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Entanglement of spin-orbit qubits induced by Coulomb interaction

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Abstract. Spin-orbit qubit (SOQ) is the dressed spin by the orbital degree of freedom through a strong spin-orbit coupling (SOC). We show that Coulomb interaction between two electrons in quantum dots located separately in two nanowires can efficiently induce quantum entanglement between two SOQs. But to achieve the highest possible value for two SOQs concurrence, strength of SOC and confining potential for the quantum dots should be tuned to an optimal ratio. The physical mechanism to achieve such quantum entanglement is based on the feasibility of the SOQ responding to the external electric field via an intrinsic electric dipole spin resonance.

1 Introduction

Entanglement of particles now has been considered as a quantum resource to implement protocols of quantum information processing [1,2]. To achieve quantum entanglement for various kinds of qubits is a central task in quantum information science and technology [3,4]. With very strong spin-orbit coupling (SOC), an electron spin becomes the so-called spin-orbit qubit (SOQ) [5,6], which has been experimentally implemented in quantum nanowires. Its bit states essentially are two dressed spin states incorporating with the orbital degree of freedom (DOF), and thus could feasibly respond to both electric and magnetic fields. In this sense quantum manipulations on SOQ can be achieved via electron-dipole spin resonance [7,8].

With those existing progresses, we will face with a crucial task: how to efficiently achieve entanglement of two SOQs? To this end we refer to our previous investigations to create inter-spin entanglement through SOC incorporating with Coulomb interaction [9]. Notice that the Coulomb interaction can be viewed as an electric field acting on one electron, depending on the state of another electron. This implies that the SOQs can also feasibly respond to the Coulomb interaction and thus cause a correlated motion of two SOQs for generating quantum entanglement between SOQs. In reference [9], a scheme was proposed to create entanglement between two local spins (rather than the SOQs) in two 2D quantum dots mediated with SOC.

In this paper, we study an alternative scheme to generate entanglement between two SOQs through Coulomb interaction. We will show that this Coulomb interaction mediated entanglement can be optimized by tuning the strength of SOC: it is not the case the stronger the SOC is, higher the entanglement is. More specifically, we consider a low dimensional system made up by two electrons which are separately confined in two paralleled nanowires and are subjected to a weakly external magnetic field. By treating the magnetic field as perturbation and modeling the Coulomb interaction linearly nearby the equilibrium point, we derive an effective Hamiltonian for two SOQs, with two qubits flip-flop induced by the Coulomb interaction. It is found that the flip-flop rate as well as two qubits entanglement (concurrence) change non-monotonically with respect to SOC strength, and they can be optimized by manipulating SOC. This discovery largely modifies the intuitionistic observation that stronger coupling is more helpful in creating highly entangled many body states.

This paper is organized as follows: in Section 2 we model the Coulomb interaction as a linear coupling for two electrons trapped in two quantum dots which are located in two parallel nanowires. In Section 3, we elucidate subspaces that can be used to encode SOQ under a limiting situation where magnetic field can be treated as perturbation to orbital motions as well as SOC. In Section 4, we discuss the generation of the correlated flip-flop processes of two qubits as mediated by the Coulomb interaction, and derive the effective Hamiltonian for SOQs encoded using nearly degenerated ground states. Then we study entanglement in the two SOQs system by calculating system dynamics as well as time evolution of concurrence. Conclusions and further remarks are given in Section 5.

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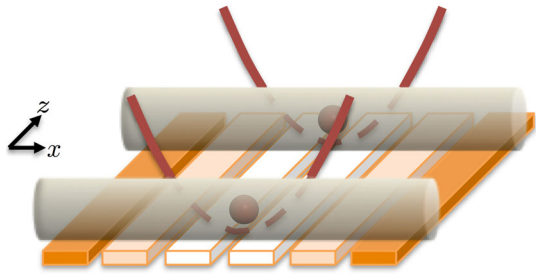

Fig. 1. Schematic of two electrons confined in quantum dots which are located separately in two 1D parallel nanowires. Electrodes beside nanowires provide harmonic traps for electrons. Those trapping potentials are represented by red parabolic curves.

2 Model setup for two parallel nanowires

We consider two parallel 1D nanowires placed along x-axis in the x-z plane. Due to strong nanowire confinement in the y-z plane, the electrons can only freely move in the x direction [10]. Two electrons are confined by biased electrodes aside each nanowire [11,12], as schematically plotted in Figure 1. By treating the confining potentials as harmonic traps, we can write down the Hamiltonian for the two electrons with SOCs as follows:

$$H = \sum_{j=1,2} \left[\frac{p_j^2}{2m} + V_j + \alpha_j \sigma_j^x p_j + \mu_j \sigma_j^z \right] + V_c. \quad (1)$$

Here, $j = 1, 2$ indicates the confined electron in nanowire at the front or the rear; m is electron effective mass; p_j and x_j are momentum and coordinate for the jth electron, respectively. We take the external trap to be of harmonic oscillator, i.e., $V_j = m\bar{\omega}_j^2 x_j^2/2$, and $\bar{\omega}_j$ is characteristic oscillator frequency of the trap; α_j is the Dresselhaus SOC strength and $2\mu_j$ is the Zeeman splitting for jth electron [13]. V_c is Coulomb interaction between two electrons

$$V_c = \frac{e^2}{4\pi\varepsilon_0\varepsilon_r\sqrt{(x_1 - x_2)^2 + (z_1 - z_2)^2}}. \quad (2)$$

Let $l_j = \sqrt{\hbar/(m\bar{\omega}_j)}$ be the characteristic length of harmonic trap and z_0 be the separation between two nanowires. In the case of strong confinement, i.e., $l_j \ll z_0$, V_c can be further simplified [9]. Specifically, we replace $z_1 - z_2$ in V_c by z_0 and expand equation (2) around $x_1 - x_2 = 0$. Then the potential is written up to second order of $x_1 - x_2$ as:

$$V_c \approx V_0 - \frac{1}{2} m\omega_c^2 (x_1 - x_2)^2, \quad (3)$$

where $V_0 = e^2/(4\pi\varepsilon_0\varepsilon_r z_0)$ and $\omega_c = \sqrt{V_0/(mz_0^2)}$.

With above approximation, the system reduces to two coupled harmonic oscillators with SOC, with the Hamiltonian

$$H = \sum_{j=1,2} \left[\frac{p_j^2}{2m} + \frac{1}{2} m\omega_j^2 x_j^2 + \alpha_j \sigma_j^x p_j + \mu_j \sigma_j^z \right] + m\omega_c^2 x_1 x_2. \quad (4)$$

Here, we have ignored the constant term V_0 and defined $\omega_j = \sqrt{\bar{\omega}_j^2 - \omega_c^2}$. For ω_j to be a physically meaningful parameter, one requires that $l_j < \sqrt{\hbar\bar{\omega}_j/V_0} z_0$ such that $\bar{\omega}_j > \omega_c$.

In terms of the bosonic creation and annihilation operators $a_j^\dagger$ and a_j, p_j and x_j are rewritten as $p_j = i\sqrt{\hbar\omega_j m/2}(a_j^\dagger - a_j)$ and $x_j = \sqrt{\hbar/(2m\omega_j)}(a_j^\dagger + a_j)$, respectively. Thus the system Hamiltonian equation (4) becomes $H = \sum_{j=1,2}(H_{j,0} + H_{j,1}) + V$, where

$$H_{j,0} = \hbar\omega_j \left(a_j^\dagger a_j + \frac{1}{2} \right) + i\xi_j \left(a_j^\dagger - a_j \right) \sigma_j^x, \quad (5)$$

$$V = g \left(a_1^\dagger + a_1 \right) \left(a_2^\dagger + a_2 \right), \quad (6)$$

and $H_{j,1} = \mu_j \sigma_j^z$. In the above equations, $\xi_j = \sqrt{m\hbar\omega_j/2}\alpha_j$ are the rescaled SOC strength and we have introduced the coupling constant $g = \hbar\omega_c^2/(2\sqrt{\omega_1\omega_2})$.

3 Dressed spin qubit in magnetic field

The existence of SOC in $H_{j,0}$ couples spatial motion of each electron to its internal spin DOF. Therefore, it provides us with an active method to manipulate spin states of electrons via controlling over their spatial DOF [14,15]. In this section, we study the driven SOC system in the limit where $\mu_j \ll \hbar\omega_j$ and $\mu_j \ll \xi_j$, i.e., the strong SOC dominates over the Zeeman effect.

First, let us analyze the energy level configuration of $H_{j,0}$. The two fold degenerated eigenvectors of $H_{j,0}$ with the corresponding eigenvalues $E_{j,n} = \hbar\omega_j(n - \eta_j^2 + 1/2)$ are $|n, \uparrow\rangle_j = |-i\eta_j, n\rangle_j|\uparrow\rangle_j$ and $|n, \downarrow\rangle_j = |i\eta_j, n\rangle_j|\downarrow\rangle_j$. Here, $|\alpha, n\rangle_j = D_j(\alpha)|n\rangle_j$ is displaced Fock state created by the displaced operator

$$D_j(\alpha) = e^{\alpha a_j^\dagger - \alpha^* a_j} \quad (7)$$

for jth electron [16]; $\eta_j = \xi_j/(\hbar\omega_j)$ is reduced SOC strength; $|\uparrow(\downarrow)\rangle_j$ are eigenstates of σ_j^x with eigenvalues ± 1, respectively. The following analyses are identical for both electrons, thus we will omit sub-index j for the clarity of presentation in rest of this section.

As shown in Figure 2a, applying a magnetic field perpendicular to the nanowire will mix $|\uparrow\rangle$ with $|\downarrow\rangle$ and lift the previously degenerated levels. Generally the level splitting will be proportional to the strength of the applied field in weak field limit. In fact, the lowest two split levels has been used to encode a SOQ [7].

In order to find out the level splitting as well as wave functions for the dressed spin, we apply degenerate state perturbation theory on H_1 by treating $\epsilon = \mu/(\hbar\omega)$ as a small parameter [17]. Since σ^z flips $|\uparrow\rangle$ to $|\downarrow\rangle$, the diagonal matrix elements $\langle n, s|H_1|n, s\rangle$ vanish, where s stands for $\uparrow$ or $\downarrow$, and off-diagonal elements are:

$$\langle n, s|H_1|n, -s\rangle = \mu e^{-2\eta^2} L_n(4\eta^2). \quad (8)$$

Here, $L_n(x)$ is Laguerre polynomials [18].

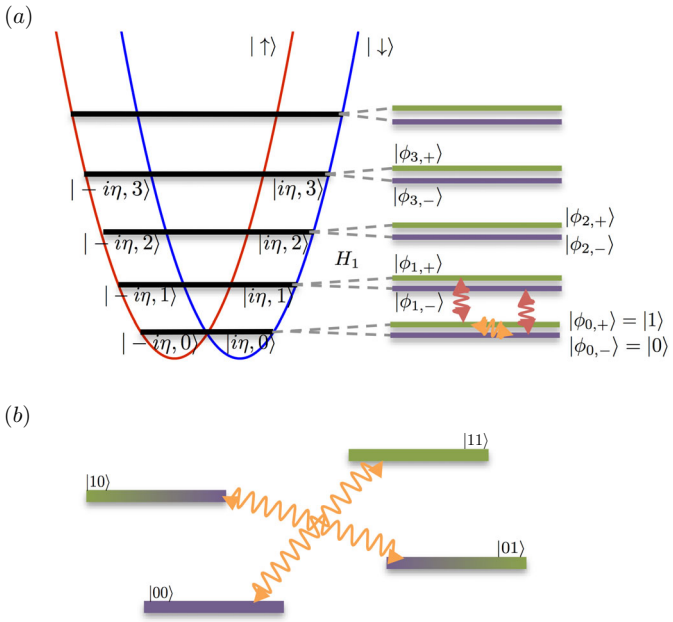

Fig. 2. (a) Energy level structures of a spin with SOC in a weak magnetic field: two parabolas indicate the different harmonic traps felt by electron with spin states $|\uparrow\rangle$ or $|\downarrow\rangle$. Black lines inside parabolas denote the eigenstates of H_0, which are the spin dependent displaced Fock states. When H_1 is introduced, each two degenerated levels split as shown on the right. An SOQ can be encoded into the nearly degenerate ground states $|\phi_{0,\pm}\rangle$. Transition between two qubit's basis can be achieved via external ac electric driving, which is shown by wavy arrows. The off-resonant transitions (red arrows) can be ignored under resonant driving. (b) Two SOQs are coupled by the linearized Coulomb interaction, the Coulomb interaction here is of the same role as the ac field in the single qubit case. However, only transitions between two pairs of basis are allowed due to generalized parity selection rules.

In the two dimensional subspace spanned by $\{|n,\uparrow\rangle, |n,\downarrow\rangle\}$, H_1 is then written as:

$$H_1 = \mu e^{-2\eta^2} L_n(4\eta^2)(|n,\uparrow\rangle\langle n,\downarrow| + h.c.). \tag{9}$$

The orthonormal eigenstates of above two-by-two matrix are $|n,\pm\rangle = \frac{1}{\sqrt{2}}(|n,\uparrow\rangle \pm |n,\downarrow\rangle)$, and the corresponding eigenvalues are:

$$\delta E_{n,\pm} = \pm\mu e^{-2\eta^2} L_n(4\eta^2), \tag{10}$$

respectively. From the zeroth order wave function $|n,\pm\rangle$, we can use the perturbation method to obtain the eigenvalues of $H_0 + H_1$ as:

$$E_{n,\pm} = E_n + \delta E_{n,\pm}, \tag{11}$$

where the first order correction $\delta E_{n,\pm}$ is given by equation (10). The explicit calculation of the first order eigenfunctions $|\phi^{(1)}_{n,\pm}\rangle$ are given in Appendix A. Two states with lowest energies $E_{0,\mp}$ are denoted by $|0\rangle \equiv |\phi^{(1)}_{0,-}\rangle$ and $|1\rangle \equiv |\phi^{(1)}_{0,+}\rangle$, which are explicitly written out in Appendix A.

Energy splitting between $|1\rangle$ and $|0\rangle$ is given by $\Delta \equiv \delta E_{0,+} - \delta E_{0,-} = 2\mu\exp(-2\eta^2) \ll \hbar\omega$ since $\epsilon \ll 1$. $\hbar\omega$ is level spacing between two nearest unperturbed levels. If all relevant interactions are approximately in resonant with $|0\rangle$ and $|1\rangle$, then the two states can be used to encode quantum information as an SOQ. Aside from this, it worths to remark another property of the SOQ. We notice that $|n,\pm\rangle$ looks similar to even and odd coherent states [19,20], which process definite parities when the arguments of displacement operator are purely imaginary. Here, as the direction of displacement in $|n,\pm\rangle$ are entangled with spin orientations, they are eigenstates of following generalized parity (GP) operator

$$\Lambda = \Pi\sigma^z, \tag{12}$$

where Π is the usual parity operator defined by $\Pi^2 = 1$ and $\Pi|x\rangle = |-x\rangle$ [21]. p, x and σ^x are odd under Λ, which means $\{\Lambda, p\} = 0$, etc., and Λ commutes with $H_0 + H_1$. Therefore, SOQ basis $|0\rangle$ and $|1\rangle$ can be simultaneously eigenstates of Λ, i.e., $\Lambda|0\rangle = -|0\rangle$ and $\Lambda|1\rangle = |1\rangle$. Furthermore, as shown in Appendix B, $|\phi^{(1)}_{n,+}\rangle$ and $|\phi^{(1)}_{n+1,-}\rangle$ have the GPs that are different from $|\phi^{(1)}_{n,-}\rangle$ and $|\phi^{(1)}_{n+1,+}\rangle$.

This property of $|\phi^{(1)}_{n,\pm}\rangle$ then indicates following selection rules. First, for ac magnetic field driving in the z direction, the below transitions are forbidden:

$$|\phi^{(1)}_{n,\pm}\rangle \leftrightarrow |\phi^{(1)}_{n+2m-1,\pm}\rangle, \ |\phi^{(1)}_{n,\pm}\rangle \leftrightarrow |\phi^{(1)}_{n+2m,\mp}\rangle \tag{13}$$

since σ^z is even under Λ, i.e., $[H_1, \Lambda] = 0$. Second, for driving that involves the parity odd operator $a^\dagger + a \propto x$, e.g., ac electric field driving in the x direction, following transitions are forbidden:

$$|\phi^{(1)}_{n,\pm}\rangle \leftrightarrow |\phi^{(1)}_{n+2m,\pm}\rangle, \ |\phi^{(1)}_{n,\pm}\rangle \leftrightarrow |\phi^{(1)}_{n+2m-1,\mp}\rangle. \tag{14}$$

4 Entanglement between two SOQS

After elucidating the level structures and defining SOQ basis in previous section, now we study how Coulomb interaction between two electrons can be used to facilitate entanglement between SOQs. To this end, we calculate the matrix elements of the linearized Coulomb interaction equation (6) in the basis of two SOQs subspace spanned by $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$, where $|MN\rangle = |M\rangle_1 \otimes |N\rangle_2$.

As shown in Appendix C, up to second order in ϵ_j, matrix elements of the Coulomb interaction are given by:

$$\langle N_1 N_2|V|M_1 M_2\rangle = (-1)^{N_1+\bar{N}_2}\delta_{N_1,\bar{M}_1}\delta_{N_2,\bar{M}_2}J, \tag{15}$$

where $\bar{N} = 0$ (1) if $N = 1$ (0), and the two qubits effective flip-flop strength J is obtained as:

$$J = 16g\epsilon_1\epsilon_2\eta_1\eta_2 e^{-2(\eta_1^2+\eta_2^2)}. \tag{16}$$

Existence of exponential decay term in J indicates a non-monotone behavior with respect to SOC. In fact, as shown

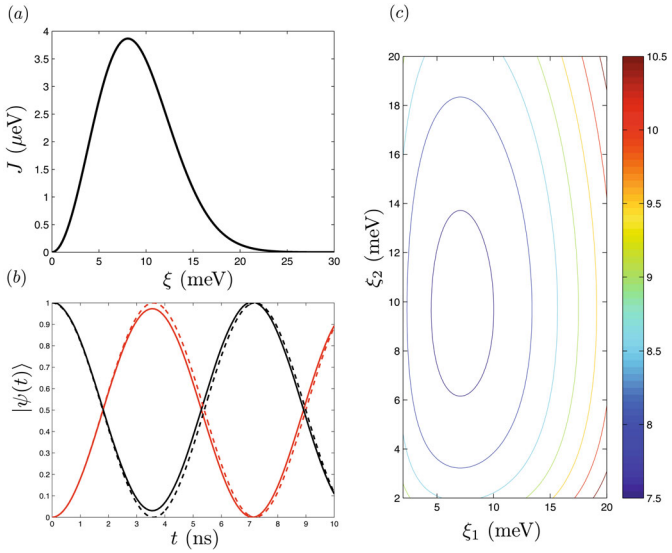

Fig. 3. (a) Coulomb interaction induced SOQs effective flip-flop strength J as a function of rescaled SOC strength $\xi_1 = \xi_2 = \xi$. (b) Time evolution of two SOQs' wave function $|\psi(t)\rangle$ under resonant condition $\Delta_1 = \Delta_2$ and $|\psi(0)\rangle = |10\rangle$. Red and black lines are $|\langle 01|\psi(t)\rangle|^2$ and $|\langle 10|\psi(t)\rangle|^2$, respectively. Solid lines are calculated from equation (17), while dashed lines are exact result from equation (4). (c) Minimal time T needed to complete a two qubits basis flip in resonant case as a function of the rescaled SOC strength ξ_1 and ξ_2. The plot is logarithm rescaled in z direction to stress the existence of minimal point. Except for (c), parameters used in calculation are: $\bar{\omega}_1 = 20$ meV, $\bar{\omega}_2 = 15$ meV, $\omega_c = 5$ meV, $\mu_1 = \mu_2 = 1$ meV and $\eta_1 = \eta_2 = 0.8$.

in Figure 3a, by setting $\eta_j = 1/\sqrt{2}$ the flip-flop strength is optimized to $J_{opt} = 8g\epsilon_1\epsilon_2 \exp(-2)$.

The possible transitions caused by the Coulomb interaction are illustrated in Figure 2b, where only transitions $|00\rangle \leftrightarrow |11\rangle$ and $|01\rangle \leftrightarrow |10\rangle$ are allowed. To illustrate why others are forbidden, we recall a definition of the parity operator as $\Pi_j = \exp(i\pi a_j^\dagger a_j)$ [22]. Using this definition, a_j and $a_j^\dagger$ can be shown to be odd under the GP operation Λ. According to selection rules similar to equation (14), GP of $|0\rangle$ and $|1\rangle$ must change after emitting or absorbing a phonon. On the other hand, at the first order the linearized Coulomb interaction results in the exchanging of a phonon for both electrons. Therefore, the GP of each SOQ states must change as a result of this interaction. Therefore, all transitions with only one SOQ changes its GP such as $|00\rangle \leftrightarrow |01\rangle$ will vanish.

Introducing the following pseudo-spin operators $S_{j,+} = |1\rangle_{jj}\langle 0|$, $S_{j,-} = S_{j,+}^\dagger$ and $S_{j,z} = [S_{j,+}, S_{j,-}]$, the Hamiltonian of the two SOQs subsystem within the qubits subspace is written as:

$$H_{eff} = \sum_{j=1,2} \Delta_j S_{j,z} - J\left(S_{1,+}S_{2,+} - S_{1,+}S_{2,-} + h.c.\right). \quad (17)$$

Here, we have ignored a constant energy shift $2(E_{1,0} + E_{2,0})$.

With qubits flip-flop processes, entanglement between two SOQs can be generated. To quantitatively discuss this, we study evolution dynamics of the two qubits system as well as their concurrence. For the two SOQs system initially in $|\psi(0)\rangle = |10\rangle$, the wave function at time t is given by:

$$|\psi(t)\rangle = (\cos\Omega t + i\sin\theta\sin\Omega t)|10\rangle + i\cos\theta\sin\Omega t|01\rangle. \quad (18)$$

Here, $\Omega = \sqrt{J^2 + (\Delta_1 - \Delta_2)^2}$ and $\theta = \arctan[(\Delta_1 - \Delta_2)/J]$ are Rabi frequency and mixing angle, respectively.

Time evolution of $|\psi(t)\rangle$ is shown in Figure 3b under resonant condition $\Delta_1 = \Delta_2$, which requires a homogenous magnetic field over two nanowires. In this case $|10\rangle$ and $|01\rangle$ are degenerated in energy. Since $\theta = 0$, the two degenerated basis flip completely. Time needed for a complete flip $T = 2\pi/J$ is plotted as function of ξ_j and shown in Figure 3c. Clearly we see that by adjusting SOC, we can achieve the shortest operation time.

We also calculated time evolution of concurrence $C(t)$ for the two SOQs to quantify degree of entanglement among them, by using a formula for two qubits in pure state [23,24], for initial state $|\psi(0)\rangle = |10\rangle$ or $|01\rangle$

$$C(t) = |\sin\Omega t|\sqrt{\sin^2 2\theta \sin^2\Omega t + 4\cos^2\theta\cos^2\Omega t}. \quad (19)$$

Notice that $C(t) = C(t + \pi/\Omega)$. When $\tan^2\theta \leq 1$, $C(t)$ always have two maximum values at:

$$t_\pm^* = \frac{1}{\Omega}\arccos\left[\pm\sqrt{\frac{1}{2}\left(1 - \tan^2\theta\right)}\right] \quad (20)$$

for $\Omega t \in [0, \pi]$. The corresponding maximum values are equal and given by $C(t_\pm^*) = 1$. On the other hand, if $\tan^2\theta > 1$, then $C(t)$ has only one maximum at $t^* = \pi/(2\Omega)$ for $\Omega t \in [0, \pi]$, and the value is $C(t^*) = |\sin 2\theta|$.

For the other two initial states $|11\rangle$ and $|00\rangle$, the concurrence $C'(t)$ is also given by equation (19) except Ω and θ being replaced by:

$$\Omega' = \sqrt{J^2 + (\Delta_1 + \Delta_2)^2}, \quad \theta' = \arctan\frac{\Delta_1 + \Delta_2}{J}. \quad (21)$$

Time evolution of $C(t)$ are shown in Figures 4a and 4b, where we see that in resonant case the system can always evolve into a maximally entangled state. For nearly resonant case, maximum value of $C(t)$ depends non-monotonically on the SOC and can be optimized by tuning the SOC. However, it is hardly to observe any entanglement if initial state is chosen as $|11\rangle$ or $|00\rangle$. Because in this case $\Delta_1 + \Delta_2 \gg J$, thus θ' approaches $\pi/2$ and $C'(t)$ becomes vanishingly small.

We found that the following intuitive argument can be helpful in illustrating the existence of the optimal choice of SOC for generating the maximum two SOQs entanglement. First consider an idea case where no SOC presents in the nanowire. In this case, the orbital and spin DOFs are separable and the SOQ basis $|0\rangle$ and $|1\rangle$ are encoded

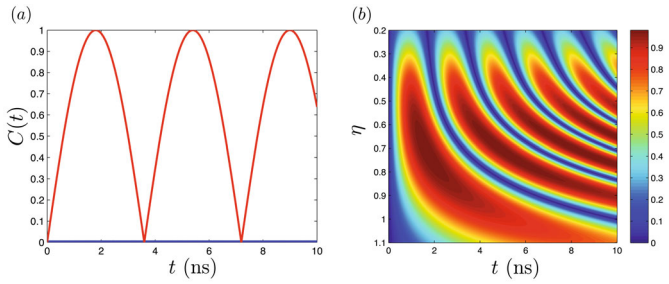

Fig. 4. (a) Time evolution of concurrence $C(t)$ for two SOQs under initial states $|10\rangle$ or $|01\rangle$ (red) and initial states $|00\rangle$ or $|11\rangle$ (blue) in resonant case, where maximum entangled states can be generated at specific time. (b) Concurrence $C(t)$ for initial state $|10\rangle$ or $|01\rangle$ as functions of time and reduced SOC strength $\eta_1 = \eta_2 = \eta$ for nearly resonant case. Here, $\mu_2 = 1.01$ meV in (b). For other parameters, see Figures 3a and 3b.

solely thought the spin DOF. Therefore, orbital manipulation such as the Coulomb interaction cannot flip the qubit states. Consequently entanglement cannot be generated for initially separable two qubits states. Next consider the opposite case where the SOC is so strong such that the level splitting between $|0\rangle$ and $|1\rangle$ is ignorable, i.e., $\mu \exp(-2\eta^2) \ll \hbar\omega$. In this case, the SOQ basis will approach the two displaced vacuum states $|-i\eta, 0\rangle|\uparrow\rangle$ and $|i\eta, 0\rangle|\downarrow\rangle$, since in this case the influence from the Zeeman term H_1 becomes exponentially small. Consequently the two qubits effective flip-flop strength $J = \langle 01|V|10\rangle$ also vanishes due to the negligible spatial overlapping between $|0\rangle$ and $|1\rangle$. The above analysis indicates that two SOQs entanglement neither prefer a very strong nor a very weak SOC. On the other hand, a non-zero J as shown in equation (16) suggests that the Coulomb interaction do help in entangling two SOQs. Thus the existence of an optimal value of SOC for entanglement generation is expectable.

5 Conclusion

In this paper, we have studied entanglement induced by the Coulomb interaction in a system of two electrons, which are separately trapped in two 1D nanowires with SOC. We have explicitly shown how the presence of magnetic field can enable the two electron spins to encode SOQs in a regime where SOC and orbital motion of electrons dominate over the Zeeman effect. The SOQ basis are shown to process definite GP, thus lead to selection rules under a large set of external driving forces including the electric field.

Based on the feasibly responding of SOQs to electric field [25], we have shown that two qubits flip-flop can be effectively created via the inter-electrons Coulomb interaction. In resonant case the flip between two SOQs is perfect and its period is found to depend strongly on SOC and changing non-monotonically, which is in contrary to an intuitive thinking. By studying the time evolution of concurrence, we have shown that entanglement among two SOQs in a nearly resonant case can be optimized by adjusting the strength of SOC. While the tuning of the SOC can be achieved, for example, by simulating the setup using other systems [26].

Finally let us remark that it is possible to generate equally efficient entanglement between $|00\rangle$ and $|11\rangle$ just by reversing the direction of the external magnetic field on one SOQ setup, i.e., using inhomogenous magnetic fields. Since the two basis become energy-degenerated if the fields on both SOQ setups are of the same magnitude. Meanwhile, the coupling between all the rest pairs of basis can be achieved by single qubit operations.

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Appendix A: SOQ wave functions by perturbation theory

The first order eigenstates are calculated according to following formula [17]

$$|\phi^{(1)}_{n,\pm}\rangle = |n,\pm\rangle + \sum_{m\neq n}\sum_{s'=\pm} \frac{\langle m,s'|H_1|n,\pm\rangle}{E_n - E_m}|m,s'\rangle. \quad \text{(A.1)}$$

Matrix elements in the denominator can be rewritten as:

$$\langle m,\pm|H_1|n,\pm\rangle = \pm\mu\langle m|D(2i\eta) + D^\dagger(2i\eta)|n\rangle \quad \text{(A.2)}$$

as well as:

$$\langle m,\mp|H_1|n,\pm\rangle = \pm\mu\langle m|D(2i\eta) - D^\dagger(2i\eta)|n\rangle. \quad \text{(A.3)}$$

Matrix elements of displacement operator under Fock states can be expressed in terms of generalized Laguerre polynomials $L_n^{(m)}(x)$ [18]. Therefore, above equations can be rewritten as:

$$\langle m|D(2i\eta) \pm D^\dagger(2i\eta)|n\rangle = \mu^{-1}(E_n - E_m)\kappa^{(\pm)}_{m,n}, \quad \text{(A.4)}$$

where

$$\kappa^{(\pm)}_{m,n} = \mu\zeta_{m,n}[1 + (-1)^{|m-n|}]/(E_n - E_m)$$

and

$$\zeta_{m,n} = \frac{1}{2}\left(\frac{n!}{m!}\right)^{f(m-n)} (2i\eta)^{|m-n|} L_n^{(|m-n|)}\left(4\eta^2\right), \quad \text{(A.5)}$$

with auxiliary function $f(x) = 1/2$ if $x \geq 0$ and $-1/2$ otherwise.

Substitute equations (A.4) and (A.5) into equation (A.1),

$$|\phi^{(1)}_{n,\pm}\rangle = |n,\pm\rangle \pm \sum_{m\neq n}(\kappa^{(+)}_{m,n}|m,\pm\rangle + \kappa^{(-)}_{m,n}|m,\mp\rangle). \quad \text{(A.6)}$$

For two lowest states where $n = 0$, coefficients $\kappa_{m,0}^{(\pm)}$ are rewritten as:

$$\kappa_{m,0}^{(\pm)} = -[1 \pm (-1)^m]\lambda_m, \tag{A.7}$$

where λ_m is defined as $\lambda_m = \epsilon\langle m|2i\eta\rangle/m$. Insert equation (A.7) to equation (A.6), two SOQ basis is written as:

$$|0\rangle = |0,-\rangle + \sum_{m>0}[\lambda_{2m-1}|2m-1,+\rangle + \lambda_{2m}|2m,-\rangle], \tag{A.8}$$

and

$$|1\rangle = |0,+\rangle - \sum_{m>0}[\lambda_{2m-1}|2m-1,-\rangle + \lambda_{2m}|2m,+\rangle]. \tag{A.9}$$

Appendix B: Generalized parity of perturbed wave functions

First notice that, by applying the generalized parity (GP) operator on the displaced Fock states

$$\begin{aligned} \Pi|\pm i\eta, 2n\rangle &= \int dx|-x\rangle\langle x|\pm i\eta, 2n\rangle \\ &= \int dx e^{\pm im\alpha x/\hbar}\varphi_{2n}(x)|-x\rangle \\ &= \int dx e^{\mp im\alpha x/\hbar}\varphi_{2n}(x)|x\rangle \\ &= |\mp i\eta, 2n\rangle, \end{aligned} \tag{B.1}$$

as well as:

$$\Pi|\pm i\eta, 2n-1\rangle = -|\mp i\eta, 2n-1\rangle, \tag{B.2}$$

where $\varphi_n(x)$ is nth eigenstates of 1D harmonic oscillator.

When spin is included, following similar relations can be derived

$$\begin{aligned} \Lambda|2n,\uparrow\rangle &= \Lambda|-i\eta, 2n\rangle|\uparrow\rangle \\ &= |i\eta, 2n\rangle|\downarrow\rangle \\ &= |2n,\downarrow\rangle, \end{aligned} \tag{B.3}$$

as well as:

$$\begin{aligned} \Lambda|2n,\downarrow\rangle = |2n,\uparrow\rangle, \Lambda|2n-1,\uparrow\rangle = -|2n-1,\downarrow\rangle \\ \Lambda|2n-1,\downarrow\rangle = -|2n-1,\uparrow\rangle. \end{aligned} \tag{B.4}$$

Those relations then indicate that $|n,\pm\rangle$ are eigenstates of the GP operator Λ

$$\begin{aligned} \Lambda|2n,+\rangle = |2n,+\rangle,\ \Lambda|2n-1,-\rangle = |2n-1,-\rangle \\ \Lambda|2n,-\rangle = -|2n,-\rangle,\ \Lambda|2n-1,+\rangle = -|2n-1,+\rangle. \end{aligned} \tag{B.5}$$

From Appendix A, $\kappa_{m,n}^{(+)}$ ($\kappa_{m,n}^{(-)}$) is non-zero only if m and n have same (opposite) oddness. Therefore, equation (A.6) is rewritten as follows for n even,

$$\begin{aligned} |\phi_{2n,\pm}^{(1)}\rangle = |2n,\pm\rangle \pm \sum_{m\neq n}(\kappa_{2m,2n}^{(+)}|2m,\pm\rangle \\ + \kappa_{2m-1,2n}^{(-)}|2m-1,\mp\rangle). \end{aligned} \tag{B.6}$$

Together with equation (B.5), we then concluded that

$$\Lambda|\phi_{2n,+}^{(1)}\rangle = |\phi_{2n,+}^{(1)}\rangle,\ \Lambda|\phi_{2n,-}^{(1)}\rangle = -|\phi_{2n,-}^{(1)}\rangle, \tag{B.7}$$

and similarly analysis for the n odd case gives

$$\Lambda|\phi_{2n-1,+}^{(1)}\rangle = -|\phi_{2n-1,+}^{(1)}\rangle,\ \Lambda|\phi_{2n-1,-}^{(1)}\rangle = |\phi_{2n-1,-}^{(1)}\rangle. \tag{B.8}$$

Appendix C: Matrix elements of the Coulomb interaction

In the subspace of two SOQs, matrix element of the linearized Coulomb interaction is written as:

$$\langle N_1N_2|V|M_1M_2\rangle = g\langle N_1|a_1^\dagger + a_1|M_1\rangle\langle N_2|a_2^\dagger + a_2|M_2\rangle. \tag{C.1}$$

Since $a_j^\dagger + a_j$ is odd under GP operation with respect to jth electron, then N_j and M_j must be different for having non-zero matrix elements (since $|0\rangle$ and $|1\rangle$ have different GP as shown in the main text). Therefore,

$$\langle N_1N_2|V|M_1M_2\rangle = \delta_{N_1,\bar{M}_1}\delta_{N_2,\bar{M}_2}J_{N_1,N_2}, \tag{C.2}$$

where $J_{mn} = g\langle m|a_1^\dagger + a_1|\bar{m}\rangle\langle n|a_2^\dagger + a_2|\bar{n}\rangle$.

Matrix element $\langle n|a^\dagger + a|\bar{n}\rangle$ can be calculated directly from equations (A.8) and (A.9). Up to first order in ϵ, it is given by:

$$\langle n|a^\dagger + a|\bar{n}\rangle = -2i(-1)^n\mathrm{Im}\lambda_1. \tag{C.3}$$

Thus J_{mn} can be rewritten as:

$$\begin{aligned} J_{mn} &= 4(-1)^{m+n-1}g\prod_{j=1,2}\mathrm{Im}\epsilon_j\langle 1|2i\eta_j\rangle \\ &= (-1)^{m+\bar{n}}16g\epsilon_1\epsilon_2\eta_1\eta_2 e^{-2\eta_1^2-2\eta_2^2}. \end{aligned} \tag{C.4}$$

Then equation (15) can be recovered by inserting equation (C.4) back to equation (C.2). Notice that although J_{mn} is second order in ϵ_j, actually we do not need to do perturbation theory on wave function to the second order. Because direct calculation shows that $\langle 0,s|a^\dagger + a|0,s'\rangle = 0$ even for $s \neq s'$. Therefore the effect from the second order correction of wave function on matrix element of the Coulomb interaction must be higher order and can be neglected.

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