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1996 Acta Phys. Sin. (Overseas Edn) 5 881

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## HIGHER-ORDER CORRECTION FOR ROTATING WAVE APPROXIMATION, RABI TRANSFORMATION, AND ITS APPLICATIONS IN THE JAYNES-CUMMINGS MODEL\*

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(Received 25 October 1995; revised manuscript received 15 July 1996)

Taking into account the consistence of the quantum adiabatic approximation and the rotating wave approximation in essence, we propose a method to study the interaction between atom and radiation field in a quantized cavity systematically. By using the exact solution of Jaynes-Cummings model as the lowest-order approximation, the effect of high frequency terms on the dynamics of atom-cavity field system is studied analytically.

PACC: 0365

### I. INTRODUCTION

In order to understand the influence of the quantized radiation field on the dynamics of two-level atom or the procession of spin  $-\frac{1}{2}$  particle, the Jaynes-Cummings model (JCM)<sup>[1]</sup> has been extensively studied in recent years.<sup>[2-6]</sup> The key point of this model is to invoke the Rotating Wave Approximation (RWA) in which the high frequency terms of the Hamiltonian in the interaction picture are neglected while the low frequency terms hold.<sup>[7]</sup> This is quite similar to that for the higher-order quantum adiabatic approximation (HOQAA), developed by one of the authors (Sun) to deal with the evolution of the quantum system governed by a slowly-changing Hamiltonian.<sup>[8-10]</sup> In the approach of HOQAA, the oscillation terms with higher frequencies are regarded as the adiabatic perturbation, and the adiabatic approximation is just the result of ignoring all the high frequency terms.

For the above consideration, it is natural to expect that a method treating the perturbation effect of high frequency terms in the dynamics of the cavity-atom system can be built by drawing an analogy to the HOQAA method. In this paper, the Rabi transformation<sup>[11]</sup> is invoked to separate the two parts of high and low frequency terms in the effective Hamiltonian

\*Project supported by the National Natural Science Foundation of China and by Fok Yin-Tung Education Foundation.

and then the central idea within the HOQAA method is generalized to build a perturbation approach for the dynamics of the atom-cavity system. This approach is called the higher-order correction for RWA. As its applications, a two-level atom interacting with cavity field beyond RWA is studied.

## II. GENERAL DESCRIPTION OF HIGHER-ORDER CORRECTION FOR RWA

We consider an  $N$ -level atom interacting with an  $M$ -mode quantized radiation field. The Hamiltonian for the problem is

$$\begin{aligned}
 H &= H_0 + V_I, \\
 H_0 &= \sum_{n=1}^N E_n |\varphi_n\rangle\langle\varphi_n| + \sum_{\alpha=1}^M \omega_\alpha a_\alpha^\dagger a_\alpha, \\
 V_I &= \sum_{m>n} \sum_{\alpha=1}^M g_{mn\alpha} (a_\alpha^\dagger + a_\alpha) (|\varphi_m\rangle\langle\varphi_n| + |\varphi_n\rangle\langle\varphi_m|). \quad (1)
 \end{aligned}$$

Here  $|\varphi_n\rangle$  is the atomic energy eigenstate corresponding to the energy level  $E_n$ ;  $a_\alpha$  and  $a_\alpha^\dagger$  are the annihilation and creation operators of  $\alpha$ -th single-mode radiation field;  $g_{mn\alpha}$  is the coupling matrix element, and  $E_1 < E_2 < \dots < E_N$ . The Hamiltonian corresponding to  $H$  in the interaction picture,

$$H_I = e^{iH_0 t} V_I e^{-iH_0 t} = H_l(t) + H_h(t), \quad (2)$$

is divided into the low frequency term

$$H_l = \sum_{m>n} \sum_{\alpha=1}^M g_{mn\alpha} (e^{-i\Delta_{mn\alpha} t} |\varphi_m\rangle\langle\varphi_n| a_\alpha + e^{i\Delta_{mn\alpha} t} |\varphi_n\rangle\langle\varphi_m| a_\alpha^\dagger), \quad (3)$$

and the high frequency term

$$H_h = \sum_{m>n} \sum_{\alpha=1}^M g_{mn\alpha} (e^{-i\Omega_{mn\alpha} t} |\varphi_m\rangle\langle\varphi_n| a_\alpha^\dagger + e^{i\Omega_{mn\alpha} t} |\varphi_n\rangle\langle\varphi_m| a_\alpha), \quad (4)$$

where

$$\Delta_{mn\alpha} = |\omega_m - \omega_n| - \omega_\alpha; \quad \Omega_{mn\alpha} = |\omega_m - \omega_n| + \omega_\alpha. \quad (5)$$

Similar to the HOQAA method, it can be concluded that the high frequency term  $H_h$  oscillates very rapidly and thus contributes much less to the wave function of evolution. In fact, the contributions of oscillating factors in  $H_h$  and  $H_l$  to the wave function can be expressed by the following integral

$$\int f(t) e^{\pm i\omega t} = \frac{f(t) e^{\pm i\omega t}}{\pm i\omega} + O\left(\frac{1}{\omega}\right)^2, \quad (6)$$

where  $f(t)$  is a slowly-varying function. For the case with  $H_l$  (or  $H_h$ ),  $\omega$  represents  $\Delta_{mna}$  (or  $\Omega_{mna}$ ). Correspondingly, the ratio of the contributions of the two parts is  $\frac{|\Delta_{mna}|}{\Omega_{mna}}$ , that is to say, in comparison with  $H_l$ , the contribution due to  $H_h$  is a small quantity for the case of high frequency field and large detuning. Then this term can be handled as a perturbation. The approximation of neglecting this small quantity is called RWA.

Let

$$U(t) = \sum_{k=0}^{\infty} U^{[k]}(t) \quad (7)$$

be the perturbation expansion of the evolution operator governed by the Hamiltonian  $H_l$ . Then, the  $k$ -th correction  $U^{[k]}(t)$  satisfies the inhomogeneous differential equation

$$i \frac{\partial U^{[k]}(t)}{\partial t} = H_l(t)U^{[k]}(t) + H_h(t)U^{[k-1]}(t), \quad (8)$$

and its solution can be obtained by the following formula

$$U^{[k]}(t) = \frac{1}{i} \int U^{[0]}(t-\tau)H_h(\tau)U^{[k-1]}(\tau)d\tau. \quad (9)$$

Here  $U^{[0]}(t)$  satisfies the lowest-order approximation equation

$$i\hbar \frac{\partial U^{[0]}(t)}{\partial t} = H_l(t)U^{[0]}(t). \quad (10)$$

In the present paper, it will be obtained by generalizing Rabi rotation transformation for different cases.

### III. RABI ROTATION TRANSFORMATION FOR THE JCM

According to the above discussion, it is necessary to give the exact solution of lowest-order approximation equation (10). By means of a symmetry analysis, the original JCM can be exactly solved.<sup>[7]</sup> In order to make a clear physical picture, we employ an alternative, but concise method, the Rabi rotation transformation technique, to solve the JCM.

The Hamiltonian of a two-level atom interacting with a single-mode radiation field is

$$H = \omega a^\dagger a + \frac{\omega_0}{2}(|2\rangle\langle 2| - |1\rangle\langle 1|) + g(a + a^\dagger)(|2\rangle\langle 1| + |1\rangle\langle 2|), \quad (11)$$

where  $|2\rangle$  and  $|1\rangle$  are the excited and ground states respectively;  $g$  is the coupling constant,  $\omega$  the frequency of the single-mode radiation field, and  $\omega_0$  the energy difference of the two-level atom.

In the interaction picture, the effective Hamiltonian is

$$\begin{aligned} H_I &= H_l + H_h, \\ H_l &= g(a|2\rangle\langle 1|e^{-i\Delta t} + a^\dagger|1\rangle\langle 2|e^{i\Delta t}), \\ H_h &= g(a^\dagger|2\rangle\langle 1|e^{-i\Omega t} + a|1\rangle\langle 2|e^{i\Omega t}), \end{aligned} \quad (12)$$

where  $\Delta = |\omega - \omega_0|$  and  $\Omega = \omega + \omega_0$ . By neglecting high frequency terms, the Hamiltonian of JCM in its interaction picture is obtained

$$H_{J-C} = g(a|2\rangle\langle 1|e^{-i\Delta t} + a^\dagger|1\rangle\langle 2|e^{i\Delta t}). \quad (13)$$

In a two-dimensional subspace spanned by vectors

$$\begin{aligned} |m, 2\rangle &= |m\rangle \otimes |2\rangle, \\ |m+1, 1\rangle &= |m+1\rangle \otimes |1\rangle, \end{aligned} \quad (14)$$

where  $|m\rangle$  is the Fock state for the cavity mode, the above Hamiltonian can be represented in terms of the Pauli matrices  $\sigma_x, \sigma_y, \sigma_z$

$$H_{J-C} = g\sqrt{m+1}(\sigma_x \cos \Delta t + \sigma_y \sin \Delta t). \quad (15)$$

Formally, it is equal to that for the procession of spin-half particles in a rotating time-dependent magnetic field

$$\mathbf{B} = g\sqrt{m+1}(\cos \Delta t, \sin \Delta t). \quad (16)$$

Therefore, we introduce a unitary transformation, Rabi rotation transformation  $V(t) = e^{-i\frac{\Delta t}{2}\sigma_z}$ , to transform the problem from the present static reference frame to the rotating reference frame, which rotates around the  $z$ -axis with an angular velocity  $\Delta$ . Thus, the effective Hamiltonian

$$H_m = g\sqrt{m+1}\sigma_x - \frac{\Delta}{2}\sigma_z \quad (17)$$

is time-independent. And then the evolution matrix of  $H_{J-C}$  is obtained as

$$\begin{aligned} U_m(t) &= V(t)e^{-iH'_m} \\ &= \begin{bmatrix} e^{-i\frac{\Delta t}{2}}(\cos \lambda_m t + i\frac{\Delta}{2\lambda_m} \sin \lambda_m t) & -e^{-i\frac{\Delta t}{2}}\frac{ig\sqrt{m+1}}{\lambda_m} \sin \lambda_m t \\ -e^{-i\frac{\Delta t}{2}}\frac{ig\sqrt{m+1}}{\lambda_m} \sin \lambda_m t & e^{i\frac{\Delta t}{2}}(\cos \lambda_m t - i\frac{\Delta}{2\lambda_m} \sin \lambda_m t) \end{bmatrix}, \end{aligned} \quad (18)$$

where

$$\lambda_m = \sqrt{g^2(m+1) + \frac{\Delta^2}{4}}. \quad (19)$$

Given an arbitrary initial state, we can discuss the dynamical problem of the atom-field system from the above solution of evolution operator.

#### IV. APPLICATION OF HIGHER-ORDER CORRECTION FOR RWA TO THE CASE WITH TWO LEVELS

In this section, we use the general higher-order correction for RWA to study the effects on the JCM (13) exercised by the high frequency terms. It has been considered in Refs. [13,14] with numerical calculations, but now we can give an explicit analytical expression of first-order correction to the original function. For the sake of simplicity, we consider the resonant case ( $\omega = \omega_0$ ). The evolution operator corresponding to  $H_{J-C}$  in the subspace spanned by  $|m, 2\rangle$  and  $|m + 1, 1\rangle$  is now reduced to

$$\begin{bmatrix} \cos g\sqrt{m+1}t & -i \sin g\sqrt{m+1}t \\ -i \sin g\sqrt{m+1}t & \cos g\sqrt{m+1}t \end{bmatrix}. \tag{20}$$

To study the dynamical behavior of atom-field system beyond the RWA, we let

$$|\psi_I(t)\rangle = \sum_{n=0}^{\infty} [C_{n,1}(t)|n, 1\rangle + C_{n,2}(t)|n, 2\rangle] \tag{21}$$

be the solution of the Schrödinger equation

$$i \frac{\partial \psi_I(t)}{\partial t} = [H_I(t) + H_h(t)]\psi_I(t). \tag{22}$$

Then, the following equations are obtained immediately

$$i \frac{\partial}{\partial t} C_{0,1} = g C_{1,1} e^{i2\Omega t},$$

$$i \frac{\partial}{\partial t} \begin{bmatrix} C_{m,2} \\ C_{m+1,1} \end{bmatrix} = g\sqrt{m+1} \sigma_x \begin{bmatrix} C_{m,2} \\ C_{m+1,1} \end{bmatrix} + \begin{bmatrix} g\sqrt{m} e^{-i2\Omega t} C_{m-1,1} \\ g\sqrt{m+2} e^{i2\Omega t} C_{m+2,2} \end{bmatrix}. \tag{23}$$

By regarding the terms including  $e^{\pm i2\Omega t}$  as perturbation, the solutions for the coefficients

$$C_{m,2} = \sum_{k=0}^{\infty} C_{m,2}^{[k]}; \quad C_{m+1,1} = \sum_{k=0}^{\infty} C_{m+1,1}^{[k]} \tag{24}$$

is solved from the lowest-order equations

$$i \frac{\partial}{\partial t} C_{0,1}^{[0]} = 0,$$

$$i \frac{\partial}{\partial t} \begin{bmatrix} C_{m,2}^{[0]} \\ C_{m+1,1}^{[0]} \end{bmatrix} = g\sqrt{m+1} \sigma_x \begin{bmatrix} C_{m,2}^{[0]} \\ C_{m+1,1}^{[0]} \end{bmatrix}, \tag{25}$$

and the  $k$ -th order equations

$$i \frac{\partial}{\partial t} C_{0,1}^{[k]} = g C_{1,1}^{[k-1]} e^{i2\Omega t},$$

$$i \frac{\partial}{\partial t} \begin{bmatrix} C_{m,2}^{[k]} \\ C_{m+1,1}^{[k]} \end{bmatrix} = g\sqrt{m+1} \sigma_x \begin{bmatrix} C_{m,2}^{[k]} \\ C_{m+1,1}^{[k]} \end{bmatrix} + \begin{bmatrix} g\sqrt{m} e^{-i2\Omega t} C_{m-1,1}^{[k-1]} \\ g\sqrt{m+2} e^{i2\Omega t} C_{m+2,2}^{[k-1]} \end{bmatrix}. \tag{26}$$

From Eq. (20), the lowest-order equation is easily obtained. If the initial condition is

$$|\psi_I(0)\rangle = \sum_{n=0}^{\infty} C_{n,2}(0) |n, 2\rangle, \tag{27}$$

then

$$\begin{aligned} C_{m,2}^{[0]}(t) &= \cos g\sqrt{m+1} t C_{m,2}(0), \\ C_{m+1,1}^{[0]}(t) &= -i \sin g\sqrt{m+1} t C_{m,2}(0). \end{aligned} \tag{28}$$

From the lowest-order solution, it is not difficult to get the solution of the first-order equation,

$$\begin{bmatrix} C_{m,2}^{[1]} \\ C_{m+1,1}^{[1]} \end{bmatrix} = \int_0^t U_m(t-\tau) f(\tau) d\tau, \tag{29}$$

where  $f(\tau)$  is the inhomogeneous term for  $k = 1$  from Eq. (26). Explicitly,

$$\begin{aligned} C_{m,2}^{[1]} &= \frac{g\sqrt{m} C_{m-2,2}(0)}{4} \left\{ \frac{e^{ig\sqrt{m+1}t} - e^{i(\Omega+g\sqrt{m-1})t}}{i(\Omega + g\sqrt{m-1} - g\sqrt{m+1})} \right. \\ &\quad - \frac{e^{ig\sqrt{m+1}t} - e^{i(\Omega-g\sqrt{m-1})t}}{i(\Omega - g\sqrt{m-1} - g\sqrt{m+1})} \\ &\quad + \frac{e^{-ig\sqrt{m+1}t} - e^{i(\Omega+g\sqrt{m-1})t}}{i(\Omega + g\sqrt{m-1} + g\sqrt{m+1})} \\ &\quad \left. - \frac{e^{-ig\sqrt{m+1}t} - e^{i(\Omega-g\sqrt{m-1})t}}{i(\Omega - g\sqrt{m-1} + g\sqrt{m+1})} \right\} \\ &+ \frac{g\sqrt{m+2} C_{m+2,2}(0)}{4} \left\{ \frac{e^{ig\sqrt{m+1}t} - e^{i(-\Omega+g\sqrt{m+3})t}}{i(-\Omega + g\sqrt{m+3} - g\sqrt{m+1})} \right. \\ &\quad + \frac{e^{ig\sqrt{m+1}t} - e^{i(-\Omega-g\sqrt{m+3})t}}{i(-\Omega - g\sqrt{m+3} - g\sqrt{m+1})} \\ &\quad - \frac{e^{-ig\sqrt{m+1}t} - e^{i(-\Omega+g\sqrt{m+3})t}}{i(-\Omega + g\sqrt{m+3} + g\sqrt{m+1})} \\ &\quad \left. - \frac{e^{-ig\sqrt{m+1}t} - e^{i(-\Omega-g\sqrt{m+3})t}}{i(-\Omega - g\sqrt{m+3} + g\sqrt{m+1})} \right\}, \end{aligned}$$

$$\begin{aligned}
 C_{m+1,1}^{[1]} = & \frac{-g\sqrt{m} C_{m-2,2}(0)}{4} \left\{ \frac{e^{ig\sqrt{m+1}t} - e^{i(\Omega+g\sqrt{m-1})t}}{i(\Omega + g\sqrt{m-1} - g\sqrt{m+1})} \right. \\
 & - \frac{e^{ig\sqrt{m+1}t} - e^{i(\Omega-g\sqrt{m-1})t}}{i(\Omega - g\sqrt{m-1} - g\sqrt{m+1})} \\
 & - \frac{e^{-ig\sqrt{m+1}t} - e^{i(\Omega+g\sqrt{m-1})t}}{i(\Omega + g\sqrt{m-1} + g\sqrt{m+1})} \\
 & \left. + \frac{e^{-ig\sqrt{m+1}t} - e^{i(\Omega-g\sqrt{m-1})t}}{i(\Omega - g\sqrt{m-1} + g\sqrt{m+1})} \right\} \\
 & + \frac{-g\sqrt{m+2} C_{m+2,2}(0)}{4} \left\{ \frac{e^{ig\sqrt{m+1}t} - e^{i(-\Omega+g\sqrt{m+3})t}}{i(-\Omega + g\sqrt{m+3} - g\sqrt{m+1})} \right. \\
 & + \frac{e^{ig\sqrt{m+1}t} - e^{i(-\Omega-g\sqrt{m+3})t}}{i(-\Omega - g\sqrt{m+3} - g\sqrt{m+1})} \\
 & + \frac{e^{-ig\sqrt{m+1}t} - e^{i(-\Omega+g\sqrt{m+3})t}}{i(-\Omega + g\sqrt{m+3} + g\sqrt{m+1})} \\
 & \left. + \frac{e^{-ig\sqrt{m+1}t} - e^{i(-\Omega-g\sqrt{m+3})t}}{i(-\Omega - g\sqrt{m+3} + g\sqrt{m+1})} \right\}. \tag{30}
 \end{aligned}$$

In order to see the effect of the high frequency terms clearly, we simplify the initial condition as

$$|\psi(0)\rangle = \frac{1}{\sqrt{2}}(|0, 2\rangle + |1, 2\rangle). \tag{31}$$

Then, the wavefunction at time  $t$  with the first-order correction is

$$\begin{aligned}
 |\psi(t)\rangle = & \frac{1}{A}(a + ib)|0, 1\rangle + \frac{1}{\sqrt{2}}(\cos gt|0, 2\rangle + \cos \sqrt{2}gt|1, 2\rangle \\
 & - i \sin gt|1, 1\rangle - i \sin \sqrt{2}gt|2, 1\rangle), \tag{32}
 \end{aligned}$$

where

$$\begin{aligned}
 a = & \frac{g}{2\sqrt{2}} \left[ \frac{\sin(2\omega + \sqrt{2}g)t}{2\omega + \sqrt{2}g} + \frac{\sin(2\omega - \sqrt{2}g)t}{2\omega - \sqrt{2}g} \right], \\
 b = & \frac{g}{2\sqrt{2}} \left[ \frac{1 - \cos(2\omega + \sqrt{2}g)t}{2\omega + \sqrt{2}g} + \frac{1 - \cos(2\omega - \sqrt{2}g)t}{2\omega - \sqrt{2}g} \right]. \tag{33}
 \end{aligned}$$



Here  $A$  is the renormalization constant  $A = \sqrt{1 + a^2 + b^2}$ . The above result is the first-order correction to the original JCM wave function. Similarly, the higher order corrections can be obtained by using our proposed method. It is clear from Eq. (32) that only when  $g$  is small enough compared with  $\omega$ , can the first-order correction be ignored; that is to say, the condition for the validity of RWA is that  $\frac{g}{\omega}$  is small. While  $g$  is large, the first term of Eq. (32) cannot be neglected and it will have effect on the measured physical quantities. Next, we give the transition probability from an excited state to the ground state beyond RWA, i.e.,

$$P(|2\rangle \rightarrow |1\rangle) = \frac{1}{A^2} \left[ a^2 + b^2 + \frac{\sin^2(gt)}{2} + \frac{\sin^2(\sqrt{2}gt)}{2} \right]. \quad (34)$$

For the case with RWA, the transition probability is obtained only by setting  $a = b = 0$  in the above expression. Now we give the numerical calculations in Fig. 1. It is clear that when  $g$  is large, the high frequency term will affect the transition probability. The transition probability with RWA will exhibit periodic oscillations. Beyond RWA it will oscillate without original period. Comparing with the chaotic behaviour originating from the semiclassical version of JCM including high frequency terms<sup>[13,14]</sup>, we see this is only a quasiperiodic oscillation. It is an intermediate state for the transition from regular oscillation to chaotic state. The high frequency terms is the source of the chaotic behaviour. If the RWA is made, only periodic oscillation can be found.

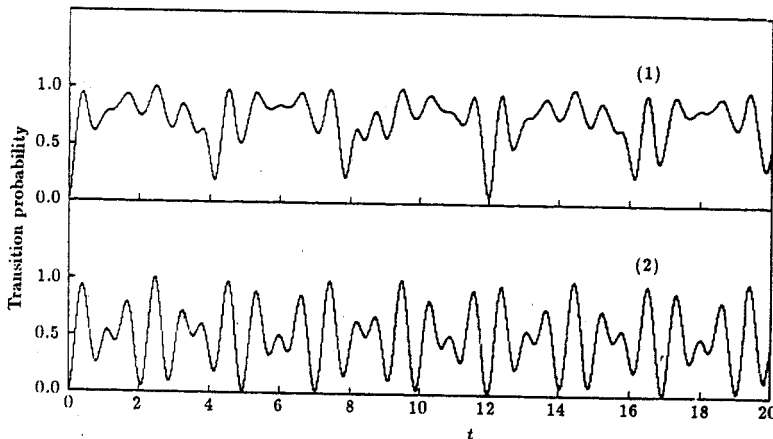


Fig. 1. Time evolution of the transition probability for the case without RWA (1) and with RWA (2), at  $\omega = 3$  and  $g = 3.15$ .

In conclusion, we have proposed a systematic method to study the high frequency effect in JCM. For the case with a two-level atom interacting with a single-mode radiation field, we find that the high frequency term is the source of chaotic behavior. If the RWA is made for the system, only periodic oscillation is observed.

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## ACKNOWLEDGEMENTS

One of us (W. X. G.) would like to acknowledge partial financial support by the Science Foundation of Jilin Education Committee for Yong Scientists.

## REFERENCES

- [1] E. T. Jaynes and F. W. Cummings, *Proc. IEEE*, **51**(1963), 89.
- [2] B. W. Shore and P. L. Knight, *J. Mod. Opt.*, **40**(1993), 1195.
- [3] G. S. Agarwal, *Phys. Rev.*, **A45**(1992), 1787,
- [4] T. Quang, P. L. Knight and V. Buzek, *Phys. Rev.*, **A44**(1991), 6092.
- [5] V. Buzek, H. Moya-Cessa, P. L. Knight and S. J. D. Phoenix, *Phys. Rev.*, **A45**(1992), 8190.
- [6] C. C. Gerry and J. H. Eberly, *Phys. Rev.*, **A42**(1990), 6805.
- [7] W. H. Louisell, *Quantum Statistical Properties of Radiations* (John. Wiley Sons, Inc. NY, 1973).
- [8] C. P. Sun, *J. Phys. A.*, **21**(1988), 1595.
- [9] C. P. Sun, *Phys. Rev.*, **D41**(1990), 1318.
- [10] C. P. Sun, *Phys. Scripta.*, **48**(1993), 393.
- [11] I. I. Rabi, *Phys. Rev.*, **51**(1973), 652.
- [12] R. London, *The Quantum Theory of Light* (Oxford University Press, 1973).
- [13] J. Seke, *J. Opt. Soc. Am.*, **B2**(1985), 968.
- [14] M. D. Crisp, *Phys. Rev.*, **A43**(1991), 2430.