It will be shown later that the higher order of E_r or the non-canonical part of ρ_{R} just determines the non-thermal property of R.

Until here, the discussion is a general one without any specification of black hole radiation. Now we will specify our system with B as black hole, R as Hawking radiation, and B′, in this case, is considered as the remaining black hole. For the black hole B with three "hairs", mass M, change Q, and angular momentum J, we let $|\omega, q, j\rangle$ being the eigenstate of radiation R with energy ω, change q, and angular momentum j. Then it follows from eq. (6) and the conservation laws for charge and angular momentum that the radiation spectrum is obtained as

$$\rho_{\mathrm{R}} = \sum_{\omega,q,j} \mathrm{e}^{-\Delta S_{\mathrm{BB}'}(\omega,q,j,M,Q,J)} |\omega,q,j\rangle\langle\omega,q,j| . \qquad (7)$$

This is the main result of this letter. To get the explicit expression of ρ_{R}, we then make use of the Bekenstein-Hawking entropy, which reads

$$S_{BH}(M,Q,J)=\frac{A_H}{4}=\pi R_H^2, \tag{8}$$

where $A_H = A_H(M,Q,J)$ and $R_H = R_H(M,Q,J)$ are the "hairs"-determined area and radius of the black hole's event horizon, respectively. It follows from eqs. (7) and (8) that the radiation spectrum of the black hole B is

$$\rho_{\mathrm{R}} = \sum_{\omega,q,j}\exp\left[\pi R_H^2(M-\omega,Q-q,J-j)\right.$$
$$\left.-\pi R_H^2(M,Q,J)\right]|\omega,q,j\rangle\langle\omega,q,j|. \tag{9}$$

As the black hole evaporates, its mass decreases, and when M becomes very small, the effect of quantum gravity gradually begins to appear. If the correction in the black hole entropy due to the quantum gravity is taken into account, the area entropy for the black hole is modified as [25]

$$S_H=\frac{A_H}{4}+\alpha\ln\frac{A_H}{4}=\pi R_H^2+\alpha\ln\left(\pi R_H^2\right), \tag{10}$$

where α is a dimensionless parameter determined by the specific quantum gravity model. The effect of quantum gravity causes eq. (9) to be rewritten as

$$\rho_{\mathrm{R}}=\sum_{\omega,q,j}\left[\frac{R_H(M-\omega,Q-q,J-j)}{R_H(M,Q,J)}\right]^{2\alpha}$$
$$\times\,\mathrm{e}^{-\left[\pi R_H^2(M,Q,J)-\pi R_H^2(M-\omega,Q-q,J-j)\right]}|\omega,q,j\rangle\langle\omega,q,j|. \tag{11}$$

This is exactly the same as the radiation spectrum obtained from the tunneling method [26]. For a given black hole, one can first get the horizon radius as the function of its external qualities with the help of its metric. And then by making use of eqs. (9) and (11), the radiation spectrum in the classical and quantum case are obtained, respectively.

Schwarzschild black hole and Reissner-Nordstrum black hole. – As the simplest black hole, there is only one hair for the Schwarzschild black hole, that is, the mass M. The radius of its event horizon is $R_H = 2M$, which together with eq. (9) give the radiation spectrum of the Schwarzschild black hole as

$$\rho_{\mathrm{R}}=\sum_{\omega}\mathrm{e}^{-8\pi\omega(M-\omega/2)}|\omega\rangle\langle\omega|, \tag{12}$$

where $|\omega\rangle$ is the eigenstate of the radiation. It is seen from eq. (12) that the probability

$$p(\omega,M)=\mathrm{e}^{-8\pi\omega(M-\omega/2)} \tag{13}$$

for the state $|\omega\rangle$ being in the distribution is the same as the tunneling probability $\Gamma(\omega,M)=\exp\left[-8\pi\omega(M-\omega/2)\right]$, which is known as the Parikh-Wilczek spectrum [10], as the result of the WKB approximation through the perspective of quantum tunneling. Note that we did not use the tunneling dynamics of the radiation process at all, but we can obtain the non-thermal spectrum eq. (12). Here, we only require the area entropy of the black hole to be the function of energy. The simple derivation of radiation shows that the entropy of the black hole is the key quantity to determine the radiation spectrum.

For the Reissner-Nordstrum (R-N), black hole with mass M and charge Q, the radius of its outer event horizon obtained from the metrics reads $R_H = M+\sqrt{M^2-Q^2}$. Substituting it into eq. (9), we obtain the non-thermal spectrum of the R-N black hole as

$$\rho_{\mathrm{R}}=\sum_{\omega,q}\exp\left\{\pi\left[(M-\omega)+\sqrt{(M-\omega)^2-(Q-q)^2}\right]^2\right.$$
$$\left.-\pi\left(M+\sqrt{M^2-Q^2}\right)^2\right\}|\omega,q\rangle\langle\omega,q|, \tag{14}$$

where $|\omega,q\rangle$ is the eigenstate of the radiation. Equation (14) is exactly consistent with the radiation probability obtained from the perspective of quantum tunneling in ref. [19]. Here, we show the density matrix for Schwarzschild and R-N black holes without referring to the dynamics of the particle tunneling. A similar process for other black holes results in the exact matching between our results and that obtained from the dynamics analysis. This remind us that while considering the non-thermal effect of the black hole radiation spectrum, there is an even deeper intrinsic relationship between the quantum tunneling approach and our statistical method. For example, in previous work, by using the radiation spectrum, we successfully obtained the number of micro-states of the black holes [27].

Information of black hole radiation. – Now we go back to eq. (6) to discuss the information carried by Hawking radiation. The corresponding Von-Neumann entropy $S_{\mathrm{R}}=-\mathrm{Tr}\left(\rho_{\mathrm{R}}\ln\rho_{\mathrm{R}}\right)$ of R reads

$$S_{\mathrm{R}}=S_{\mathrm{B}}-\sum_r\mathrm{e}^{-\Delta S_{\mathrm{BB}'}(E_r,E)}S_{\mathrm{B}'}(E-E_r). \tag{15}$$

Namely, $S_{\mathrm{R}} = S_{\mathrm{B}} - S(\mathrm{B}'|\mathrm{R})$, where $S(\mathrm{B}'|\mathrm{R}) = \sum_r \mathrm{e}^{-\Delta S_{\mathrm{BB}'}(E_r,E)}S_{\mathrm{B}'}(E-E_r)$ is the conditional entropy of B'. This shows that there exists a correlation between R and B', as a result of energy conservation. In the low energy limit, *i.e.*, $E_r \ll E$, the conditional entropy is approximated, by keeping the first order of E_r, as $S(\mathrm{B}'|\mathrm{R}) \approx S_{\mathrm{B}'}(E_{\mathrm{B}'})$. Here, $E_{\mathrm{B}'} = E - E_{\mathrm{R}}$ and $E_{\mathrm{R}} = \sum_r \exp\left[-\Delta S_{\mathrm{BB}'}(E_r,E)\right]E_r$ are the internal energy of B' and R, respectively. This indicates that the correlation between R and B could be ignored when the energy of R is much smaller than that of B. At this time, the radiation spectrum of the Schwarzschild black hole in eq. (12) can be approximated as $\rho_{\mathrm{R}} = \sum_\omega \exp(-8\pi M\omega)|\omega\rangle\langle\omega|$, which is just the thermal radiation discovered by Hawking.

So far it becomes clear that the thermal spectrum of the black hole radiation is due to the ignorance of the correlation information between the black hole and its radiation. This is also the primary cause of the black hole information loss. Analogously, it can be proved that the conditional entropy for B' of an arbitrary black hole system is $S\left(\mathrm{B}'|\mathrm{R}\right) \approx S_{\mathrm{B}'}\left(M', Q', J'\right)$, where $M' = M - \sum_{\omega} p\left(\omega\right)\omega$, $Q' = Q - \sum_{q} p\left(q\right) q$, and $J' = J - \sum_{j} p\left(j\right) j$ are the average of the mass, charge, and angular momentum of the black hole B'. The conditions for this approximation are $\sum_{\omega} p\left(\omega\right)\omega \ll M$, $\sum_{q} p\left(q\right) q \ll Q$, and $\sum_{j} p\left(j\right) j \ll J$.

The existence of a correlation between sequences of emitted particles has been proved via mutual information [11,28,29]. Yet, the proofs are based on case-by-case studies. We revisit this proof with our general formalism. The mutual information for two emissions ω_1 and ω_2 can be written, by definition [30], as $I = \sum_{\omega_1,\omega_2} p_{1,2} \ln\left[p_{1,2}/\left(p_1 p_2\right)\right]$, where p_1 (p_2) is the possibility for ω_1 (ω_2) in the distribution, and $p_{1,2}$ is the joint probability of the two emissions. In our case, these possibilities are given by eq. (13) as $p_1 = p\left(\omega_1, M\right)$, $p_2 = p\left(\omega_2, M\right)$, and $p_{1,2} = p\left(\omega_1 + \omega_2, M\right)$. Moreover, it is easily checked that $p\left(\omega_1 + \omega_2, M\right) = p\left(\omega_1, M\right) p\left(\omega_2, M - \omega_1\right)$, with the help of which, by straightforward calculation, we obtain the mutual information $I = 8\pi \left\langle \omega_1 \right\rangle \left\langle \omega_2 \right\rangle$, where $\left\langle \omega_i \right\rangle = \sum \omega_i p\left(\omega_i, M\right)$ is the average energy of ω_i ($i = 1, 2$). This shows that the mutual information between the emissions is proportional to their internal energy, and coincides with the result in ref. [11], where the authors proved that the sum of these mutual information is just the total information of the initial black hole. From this point of view, our result can naturally give the conclusion that the black hole information is not lost, if the information correlation between the radiation particles is taken into account. The derivation of this conclusion, which needs to be emphasized here, is now dynamics independent.

Conclusion. – In summary, we straightforwardly derived the non-thermal spectrum of the black hole radiation from the area law of entropy, which only depends on a few external qualities of the black hole (known as hairs). The derivation is based on the principle of canonical typicality, without referring to the dynamics of quantum tunneling across the horizon. We showed that there exists information correlation between the black hole and its radiation, and thus the fact that the information is not lost with the black hole's evaporating is clarified. Taking into account the effect of quantum gravity, we achieved the modified radiation spectrum through our universal protocol succinctly. Since the general formalism we developed does not require the specific form of the system and has been successfully applied to the black hole to obtain its non thermal radiation spectrum, we therefore conjecture it can also be applied to some non-relativistic systems, such as ideal gas, black-body, etc., to obtain their radiation spectra. The advantage of this approach is that there is no need to analyze the particle dynamics in the radiation process, but it only refers to the dependence of the entropy on the macro-conserved qualities of the whole system.

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Note added in proofs: Recently, we applied the canonical typicality approach developed in this paper to obtain the radiation spectrum of Schwarzschild black hole in the case with dark energy [31]. Some interesting results have been reported in this relevant work, showing the influences of dark energy on the non-thermal radiation of black holes.

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