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APPROXIMATION THEORY OF OSCILLATING FACTOR SUPPRESSING AMPLITUDE IN QUANTUM PROCESS AND ITS APPLICATIONS *

SUN CHANG-PU (孙昌璞) and LIU XIA-JI (刘夏姬)
Institute of Theoretical Physics, Northeast Normal University, Changchun 130024, China

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The oscillating factor suppressing amplitude approximation is proposed in this paper as a basic method to study the evolution of a quantum system. By making use of this method, not only the quantum adiabatic approximation is described again, but also both the higher-order corrections for the rotation wave approximation and the influences of other levels on two-level approximation can be analytically investigated with the explicit discussion for the quantitative conditions under which the two approximations hold.

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I. INTRODUCTION

It is well known that the practical problems are frequently not solved in an exact form for physical theories. Besides using approximate model to simplify the problems, various approximation methods are needed to deal with these problems. Especially in quantum physics, the approximation methods, such as perturbation theory, the variation method, the WKB approximation, the adiabatic aproximation, the rotation wave approximation (RWA) and the two-level (or the few level) approximation (TLA) and so on, are the important ways linking the physical theories and the practical problems.

In recent years, the quantum adiabatic approximation (QAA) method has attracted physicist's notice again and evoked new developments because of the discovery of the Berry's geometric phase factor.[1-4] For researches in cavity quantum electrodynamics (CQED) such as the atomic cooling and trapping in cavities[5,6], the RWA and TLA are still used widely as basic tools.[7,8] However, the QAA, RWA and TLA cannot hold strictly in many practical problems and the effects of the non-QAA, the non-TLA and so on are often substantial to have influence on physical phenomenon, but there are few systematic analyses for them (especially for the TLA) except for the QAA. Therefore, it is necessary to build a systematic method to deal with them uniquely.

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Based on the above considerations, the oscillating factor suppressing amplitude approximation (OFSAA) is proposed in this paper as a basic method to study the evolution of a quantum system. Notice that the perturbation in this perturbation scheme is not caused by the small variation of the Hamiltonian, but is associated with the repaid-oscillating part of the Hamiltonian that suppresses the contribution to the wave function. With the application of this method, the influence of the third level on the TLA is studied in detail and the analytic condition is given, under which the TLA holds. This method is also used to discuss the effects of the non-RWA on the Jaynes-Cummigs (JC) model. In the appendix A the high-order QAA method is proved to be a special example of this approximation method.

II. OSCILLATING FACTOR SUPPRESSING AMPLITUDE APPROXIMATION

Let the Hamiltonian or its effective form describing the evolution of a quantum system be

$$\hat{H} = \hat{H}_0(t) + \hat{H}'(t), \tag{1}$$

which can be separated into the slowly-changing part $\hat{H}_0(t)$ and the fast-changing part $\hat{H}'(t)$

$$\hat{H}' = \hat{V}(t)\mathrm{e}^{-i\Gamma(t)} + \mathrm{H.c.}, \tag{2}$$

where $\hat{V}(t)$ changes slowly with time, $\Gamma(t)$ is the phase of oscillation and $\Omega(t) = \dot{\Gamma}(t)$. The much large value of $|\Omega(t)| \equiv |\dot{\Gamma}(t)|$ means a fast oscillation of $\hat{H}'$.

The Schrödinger equation

$$\mathrm{i}\hbar\frac{\partial}{\partial t}|\Psi(t)\rangle = \hat{H}(t)|\Psi(t)\rangle, \tag{3}$$

has its integral form

$$\mathrm{i}\hbar(|\Psi(\tau)\rangle - |\Psi(0)\rangle) = \int_0^\tau \hat{H}_0(t)|\Psi(t)\rangle\mathrm{d}t + \Big(\int_0^\tau \mathrm{e}^{-\mathrm{i}\Gamma(t)}\hat{V}(t)|\Psi(t)\rangle\mathrm{d}t + \mathrm{H.c.}\Big). \tag{4}$$

It is shown that $\mathrm{e}^{\pm\mathrm{i}\Gamma(t)}$ in the second integration in Eq. (4) is a fast-changing oscillation factor, which can cancel the integral of a slowly-changing function. By integrating it by part successively, the second term on the right-hand side of Eq. (4) can be expanded into a converging power series

$$\int_0^\tau \mathrm{e}^{-\mathrm{i}\Gamma(t)}\hat{V}(t)|\Psi(t)\rangle\mathrm{d}t + \mathrm{H.c.} = \frac{\mathrm{i}\,\mathrm{e}^{-\mathrm{i}\Gamma(t)}}{\Omega(t)}\sum_{k=0}^{\infty}\Big(\frac{\hat{O}}{\Omega(t)}\Big)^k\hat{V}(t)|\Psi(t)\rangle\Big|_0^\tau + \mathrm{H.c.}, \tag{5}$$

where

$$\hat{O} = -\mathrm{i}\Big(\frac{\dot{\Omega}(t)}{\Omega(t)} + \frac{\mathrm{d}}{\mathrm{d}t}\Big) \tag{6}$$

is the "covariant" differential for time acting on $|\Psi(t)\rangle$. This fact is proved in appendix B in detail. For the case $0 \leq t \leq \tau$ with very large $\Omega(t)$, this term gives a very small contribution to the wave function and the expression (5) is a rapidly converging power series.

According to the above analysis, a Schrödinger equation in differential form is obtained as

$$\mathrm{i}\hbar\frac{\mathrm{d}}{\mathrm{d}\tau}|\Psi(\tau)\rangle = \hat{H}_0(\tau)|\Psi(\tau)\rangle + \frac{\mathrm{d}}{\mathrm{d}\tau}\Big[\frac{\mathrm{i}\,\mathrm{e}^{-\mathrm{i}\Gamma(\tau)}}{\Omega(\tau)}\sum_{k=0}^{\infty}\Big(\frac{\hat{O}}{\Omega(\tau)}\Big)^k \varepsilon^{k+1}\hat{V}(\tau)|\Psi(\tau)\rangle + \mathrm{H.c.}\Big], \quad (7)$$

where ε is a perturbation parameter accompanying the small quantity $(\Omega(\tau))^{-1}$. Similarly to that in the usual peturbation theory, we assume that $|\Psi(\tau)\rangle = \sum_{k=0}^{\infty}\varepsilon^k|\Psi^{[k]}(\tau)\rangle$ (ε will be taken to be 1 at the end of the calculation) and then substitute it into Eq. (7). Comparing the coefficients of the terms with the same power of ε on the twosides of Eq. (7), one can obtain the approximate equations of every order

$$\begin{aligned}
&\mathrm{i}\hbar\frac{\mathrm{d}}{\mathrm{d}\tau}|\Psi^{[0]}(\tau)\rangle = \hat{H}_0(\tau)|\Psi^{[0]}(\tau)\rangle,\\
&\mathrm{i}\hbar\frac{\mathrm{d}}{\mathrm{d}\tau}|\Psi^{[1]}(\tau)\rangle = \hat{H}_0(\tau)|\Psi^{[1]}(\tau)\rangle + \frac{\mathrm{d}}{\mathrm{d}\tau}\Big[\mathrm{i}\,\frac{\mathrm{e}^{-\mathrm{i}\Gamma(\tau)}}{\Omega(\tau)}\hat{V}(\tau)|\Psi^{[0]}(\tau)\rangle + \mathrm{H.c.}\Big], \qquad (8)\\
&\qquad\qquad\cdots\cdots\cdots\\
&\mathrm{i}\hbar\frac{\mathrm{d}}{\mathrm{d}\tau}|\Psi^{[n]}(\tau)\rangle = \hat{H}_0(\tau)|\Psi^{[n]}(\tau)\rangle + \frac{\mathrm{d}}{\mathrm{d}\tau}\Big[\frac{\mathrm{i}\,\mathrm{e}^{-\mathrm{i}\Gamma(\tau)}}{\Omega(\tau)}\sum_{k=0}^{n-1}\Big(\frac{\hat{O}}{\Omega(\tau)}\Big)^k\hat{V}(\tau)|\Psi^{[n-k-1]}(\tau)\rangle + \mathrm{H.c.}\Big].
\end{aligned}$$

From the above equations, it is observed that the n-th order equation only includes $|\Psi^{[0]}(\tau)\rangle$, $|\Psi^{[1]}(\tau)\rangle, \cdots, |\Psi^{[n-1]}(\tau)\rangle$ and so it can be solved successively order by order. In the following discussions, we take $\hbar = 1$.

III. TWO-LEVEL APPROXIMATIONS AND THE INFLUENCE FROM OTHER LEVELS

In this section the TLA is studied as an application of the OFSAA. As the spectral structure of atoms is very complicated in practical situation, only a few levels with the level spacing approaching the frequency of the external field are used to study the interaction between the atom and light field. This is because the transitions near resonance are much stronger than that of off-resonance under the action of external field. These levels form an ideal model of few-level-atom. However, the influences of that besides these levels on the atomic dynamics must be considered in concrete problems such as atomic trapping and then the condition for the TLA can be analyzed quantitatively. To discuss the above-mentioned problems in a simple way, let us examine a V–type atom with three levels, as illustrated in

Fig. 1.

In this figure, the level-spacings between $|e_1\rangle, |e_1\rangle$ and $|g\rangle$ are respectively ω_1 and ω_2.Under dipole approximation and the RWA, the Hamiltonian of the atom-field system is

$$\begin{aligned}\hat{H} =&\omega a^+a + \omega_1|e_1\rangle\langle e_1| + \omega_2|e_2\rangle\langle e_2| + g[a^+(|g\rangle\langle e_1| + \mu|g\rangle\langle e_2|)\\ &+ a(|e_1\rangle\langle g| + \mu|e_2\rangle\langle g|)], \quad \mu \neq 1,\end{aligned} \tag{9}$$

where the requirement that $\mu \neq 1$ means different strengths of the coupling of light to $|e_1\rangle$ and $|e_2\rangle$, respectively.

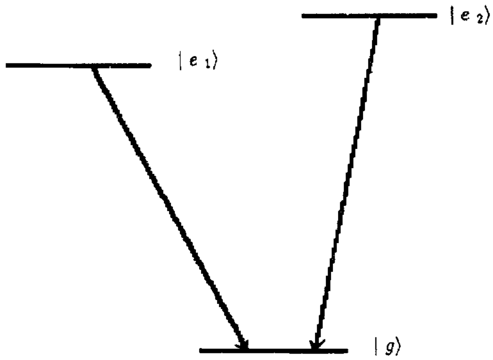

Fig. 1. The V–type three-level atom.

In the interaction picture, the interacting Hamiltonian can be divided into two parts, the low-frequency part

$$\hat{H}_\mathrm{l} = g(\mathrm{e}^{-\mathrm{i}\,\delta t}a^+|g\rangle\langle e_1| + \mathrm{e}^{\mathrm{i}\,\delta t}a|e_1\rangle\langle g|) \tag{10}$$

with the frequency

$$\delta = \omega_1 - \omega,$$

and the high-frequency part

$$\hat{H}_\mathrm{h} = g\mu(\mathrm{e}^{-\mathrm{i}\,\Omega t}a^+|g\rangle\langle e_2| + \mathrm{e}^{\mathrm{i}\,\Omega t}a|e_2\rangle\langle g|) \tag{11}$$

with the frequency

$$\Omega = \omega_2 - \omega.$$

Notice that the vectors $\{|g, n+1\rangle, |e_1, n\rangle, |e_2, n\rangle\}$ span an invariant subspace for the evolution of the system. In this space, the matrix representation of the Hamiltonian is

$$\hat{H}_\mathrm{l}(n) = g\sqrt{n+1}\begin{pmatrix} e & \mathrm{e}^{-\mathrm{i}\,\delta t} & 0 \\ \mathrm{e}^{\mathrm{i}\,\delta t} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\hat{H}_{\mathrm{h}}(n) = \mu g\sqrt{n+1}\begin{pmatrix} 0 & 0 & \mathrm{e}^{-\mathrm{i}\,\Omega t} \\ 0 & 0 & 0 \\ \mathrm{e}^{\mathrm{i}\,\Omega t} & 0 & 0 \end{pmatrix}. \tag{12}$$

It can be found from the above expressions that the evolution matrix governed by $\hat{H}_{\mathrm{l}}$ is

$$U^{[0]}(t) = \sum_{n=1}^{\infty} \oplus U_n(t) = \begin{pmatrix} 1 & 0 & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & U_0^{[0]} & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & 0 & \cdot & \cdots & \cdots & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdot & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdot & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & \cdots & \cdots & U_N^{[0]} \end{pmatrix}, \tag{13}$$

where

$$U_n^{[0]}(t) = \begin{pmatrix} A_n(t) & B_n(t) & 0 \\ -B_n^*(t) & A_n^*(t) & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$A_n(t) = \mathrm{e}^{-\mathrm{i}\,\delta t/2}\Big[\cos(\Omega_n t) + \mathrm{i}\,\frac{\delta}{2\Omega_n}\sin(\Omega_n t)\Big],$$

$$B_n(t) = -\frac{\mathrm{i}\,g\sqrt{n+1}}{\Omega_n}\mathrm{e}^{-\mathrm{i}\,\delta t/2}\sin(\Omega_n t), \tag{14}$$

$$\Omega_n = \sqrt{\delta^2/4 + g^2(n+1)}.$$

Then, the zeroth order approximate wave function is obtained as

$$|\,\Psi^{[0]}(t)\rangle = U^{[0]}(t)|\,\Psi(0)\rangle.$$

It is known from the discussion in the last section that the first-order correction $|\,\Psi^{[1]}(t)\rangle$ obeys

$$\mathrm{i}\,\frac{\mathrm{d}}{\mathrm{d}\,t}|\,\Psi^{[1]}(t)\rangle = \hat{H}_{\mathrm{l}}\,|\,\Psi^{[1]}(t)\rangle + \frac{\mathrm{d}}{\mathrm{d}\,t}\Big[\frac{\mathrm{i}\,\mathrm{e}^{-\mathrm{i}\,\Omega t}}{\Omega}\mu g a^{+}|g\rangle\langle e_2||\,\Psi^{[0]}(t)\rangle + \mathrm{H.c.}\Big], \tag{15}$$

with the initial condition

$$|\,\Psi_n(0)\rangle = \begin{pmatrix} C_1 \\ C_2 \\ 0 \end{pmatrix}, \tag{16}$$

where C_1 and C_2 satisfy $|C_1|^2 + |C_2|^2 = 1$. Then we have zeroth order solution

$$|\,\Psi_n^{[0]}(t)\rangle = \begin{pmatrix} C_1 A_n(t) + C_2 B_n \\ -C_1 B_n^*(t) + C_2 A_n^*(t) \\ 0 \end{pmatrix}, \tag{17}$$

and the first-order correction

$$|\Psi_n^{[1]}(t)\rangle = \begin{pmatrix} 0 \\ 0 \\ -\mu g\sqrt{n+1}\dfrac{e^{i\Omega t}}{\Omega}(C_1 A_n(t) + C_2 B_n(t)) \end{pmatrix}. \tag{18}$$

From the above results it can be observed that for

$$\left|\frac{g\sqrt{n+1}}{\Omega}\right| \ll 1, \tag{19}$$

the first-order correction can be neglected and the TLA holds. Then, Eq. (19) can be thought of as the dynamic condition for the TLA. The condition (19) means a weak light field, the third level with much higher energy than that of the resonant photon since the strength of the light field is proportional to the photon number n; and small g corresponds to the weak coupling. Physically, the strong light field and strong interaction may cause multi-photon transition, hence the probability of off-resonant transition cannot be neglected in this case. Therefore, the condition (19) is reasonable.

IV. ROTATING WAVE APPROXIMATION AND ITS HIGHER-ORDER GENERALIZATION

The RWA is an aprroximation method used frequently in quantum optics and atomic physics. The Dick Hamiltonian in this approximation ignores the process of creating-annihilating virtual photon and effectively describes the collapse and revival of atomic population, the resonant fluoescence and super-fluoescence. In practical problems, however, one should quantitatively understand the deviation from the RWA and the condition under which the RWA holds. In the following discussion the general theory of the OFSAA will be applied to investigate this problem. For simplicity, the case of a two-level atom interacting with a single-mode light field is considered in this section. In the dipole approximation the Hamiltonian of the field-atom system

$$\hat{H} = \omega a^+ a - \frac{1}{2}\omega_1\sigma_3 + g\sigma_1(a + a^+) \tag{20}$$

can be divided into the part of RWA–the Jaynes-Cummings Hamiltonian

$$\hat{H}_0 = \omega a^+ a - \frac{1}{2}\omega_1\sigma_3 + g(\sigma_+ a + \sigma_- a^+) \tag{21}$$

and the part of non-RWA

$$\hat{H}' = g(\sigma_+ a^+ + \sigma_- a), \tag{22}$$

where the quasi-spin operators σ_+, σ_- and σ_3 are defined in terms of the ground state $|g\rangle$ and the excited state $|e\rangle$

$$\sigma_3 = |g\rangle\langle g| - |e\rangle\langle e|, \quad \sigma_+ = |e\rangle\langle g|, \quad \sigma_- = |g\rangle\langle e|, \tag{23}$$

where a^+ and a denote the creation and annihilation operator of the cavity mode. In the interacting picture the interacting part $g(\sigma_+ a + \sigma_- a^+)$ can be written as a combination of the high-and low-frequency parts,

$$\hat{H}_{\rm I} = (g\sigma_+ a {\rm e}^{{\rm i}\,\delta t} + {\rm H.c.}) + (g\sigma_+ a^+ {\rm e}^{{\rm i}\,\Omega t} + {\rm H.c.}) \equiv \tilde{H}_0(t) + [g\sigma_+ a^+ {\rm e}^{{\rm i}\,\Omega t} + {\rm H.c.}], \tag{24}$$

where $\delta = \omega_1 - \omega$, $\Omega = \omega_1 + \omega$. In this case, the differential equation (7) is expanded in terms of $(\omega_0 + \omega)^{-1}$ as

$${\rm i}\frac{\rm d}{{\rm d}t}|\Psi(t)\rangle = \tilde{H}_0(t)|\Psi(t)\rangle + \frac{\rm d}{{\rm d}t}\Big[\frac{-{\rm i}\,{\rm e}^{{\rm i}\,\Omega t}}{\Omega}\sum_{k=0}^{\infty}\Big(\frac{-{\rm i}}{\Omega}\frac{\rm d}{{\rm d}t}\Big)^k g\sigma_+ a^+|\Psi(\tau)\rangle\Big|_0^t + {\rm H.c.}\Big] \tag{25}$$

and the corresponding perturbation approximate equations from Eq. (8) are specified as

$$\begin{aligned}
&{\rm i}\frac{\rm d}{{\rm d}t}|\Psi^{[0]}(t)\rangle = \tilde{H}_0(t)|\Psi^{[0]}(t)\rangle,\\
&{\rm i}\frac{\rm d}{{\rm d}t}|\Psi^{[1]}(t)\rangle = \tilde{H}_0(t)|\Psi^{[1]}(t)\rangle + \frac{\rm d}{{\rm d}t}\Big[\frac{(-{\rm i}){\rm e}^{{\rm i}\,\Omega t}}{\Omega}g\sigma_+ a^+|\Psi^{[0]}(t)\rangle + {\rm H.c.}\Big],\\
&\qquad\qquad\cdots\cdots\cdots\\
&{\rm i}\frac{\rm d}{{\rm d}t}|\Psi^{[n]}(t)\rangle = \hat{H}_n(t)|\Psi^{[n]}(t)\rangle\\
&\qquad + \frac{\rm d}{{\rm d}t}\Big[\frac{-{\rm i}\,{\rm e}^{{\rm i}\,\Omega t}}{\Omega}\sum_{k=0}^{n-1}\Big(\frac{-{\rm i}}{\Omega}\frac{\rm d}{{\rm d}t}\Big)^k g\sigma_+ a^+|\Psi^{[n-k-1]}(t)\rangle + {\rm H.c.}\Big].
\end{aligned}\tag{26}$$

The zeroth order solutions are obtained as

$$\begin{aligned}
&C^{[0]}_{g,0} = C^{[0]}_{g,0}(0),\\
&C^{[0]}_{g,n+1} = A_n(t)C^{[0]}_{g,n+1}(0) + B_n(t)C^{[0]}_{e,n}(0),\\
&C^{[0]}_{g,n+1} = -B^*_n(t)C^{[0]}_{g,n+1}(0) + A^*_n(t)C^{[0]}_{e,n}(0),
\end{aligned}\tag{27}$$

where $A_n(t)$, $B_n(t)$ and Ω_n are defined same as in section III. Notice only that $\Omega = \omega_1 + \omega$ rather than $\Omega = \omega_1 - \omega$; $C^{[0]}_{g,n+1}(0)$ and $C^{[0]}_{e,n}(0)$ are determined by the initial conditions.

Let us define

$$a_n = \Omega_n + \delta/2, \quad b_n = -\Omega_n + \delta/2,$$

$$\alpha = 1/2 - \mathrm{i}\,\delta/4\Omega_n, \quad \beta_n = \mathrm{i}\,(g\sqrt{n+1})/2\Omega_n,$$

$$D_{n_1} = \sqrt{n}[\beta^*_{n-2}C^{[0]}_{e,n-2}(0) + \alpha^*_{n-2}C^{[0]}_{g,n-1}(0)],$$

$$D_{n_2} = \sqrt{n}[\beta_{n-2}C^{[0]}_{e,n-2}(0) + \alpha_{n-2}C^{[0]}_{g,n-1}(0)],$$

$$D_{n_3} = \sqrt{n+2}[\alpha_{n+2}C^{[0]}_{e,n+2}(0) + \beta^*_{n+2}C^{[0]}_{g,n+3}(0)],$$

$$D_{n_4} = \sqrt{n+2}[\alpha_{n+2}C^{[0]}_{e,n+2}(0) + \beta_n C^{[0]}_{g,n+3}(0)].$$

Then, the first-order approximate solutions are obtained as

$$C^{[1]}_{g,0}(t) = g\Big[\frac{\alpha_1 C^{[0]}_{e,1}(0) + \beta^*_1 C^{[0]}_{g,2}(0)}{\Omega - a_1}(\mathrm{e}^{\,\mathrm{i}\,(a_1-\Omega)t} - 1)\Big] + \frac{\alpha^*_1 C^{[0]}_{e,1}(0) + \beta_1 C^{[0]}_{g,2}(0)}{\Omega - b_1}(\mathrm{e}^{\,\mathrm{i}\,(b_1-\Omega)t} - 1)\Big],$$

$$C^{[1]}_{e,0}(t) = \frac{\beta^*_0 D_{0_3} g}{\Omega + a_0 - a_2}(\mathrm{e}^{\,\mathrm{i}\,(a_1-\Omega)t} - \mathrm{e}^{\,\mathrm{i}\,a_0 t}) + \frac{\beta_0 D_{0_3} g}{\Omega + b_0 - a_2}(\mathrm{e}^{\,\mathrm{i}\,(a_2-\Omega)t} - \mathrm{e}^{\,\mathrm{i}\,b_0 t})$$

$$+ \frac{\beta^*_0 D_{0_4} g}{\Omega + a_0 - b_2}(\mathrm{e}^{\,\mathrm{i}\,(b_2-\Omega)t} - \mathrm{e}^{\,\mathrm{i}\,a_0 t}) + \frac{\beta_0 D_{0_3} g}{\Omega + b_0 - b_2}(\mathrm{e}^{\,\mathrm{i}\,(b_2-\Omega)t} - \mathrm{e}^{\,\mathrm{i}\,b_0 t}), \text{ for } n = 0;$$

$$C^{[1]}_{g,1}(t) = \frac{\alpha^*_0 D_{0_3} g}{\Omega + b_0 - a_2}(\mathrm{e}^{\,\mathrm{i}\,(a_2-\Omega)t} - \mathrm{e}^{\,\mathrm{i}\,a_0 t}) + \frac{\alpha_0 D_{0_3} g}{\Omega + a_0 - a_2}(\mathrm{e}^{\,\mathrm{i}\,(a_2-\Omega)t} - e^{\mathrm{i}a_0 t})$$

$$+ \frac{\alpha^*_0 D_{0_4} g}{\Omega + b_0 - b_2}(\mathrm{e}^{\,\mathrm{i}\,(b_2-\Omega)t} - \mathrm{e}^{\,\mathrm{i}\,b_0 t}) + \frac{\alpha_0 D_{0_4} g}{\Omega + a_0 - b_2}(\mathrm{e}^{\,\mathrm{i}\,(b_2-\Omega)t} - \mathrm{e}^{\,\mathrm{i}\,a_0 t}),$$

$$C^{[1]}_{e,1}(t) = \frac{\alpha^*_0 D_1 g}{\Omega - a_1}(\mathrm{e}^{\,\mathrm{i}\,a_1 t} - \mathrm{e}^{\,\mathrm{i}\,\Omega t}) + \frac{\alpha^*_0 D_1 g}{\Omega - b_1}(\mathrm{e}^{\,\mathrm{i}\,b_m t} - \mathrm{e}^{\,\mathrm{i}\,\Omega t})$$

$$+ \frac{\beta^*_1 D_{1_3} g}{\Omega + a_1 - a_3}(\mathrm{e}^{\,\mathrm{i}\,(a_3-\Omega)t} - \mathrm{e}^{\,\mathrm{i}\,a_1 t}) + \frac{\beta_1 D_{1_3} g}{\Omega + b_1 - a_3}(\mathrm{e}^{\,\mathrm{i}\,(a_3-\Omega)t} - \mathrm{e}^{\,\mathrm{i}\,b_1 t})$$

$$+ \frac{\beta^*_1 D_{1_4} g}{\Omega + a_1 - b_3}(\mathrm{e}^{\,\mathrm{i}\,(a_3-\Omega)t} - \mathrm{e}^{\,\mathrm{i}\,b_1 t}) + \frac{\beta_1 D_{1_4} g}{\Omega + b_1 - b_3}(\mathrm{e}^{\,\mathrm{i}\,(b_3-\Omega)t} - \mathrm{e}^{\,\mathrm{i}\,b_1 t}),$$

$$C^{[1]}_{g,2}(t) = \frac{\beta^*_1 D_1 g}{\Omega + b_1}(\mathrm{e}^{\,-\mathrm{i}\,b_1 t} - \mathrm{e}^{\,\mathrm{i}\,\Omega t}) + \frac{\beta_1 D_1 g}{\Omega + a_1}(\mathrm{e}^{\,\mathrm{i}\,a_1 t} - \mathrm{e}^{\,\mathrm{i}\,\Omega t})$$

$$+ \frac{\alpha^*_1 D_{1_3} g}{\Omega - b_1 - a_3}(\mathrm{e}^{\,\mathrm{i}\,(a_3-\Omega)t} - \mathrm{e}^{\,-\mathrm{i}\,b_1 t}) + \frac{\alpha_1 D_{1_3} g}{\Omega - a_1 - a_3}(\mathrm{e}^{\,\mathrm{i}\,(a_3-\Omega)t} - \mathrm{e}^{\,-\mathrm{i}\,a_1 t})$$

$$+ \frac{\alpha^*_1 D_{1_4} g}{\Omega - b_1 - b_3}(\mathrm{e}^{\,\mathrm{i}\,(b_3-\Omega)t} - \mathrm{e}^{\,\mathrm{i}\,b_1 t}) + \frac{\alpha_1 D_{1_4} g}{\Omega - a_1 - b_3}(\mathrm{e}^{\,\mathrm{i}\,(b_3-\Omega)t} - \mathrm{e}^{\,-\mathrm{i}\,a_1 t}),$$

$$C^{[1]}_{e,n}(t) = \frac{\alpha_n D_{n_1} g}{\Omega - b_{n-2} - a_n}(\mathrm{e}^{\,a_n t} - \mathrm{e}^{\,\mathrm{i}\,(\Omega - b_{n-2})t}) + \frac{\alpha^*_n D_{n_1} g}{\Omega - b_{n-2} - b_n}(\mathrm{e}^{\,b_n t} - \mathrm{e}^{\,\mathrm{i}\,(\Omega - b_{n-2})t})$$

$$+\frac{\alpha_n^* D_{n_2} g}{\Omega - a_{n-2} - a_n}(\mathrm{e}^{a_n t} - \mathrm{e}^{\mathrm{i}(\Omega - a_{n-2})t}) + \frac{\alpha_n^* D_{n_2} g}{\Omega + a_{n-2} - b_n}(\mathrm{e}^{b_n t} - \mathrm{e}^{\mathrm{i}(\Omega - a_{n-2})t})$$

$$+\frac{\beta_n^* D_{n_3} g}{\Omega - a_{n+2} + a_n}(\mathrm{e}^{a_n t} - \mathrm{e}^{\mathrm{i}(-\Omega + a_{n+2})t}) + \frac{\beta_n D_{n_3} g}{\Omega - a_{n+2} + b_n}(\mathrm{e}^{b_n t} - \mathrm{e}^{\mathrm{i}(-\Omega + a_{n+2})t})$$

$$+\frac{\beta_n^* D_{n_4} g}{\Omega - b_{n+2} + a_n}(\mathrm{e}^{a_n t} - \mathrm{e}^{\mathrm{i}(-\Omega + b_{n+2})t}) + \frac{\beta_n D_{n_4} g}{\Omega - b_{n+2} + b_n}(\mathrm{e}^{b_n t} - \mathrm{e}^{\mathrm{i}(-\Omega + b_{n+2})t}),$$

$$C_{g,n+1}^{[1]}(t) = \frac{\beta_n D_{n_1} g}{\Omega - b_{n-2} + b_n}(\mathrm{e}^{-b_n t} - \mathrm{e}^{\mathrm{i}(\Omega - b_{n-2})t}) + \frac{\beta_n D_{n_1} g}{\Omega - b_{n-2} + a_n}(\mathrm{e}^{-a_n t} - \mathrm{e}^{\mathrm{i}(\Omega - b_{n-2})t})$$

$$+\frac{\beta_n^* D_{n_2} g}{\Omega - a_{n-2} + b_n}(\mathrm{e}^{-b_n t} - \mathrm{e}^{\mathrm{i}(\Omega - a_{n-2})t}) + \frac{\beta_n D_{n_2} g}{\Omega + a_{n-2} + b_n}(\mathrm{e}^{-a_n t} - \mathrm{e}^{\mathrm{i}(\Omega - a_{n-2})t})$$

$$+\frac{\alpha_n^* D_{n_3} g}{\Omega + a_{n+2} - b_n}(\mathrm{e}^{-b_n t} - \mathrm{e}^{\mathrm{i}(\Omega - a_{n+2})t}) + \frac{\alpha_n D_{n_3} g}{\Omega - a_{n+2} - a_n}(\mathrm{e}^{-a_n t} - \mathrm{e}^{\mathrm{i}(-\Omega + a_{n+2})t})$$

$$+\frac{\alpha_n^* D_{n_4} g}{\Omega - b_{n-2} - b_n}(\mathrm{e}^{-b_n t} - \mathrm{e}^{\mathrm{i}(-\Omega + b_{n+2})t}) + \frac{\alpha_n D_{n_4} g}{\Omega - b_{n+2} - a_n}(\mathrm{e}^{-a_n t} - \mathrm{e}^{\mathrm{i}(-\Omega + b_{n+2})t})$$

$$+\frac{\alpha_n D_{m_1} g}{\Omega - b_{n-2} - a_n}(\mathrm{e}^{a_n t} - \mathrm{e}^{\mathrm{i}(\Omega - b_{n-2})t}), \qquad \text{for } n = 1. \tag{28}$$

From the above equations the conditions of the RWA are known as

$$(g\sqrt{n+1})/\Omega \ll 1, \quad \delta/\Omega \ll 1. \tag{29}$$

These conditions show that the RWA holds when variance rate of the high-frequency terms is much larger than that of the low-frequency one in the case of weak coupling and weak light field. In the view of energy conservation, the essence of RWA is an energy-conservation approximation.[7,8] The above results are the quantitative conditions of the energy-conservation approximation for a specific model.[9]

APPENDIX A

The QAA method is mainly used to describe the dynamic process governed by a Hamiltonian that changes slowly, but with a finite variance. It is proved in the following that the essence of the QAA is still that the rapidly-oscillating factor suppresses the contribution of a slow-changing function to an integral. Let $\hat{H}(t)$ be an Hermitian Hamiltonian with a discrete system of instantaneous eigenstate $\{|n(t)\rangle|\ 0 \le n \le N\}$ and corresponding eigenvalues $E_n(t)$. With the "rotating axis" representation, we let

$$|\Psi(t)\rangle = \sum_n C_n(t)\mathrm{e}^{(\mathrm{i}\hbar)^{-1}\int_0^t E_n(\tau)\mathrm{d}\tau}|n(t)\rangle \tag{A.1}$$

be a solution of the Schrödinger equation. The coefficients $C_n(t)$ satisfy an effective Schrödinger equation

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\tilde{C}(t) = (\hat{H}(t) + \tilde{W}(t))\tilde{C}(t), \tag{A.2}$$

where

$$\tilde{C}(t) = \begin{pmatrix} \tilde{C}_1(t) \\ \tilde{C}_2(t) \\ \vdots \\ \tilde{C}_N(t) \\ \vdots \end{pmatrix},$$

$$\tilde{H}_{n,m}(t) = \begin{cases} -\mathrm{i} < n(t)|\frac{\mathrm{d}}{\mathrm{d}\,t}|n(t)\rangle & \text{if } m = n, \\ 0 & \text{if } m \neq n. \end{cases}$$

$$\tilde{W}_{n,m} = \begin{cases} -\mathrm{i}\ \exp\{-\mathrm{i}\,\Gamma_{n,m}(t)\} < m(t)\left|\frac{d}{dt}\right|n(t)\rangle & \text{if } m \neq n, \\ 0 & \text{if } m = n, \end{cases}$$

$$\Gamma_{n,m} = \int_0^t (E_n(\tau) - E_m(\tau))\mathrm{d}\,\tau.$$

It is clear that when

$$F_{n,m}(t) = \left|\frac{\langle m(t)|\frac{\mathrm{d}}{\mathrm{d}\,t}|n(t)\rangle}{E_m(t) - E_n(t)}\right| \quad (m \neq n) \tag{A.3}$$

is very small, $\tilde{W}_{n,m}$ is a high-frequency oscillation term. When the adiabatic condition $F_{n,m}(t) \ll 1$ is satisfied, the contributions of the term $W(t)$ can be neglected. Then, we obtain the adiabatic approximate solution

$$C_n^{[0]}(t) = \exp\{\mathrm{i}\,\nu_n(t)\}C_n(0),$$

where

$$\nu_n(t) = \mathrm{i}\int_0^t \langle n(\tau)|\frac{\mathrm{d}}{\mathrm{d}\,\tau}|n(\tau)\rangle\mathrm{d}\,\tau \tag{A.4}$$

is the Berry's geometric phase factor. Based on the approximate solution (A.4) the approximate solutions of each order can be obtained by making use of the scheme of the OFSAA. This method and its applications have been systematically studied by one (CPS) of the authors[2-4] and so the details are not described here. The purpose of this appendix is to show that the QAA is only a special example of the OFSAA in the rotating axis representation.

APPENDIX B

Let us prove formula (5)

$$\int_0^\tau \mathrm{e}^{-\mathrm{i}\,\Gamma(t)}\hat{V}(t)|\Psi(t)\rangle\mathrm{d}\,t + \text{H.c.} = \frac{\mathrm{i}\,\mathrm{e}^{-\mathrm{i}\,\Gamma(t)}}{\Omega(t)}\sum_{k=0}^{\infty}\left(\frac{\hat{O}}{\Omega(t)}\right)^k\hat{V}(t)|\Psi(t)\rangle\Big|_0^\tau + \text{H.c.}, \tag{5}$$

where

$$\hat{O} = -\mathrm{i}\left(\frac{\dot{\Omega}(t)}{\Omega(t)} + \frac{\mathrm{d}}{\mathrm{d}\,t}\right). \tag{6}$$

Defining

$$\hat{A}(t) = \hat{V}(t)|\Psi(t)\rangle,$$

we obtain the integral by part

$$\int_0^\tau \mathrm{e}^{-\mathrm{i}\,\Gamma(t)}\hat{A}(t)\mathrm{d}\,t = \frac{\mathrm{i}}{\Omega(t)}\hat{A}(t)\mathrm{e}^{-\mathrm{i}\,\Gamma(t)}\Big|_0^\tau + \int_0^\tau \left(\frac{\hat{O}}{\Omega(t)}\hat{A}(t)\right)\mathrm{e}^{-\mathrm{i}\,\Gamma(t)}\mathrm{d}\,t, \tag{B.1}$$

and then ntegrate the second part on the right hand side of Eq. (B.1)

$$\int_0^T \left(\frac{\hat{O}}{\Omega(t)}\hat{A}(t)\right) e^{-i\Gamma(t)} dt = \frac{i e^{-i\Gamma(t)}}{\Omega(t)} \left(\frac{\hat{O}}{\Omega(t)}\hat{A}(t)\right)\Big|_0^T + \int_0^T \frac{d}{dt}\left[\frac{-i}{\Omega(t)}\left(\frac{\hat{O}}{\Omega(t)}\hat{A}(t)\right)\right] e^{-i\Gamma(t)} dt. \quad (B.2)$$

From

$$\frac{d}{dt}\left[\frac{-i}{\Omega(t)}\left(\frac{\hat{O}}{\Omega(t)}\hat{A}(t)\right)\right] = \frac{i}{\Omega(t)}\left(\frac{\dot{\Omega}(t)}{\Omega(t)} - \frac{d}{dt}\right)\left(\frac{\hat{O}}{\Omega(t)}\hat{A}(t)\right) - \left(\frac{\hat{O}}{\Omega(t)}\hat{A}(t)\right),$$

we obtain

$$\int_0^T \left(\frac{\hat{O}}{\Omega(t)}\hat{A}(t)\right) e^{-i\Gamma(t)} dt = \frac{i e^{-i\Gamma(t)}}{\Omega(t)} \left(\frac{\hat{O}}{\Omega(t)}\hat{A}(t)\right)\Big|_0^T + \int_0^T \left(\frac{\hat{O}}{\Omega(t)}\right)^2 \hat{A}(t) e^{-i\Gamma(t)} dt. \quad (B.3)$$

Finally, by recurrence, we prove that

$$\int_0^T e^{-i\Gamma(t)} \hat{V}(t)|\Psi(t)\rangle dt + \text{H.c.} = \frac{i e^{-i\Gamma(t)}}{\Omega(t)} \sum_{k=0}^{\infty} \left(\frac{\hat{O}}{\Omega(t)}\right)^k \hat{V}(t)|\Psi(t)\rangle\Big|_0^T + \text{H.c.} \quad (B.4)$$

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