CHINESE JOURNAL OF PHYSICS VOL. 33, NO. 1 FEBRUARY 1994

# Dynamical Realizability for Quantum Measurement and Factorization of the Evolution Operator

Chang-Pu Sun
*Department of Physics, Northeast Normal University, Changchun 130024, China*

(Received September 11, 1993; revised manuscript received November 3, 1993)

By building a general dynamical model for the quantum measurement process, it is shown that factorization of the reduced evolution operator effectively results in the quantum mechanical realization of wave packet collapse and state correlation between the measured system and the measuring instrument-detector. This realizability is largely independent of the details of both the interaction and the Hamiltonian of the detector. The Coleman-Hepp model and all of its generalizations are only special cases of the more universal model given in this letter. Finally an explicit example of this model is given in connection with a coherent state.

PACS. 03.65.Bz – Foundations, theory of measurement, miscellaneous theories.
PACS. 03.80.+r – General theory of scattering.

It is well-known that, although the theory of quantum mechanics and its applications are extremely successful, its interpretation in connection with the corresponding measurement is still a valid problem that the physicist must face [1-3]. Since an exactly-solvable model was presented twenty years ago [4] to describe von Neumann's wave packet collapse (WPC) in measurement as a quantum dynamical proccess caused by the interaction between the measured system (S) and the measuring instrument-detector (D), considerable studies have been focused on this model [5-9], which is now called the Coleman-Hepp (CH) model. More recently, it was respectively generalized to the case with energy exchange between S and D [8] and to the case with simultaneously the classical limit – the large quantum number limit and the macroscopic limit – the large particle number limit [9]. Notice that another important problem in quantum measurement, the state correlation between S and D (SCB) can also be studied by making use of other solvable-models, e.g., the Cini model in Ref. IO.

However, all the investigations mentioned above use the concrete forms of the interaction and thus the main conclusions seem to depend on the selection of the specific form of the interactions. There is no doubt that a model-independent study of this problem would be much appreciated. The present studies are dedicated to seek the essence of the quantum machanical realization of the WPC and SCB in the CH-model and its generalizations. To this end a more universal dynamical model of quantum measurement is proposed which is as interaction -independent as possible. Based on this model, we will show that the realizability of the WPC and SCB as a quantum dynamical process mainly depends on the factorizability of the reduced evolution operator for the system. This crucial observation not only reveals the essence rooted in those well-established exactly-solvable models for quantum measurement, but also provides us with guidance to find new exactly-solvable models. Finally, as an explicit example, an exactly-solvable model associated with a coherent state is analysed in detail.

## I. THE GENERAL MODEL

Our model can be regarded as an universal promotion of the original CH model. The measured system S is still represented by an ultrarelativisic particle with the free Hamiltonian $H_0 = c\hat{P}$, but the detector D is made of N particles each with a single-particle Hamiltonian $h_k(x_k), (k = 1, 2, \ldots, N)$, which is Hermitian. S is assumed to be independently subjected to the interaction $V_k(x, x_k)$ of each particle k. Here, $x$ and $x_k$ are the coordinates of S and the single particle $k$ in D respectively; the $k$'th interaction potential $V_k(x, x_k)$ only depends on $x$ and $x_k$ and $h_k(x_k)$ only depends on the single particle coordinate $x_k$ and the corresponding momentum. Then, we can write down the total Hamiltonian for the 'universe' = S + D

$$H = H_0 + H' = H_0 + H_I + H_D :$$
$$H_I = \sum_{k=1}^{N} V_k(x, x_k), H_D = \sum_{k=1}^{N} h_k(x_k), \tag{1}$$

where

$$H' = H_I + H_D = \sum_{k=1}^{N} [h_k(x_k) + V_k(x, x_k)], \tag{2}$$

is a direct *sum decomposition* of single-particle forms. This fact, associated with the fact that the $H_0$ is of the first order in $\hat{P}$, will lead to the factorization of the effective (reduced) evolution operator and thereby produces WPC in quantum measurement. This factorizability is also closely related to SCB. Fortunately, to prove it we do not need the specific forms of both $h_k(x_k)$ and $V_k(x, x_k)$. In this sense we say this model is more universal. It is worth noting that the original IIC model and its generalizations are special examples of this universal model.

## II. THE EVOLUTION OPERATOR

In order to interpretate the WPC and SCB as a consequence of the Schrödinger evolution of the universe (S+D), we should consider the properties of the evolution operator defined by the general Hamiltonian (1). Following Hepp [4], we first transform into the interaction representation by assuming the evolution operator to be of the following form

$$U(t) = e^{-ict\hat{P}/\hbar} U_e(t), \tag{3}$$

Obviously, the reduced evolution operator $U_e(t)$ obeys an effective Schrödinger equation

$$i\hbar \frac{\partial}{\partial t} U_e(t) = H_e(t) U_e(t), \tag{4}$$

with the effective Hamiltonian

$$H_e(t) = \sum_{k=1}^{N} h_{ek}(t) = \sum_{k=1}^{N} [h_k(x_k) + V_k(x + ct, x_k)], \tag{5}$$

depending on time. Since $H_e(t)$ is a direct sum of the time-dependent Hamiltonians $h_{ek}(t)(k = 1, 2, \ldots, N)$ parameterized by $x$, the $x$-depenedent evolution operator, as the formal solution to Eq. (-1)

$$U_e(t) = \prod_{k=1}^{N} \otimes U^{[k]}(t) = U^{[1]}(t) \otimes U^{[2]}(t) \otimes \cdots \otimes U^{[N]}(t), \tag{6}$$

is factorizable, that is to say, $U_e(t)$ is a direct product of the single-particle evolution operators

$$U^{[k]}(t) = \Im \exp[(1/i\hbar \int_0^t h_{ek}(t) dt], \tag{7}$$

where $\Im$ denotes the time-order operation. As proved below, it is just the above factorizable property of the reduced evolution operator that results in the quantum dynamical realization of WPC and is closely related to SCB in quantum measurement.

## III. WAVE PACKET COLLAPSE AS A QUANTUM DYNAMICAL PROCESS

Let us use the ideal double-slit interference experiment to show that WPC is the consequence of the quantum dynamical evolution of the universe S+D described by the above factorizable evolution operator. An incident wave is split by a divider into two branches $|\psi_1>$ and $|\psi_2>$ and the detector D is in the ground state

$$|0>=|0_1>\otimes|0_2>\otimes\cdots|0_N>, \tag{8}$$

at the same time where $|0_k>$ is the ground state of $h_k(x_k)$. Then, the initial state for the universe S+D is

$$|\psi(0)>=(C_1|\psi_1>+C_2|\psi_2>)\otimes|0>, \tag{9}$$

Notice that the ground state is required by a stable measuring instrument. Starting with this initial state the universe S+D will evolve according to the wave function

$$|\psi(t)>=C_1|\psi_1>\otimes|0>+C_2|\psi_2>\otimes U_e(t)|0>, \tag{10}$$

Here, like the authors in Ref. 4-9, we have supposed that only the second branch wave $|\psi_2>$ interacts with D so that the double-slit interference experiment can be realized. For example, such a partiality of interaction for different states of S can be automatically given in the 'momentum' (p-) representation with the basis $|p>$ if we use an improved Hamiltonian

$$H=H_0+H'=H_0+H_I\delta_{p_0}(\hat{P})+H_D, \tag{11}$$

obtained by introducing an operator function $\delta_{p_0}(\hat{P})$:

$$\delta_{p_0}(\hat{P})|p>=\delta_{p_0,p}|p>$$

into the original Hamiltonian (1) and take

$$|\psi_1>=\sum_{p'\neq p_0}C'_{p'}|p'>,|\psi_2>=|p_0>, \tag{12}$$

From the final state (10), we explicitly write down the density matrix for the universe S+D

$$\begin{aligned}\rho(t)=&|\psi(t)><\psi(t)|=|C_1|^2|\psi_1(t)><\psi_1(t)|\otimes|0><0|\\&+|C_2|^2|\psi_2(t)><\psi_2(t)|\otimes U_e(t)|0><0|U_e(t)^+\\&+C_1C_2^*|\psi_1(t)><\psi_2(t)|\otimes U_e(t)|0><0|\\&+C_2C_1^*|\psi_2(t)><\psi_1(t)|)|\otimes|0><0|U_e(t)^+\end{aligned} \tag{13}$$

In the problem of WPC, because we are only interested in the behaviours of the system S and the effect of the detector D on it, we only need the reduced density matrix for S

$$\begin{aligned}\rho(t)_S=&Tr_D\rho(t)=|C_1|^2|\psi_1(t)><\psi_1(t)|+|C_2|^2|\psi_2(t)><\psi_2(t)|\\&+(C_1C_2^*|\psi_1(t)><\psi_2(t)|+C_2C_1^*|\psi_2(t)><\psi_1(t)|)<0|U_e(t)|0>,\end{aligned} \tag{14}$$

where $Tr_D$ represents the trace over the variables of the detector D. Let us recall that the WPC postulate means the reduction to a pure state density matrix

$$\rho(t)_S \rightarrow \rho(t) = |C_1|^2|\psi_1(t)><\psi_1(t)| \text{ t } |C_2|^2|\psi_2(t)><\psi_2(t)|, \tag{15}$$

Obviously, under a certain condition to be determined, if $<0|U_e(t)|0>=0$, then the coherent terms in Eq. (14) vanish and the quantum dynamics automatically leads to this reduction, i.e., the WPC! Now, let us prove that this condition is just the macroscopic limit which is defined by a very large particle number N of D, i.e., $N\rightarrow\infty$. In fact, due to the factorization of the reduced evolution operator $U_e(t)$, the norm of $<0|U_e(t)|0>$ is

$$|<0|U_e(t)|0>| = \prod_{k=1}^{N} |<0_k|U^{[k]}|0_k>| = \exp\left[-\sum_{k=1}^{N}\Delta_k(t)\right], \tag{16}$$

where

$$e^{-\Delta_k(t)} = |<0_k|U^{[k]}|0_k>| = \left[1-\sum_{n\neq 0}|<n|U^{[k]}|0_k>|^2\right]^{1/2} \leq 1, \tag{17}$$

Usually, $\Delta_k(t)$ is non-zero and positive and thus the series $\Sigma_{k=1}^{\infty}\Delta_k(t)$ diverges to infinity, that is to say, the factor $<0|U_e(t)|0>$ as well as its norm approach zero as $N\rightarrow\infty$. This just proves the central conclusion that WPC can appear as a quantum dynamical process for the universal model (1) in the macroscopic limit as long as the dynamical models are selected to have factorizable evolution operators. However, in this case, there is no interactions among the particles in the detector. We understand it as an ideal case. Because the particles in a realistic measuring instrument must interact with each other, it is necessary to build an exactly-solvable dynamic model of quantum measurement with a self-interacting detector. We belive the above -mentioned factorization property probably is also a clue to finding such a model.

In terms of the above model, we can also discuss the energy-exchange process and the delicate behaviour as $N\rightarrow\infty$ such as in Ref. S. For the latter, we have

$$|<0|U_e(t)|0>| \sim e^{-N\bar{\Delta}_k(t)}$$

where $\bar{\Delta}_k(t)$ represents the average value of $\Delta_k(t)$. The above formula shows the gradual disappearance of the interference.

## IV. CORRELATION BETWEEN STATES OF THE SYSTEM AND DETECTORS

Physically, the measurement is a. scheme using the counting number of the measuring instrument D to manifest the state of the measured system S. The state correlation between S and D (SCB) will shows this manifestation. Now, we show how this correlation occurs for the above dynamical model (1) in a certain limit. To simplify the problem, we also use the improved Hamiltonian (11). Let $c_{m_k}$ be the coefficient with the largest norm among $c_{n_k} =< n_k|U^{[k]}(t)|0_k >$ (k = $1,2\ldots$) and assume that the corresponding state $|m_k>$ is not degenerate. Then,

$$U^{[k]}(t)|0>= c_{m_k}\{|m_k> + \sum_{n\neq m}[c_n/c_{m_k}]|n>\}. \tag{18}$$

Except for the coefficient of $|m_k>$, each of the terms in $c_{m_k}^{-1}U^{[k]}(t)|0>$ has a norm less than 1. Because of the factorization of the reduced evolution operator, the wave function $U(t)|0>=\prod_{k=1}^{N} U^{[k]}|0_k>$ will be strongly peaked around the state

$$|m>=|m_1>\otimes|m_2>\otimes|m_3>\otimes\cdots\otimes|m_N>.$$

If the universe S+D with the Hamiltonian (6) initially is in the state

$$|\psi(0)>=[|p_0>+|p>]\otimes|0>, p\neq p_0.$$

Then, it will evolve into a state near the state

$$|\psi(t)>=|p_0>\otimes\prod_{k=1}^{N}c_{m_k}|m>+|p>\otimes|0>. \tag{19}$$

This just manifests the correlation between the state $|p_0>$ of the system and $|m>$ of the detector. Notice that SCB can exactly appear only for the 'classical' limit, in which some parameters or internal quantum numbers take their limit values (e.g., in Ref. 9, 10, this is the limit with infinite spin). In fact, in a realistic problem, the correlation often occurs as a good approximation valid to a quite high degree. A special example of such a correlation problem was discussed in Ref. 10. In accord with the above general arguments, we will give a new example to deal with both WPC and SCB.

## V. A NEW EXACTLY-SOLVABLE DYNAMICAL MODEL FOR QUANTUM MEASUREMENT AND COHERENT STATES

Up to now we have described the general form of the Hamiltonian as Eq. (1) for the dynamics of quantum measurement. It is easy to see that the CH model and its various generalizations are only some special and explicit examples of the above more universal model. Now, let us apply the above general results, as a guidance rule, to build a new exactly-solvable model for quantum measurement. In this model, WPC and SCBSD can be simultaneously described as a quantum dynamical process. The model Hamiltonian is

$$H = c\hat{P} + \sum_{k=1}^{N} f_k(x)[a_k^+ + a_k] + \sum_{k=1}^{N} \hbar\omega_k a_k^+ a_k, \tag{20}$$

Here, the detector is made of N harmonic oscillators linearly coupled to an ultra-relativistic particle as the measured system; $a_k^+$ and $a_k$ are the creation and annihilation operators for boson states respectively. The coupling function $f_k(x) = c_k x$ only depends on the coordinate of the ultrarelativistic particle. Because the effective Hamiltonian

$$H_e(t) = \sum_{k=1}^{N} (f_k(x+ct)[a_k^+ + a_k] + \hbar\omega_k a_k^+ a_k), \tag{21}$$

is completely decomposable, the reduced evolution operator $U_e$ is factorizable, i.e., $U_e = \prod_k U^{[k]}(t)$, and its factors are [11]

$$U^{[k]}(t) = e^{-i\omega_k a_k^+ a_k t} e^{F_k(t)} e^{A_k(t) a_k^+} e^{B_k(t) a_k}, \tag{22}$$

where the functions $F_k(t), A_k(t), B_k(t)$ are defined by

$$\begin{aligned} A_k(t) &= -B_k(t)^* = -\frac{c_k}{\hbar\omega_k^2}[(x + ct + ic/\omega_k)e^{i\omega_k t} - ic/\omega_k - x], \\ F_k(t) &= \int_0^t A_k(s)\frac{\partial B_k(s)}{\partial s} ds \end{aligned} \tag{23}$$

Notice that the real part of $F_k(t)$ is $-\eta_k(t)$:

$$\begin{aligned} \eta_k(t) &= \frac{1}{2}|A(t)|^2 \\ &= \frac{c_k^2}{\hbar^2\omega_k^2}\left[\frac{1}{2}c^2t^2 + xct + x^2 + \frac{c^2}{\omega_k^2} - \frac{c^2 t}{\omega_k}\sin\omega_k t - \cos\omega_k t\left(x^2 + xct + \frac{c^2}{\omega_k^2}\right)\right] \end{aligned} \tag{24}$$

Since $\eta_k(t)$ is larger than zero after an interval of time, the norm

$$| < 0|U(t)|0 > | = e^{-\sum_{k=1}^{N} \eta_k(t)}$$

must approach zero as $N \to \infty$. This implies the dynamical realization of WPC for the quantum measurement.

Let us show how the correlation between the states of the system and the detector appears as a quantum dynamical process. If the detector is initially in its ground state $|0> = |0_1> \otimes |0_2> \otimes \cdots |0_N>$, the state at $t$ is a direct product of the coherent states.

$$|\psi^k(t)> = U^{[k]}(t)|0> = e^{F_k(t)} \sum_{n=0}^{\infty} e^{-in\omega_k t} \frac{A_k^n(t)}{n!} a_k^{+n}|0_k>, \tag{25}$$

Using the Stirling formula, we immediately determine the value $\bar{n}_k$ of quantum number $n$ for which the norm of the coefficient of the Fock state $|n>_f = 1/(n!)^{1/2} a_k^{+n}|0_k>$ in Eq. (25) is maximum, obtaining

$$\bar{n}_k = |A_k(t)|^2, \tag{26}$$

At time $t$ the validity of Stirling's formula requires $\bar{n}_k = |A_k(t)|^2$ to be sufficiently large. This means that the counting number of the detector is macroscopically large. It is just what we expect for a measuring instrument. If we take the initial state of the system to be $|\psi(0)> = (v|p_0> + w|p \neq p_0>) \otimes |0>$, then the correlation is enjoyed by the Schrödinger evoluting state

$$|\psi(t)> = v|p_0> \otimes |\bar{n}_1> \otimes |\bar{n}_2> \otimes \cdots \otimes |\bar{n}_N> + w|p> \otimes |0> . \tag{27}$$

## VI. FINAL REMARKS

Finally, we should point out that in practical problems there must exist interactions among the particles constituting the detector D. They seem to break the factorization of the reduced evolution operator. How to realize the quantum measurement both for WPC and SCB in this case is an open question that we must face. It is expected, at least for some special cases, that certain canonical (or unitary) transformations possibly enable these particles to become quasi-free ones. This is just similar to the system of harmonic oscillators with quadratic coupling. In this case, we can imagine that the detector is made of free quasi-particles that do not interact with each other. If each quasi-particle interacts with the system independently, then the factorizability of the evolution operator can be preserved in solvable models for quantum measurement.

We also remark on the realization of the double-slit type experiment where the interaction selects only one of the two branch wave functions. Concerning the introduction of $\delta_{p_0}(\hat{p})$ in the Hamiltonian (11) (to realize the partiality of interaction), some may not feel content. In fact, we can also get this selection in a quite natural way. If the system is a spin-1/2 with the free Hamiltonian $H_0 = \hbar\omega\sigma_3$ and the detector is still defined by the general Hamiltonian in Eq. (2), the following interaction

$$H_I = g(1+\sigma_2)\sum_{k=1}^{N} h_k(x_k), \tag{28}$$

naturally results in a selective interaction. Namely, the detector only acts on the spin-up state. The spin-down state is free of interaction. Thus, Eq. (28) defines a new dynamical model for quantum measurement, which is an extensive generalization of Cini's model [10].

## ACKNOWLEDGMENTS

The author wishes to express his sincere thanks to Prof. C. N. Yang for drawing his attentions to new progress in the quantum measurement. He also thank D. H. Feng, R. M. Xu, L. H. Yu and W. M. Zhang for many useful discussions. He is supported by Cha Chi-Ming fellowship through the CEEC in State University of New York at Stony Brook and in part by the NFS of China through Northeast Normal University.

## REFERENCES

[1] J. von Neumann, *Mathematische Grundlage de Quantummechanik,* Springer, Berlin, 1933.

[2] V. B. Braginsky, F. Y. Khalili, *Quantum Measurement,* Cambridge, 1992.

[3] R. Omnes, Rev. Mod. Phys. 64, 339 (1992).

[4] K. Hepp, Helv. Phys. Acta. 45, 237 (1972).

[5] J. S. Bell, Helv. Phys. Acta. 48, 93 (1975).

[6] T. Kobayashi, Found. Phys. Lett. 5, 265 (1992).

[7] M. Namik, S. Pascazio, Phys. Rev. A44, 39 (1991).

[8] H. Nakazato, S. Pascazio, Phys. Rev. Lett. 70, 1 (1993).

[9] C. P. Sun, Phys. Rev. A48, 898 (1993).

[10] M. Cini, Nuovo Cimento, 73B, 27 (1983).

[11] C. P. Sun, Q. Xiao, Commun. Theor. Phys., 16, 359 (1991); W. M. Zhang, D. H. Feng, R. Gilmore, Rev. Mod. Phys. 62, 867 (1990).