QUANTUM DOUBLE CONSTRUCTION OF NEW SOLUTIONS TO THE QUANTUM YANG-BAXTER EQUATION*

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ABSTRACT

By establishing a new Hopf algebra and the corresponding quantum double, we construct a new universal R-matrix. Under concrete representations, it produces a family of new solutions to the quantum Yang-Baxter equation without a parameter, in which the so-called non-standard solution associated with the Lie algebra $sl(3)$ is included.

Keywords: Hopf algebra, quantum double, Yang-Baxter equation, quantum algebra.

I. INTRODUCTION

Drinfeld's quantum double theory[1] provides a general method to construct solutions to the quantum Yang-Baxter equation (QYBE). It tells us that given a Hopf algebra A and its dual algebra A^0, one can find a unique quasi-triangular Hopf algebra $D(A)$ (quantum double) such that both A and A^0 become the subalgebras of $D(A)$ and the linear mapping $A\otimes A^0 \to D(A)$ is bijective. In this sense (identifying $A\otimes A^0$ with $D(A)$ as a vector space), if $\{a_i\}$ and $\{b_i\}$ are dual bases of A and A^0, then the universal R-matrix

$$\mathscr{R} = \sum a_i \otimes b_i \in A\otimes A^0$$

satisfies the universal Yang-Baxter equation

$$\mathscr{R}_{12}\mathscr{R}_{13}\mathscr{R}_{23} = \mathscr{R}_{23}\mathscr{R}_{13}\mathscr{R}_{12},$$

where $\mathscr{R}_{12} = \sum a_i\otimes b_i\otimes I$, $\mathscr{R}_{13} = \sum a_i\otimes I\otimes b_i$ and $\mathscr{R}_{23} = \sum I\otimes a_i\otimes b_i$.

Up to now, the concrete realizations of the quantum double that people have obtained mainly include the q-deformations of the universal enveloping algebras of simple Lie (super) algebras and Kac-Moody algebras, which are called quantum (super) algebras or quantum groups[2,3]. So it is natural to ask whether one can find a new realization of the quantum double and get new solutions to QYBE from it. The purpose of this paper is to answer this question. We first introduce a param-

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eterized Hopf algebra and establish the quantum double structure, and then construct the corresponding universal R-matrix. Finally, from a special representation, through the universal R-matrix we get a family of new solutions to QYBE which include the non-standard solution associated with sl (3) as a special case (without central elements). Another solution in the family is the so-called colored one. It is completely new, for people have only obtained the colored solutions associated with the fundamental representation of $sl(2)$ before[5].

It is worth pointing out that although a Hopf algebra structure[6] has been found for the q-boson algebra[7], the corresponding universal R-matrix only gives the standard solutions associated with $sl_q(2)$. The non-standard solutions associated with $sl_q(2)$ can be constructed from its non-generic representations ($q^p = 1$).

II. Hopf Algebra $\mathscr{A} = \mathscr{A}_2(\lambda)$

Suppose that $\mathscr{A}_2(\lambda) = \mathscr{A}$ is an associative $\mathscr{C}$-algebra whose generators $\{X_i^+, H_i | i = 1,2\}$ satisfy

$$
\begin{aligned}
&[H_1,H_2] = 0,\ [H_1,X_1^+] = 2X_1^+,\ [H_2,X_2^+] = 2X_2^+,\\
&[H_1,X_2^+] = \lambda X_2^+,\ [H_2,X_1^+] = \lambda X_1^+,\\
&(X_1^+)^2 = (X_2^+)^2 = 0,
\end{aligned}
\tag{2.1}
$$

where λ belongs to the complex number field $\mathscr{C}$. We define the following mappings:

1) $\Delta: \mathscr{A} \to \mathscr{A} \otimes \mathscr{A}$:

$$
\begin{aligned}
&\Delta H_i = H_i \otimes 1 + I \otimes H_i,\\
&\Delta X_i^+ = X_i^+ \otimes \exp\left(\frac{h}{4} H_i\right) + \exp\left(-\frac{h}{4} H_i\right) \otimes X_i^+;
\end{aligned}
\tag{2.2}
$$

2) $\varepsilon: \mathscr{A} \to \mathscr{C}$:

$$\varepsilon(H_i) = \varepsilon(X_i^+) = 0, \varepsilon(I) = I; \tag{2.3}$$

3) $S: \mathscr{A} \to \mathscr{A}$:

$$S(H_i) = -H_i,\ S(X_i^+) = -e^{h/2} X_i^+. \tag{2.4}$$

Lemma 1. (i) *If Δ is an algebraic homomorphism, then $h = \pm i\pi(2K+1)$ or $q^4 = 1 (q = e^{h/2})$. Here $K \in \mathbb{Z} = \{0, \pm 1, \pm 2, \cdots\}$.*

(ii) *S is an algebraic antimorphism.*

Proof. (i) From the condition that Δ is an algebraic homomorphism and the equation $(X_i^+)^2 = 0$, we have

$$0 = \Delta(X_i^+)^2 = \{\Delta(X_i^+)\}^2 = (e^{h/2} + e^{-h/2}) e^{-\frac{h}{4}H_i} X_i^+,$$

that is, $e^{h/2} + e^{-h/2} = 0$ or $e^{2h} = 1$.

(ii) Using the definition one can directly verify that S is an algebra antimorphism.

Due to Lemma 1, in the following discussion we take $h = \pm i(2K+1)\pi$ ($K \in \mathbf{Z}$).

Then, by some straightforward calculation we can prove

Theorem 1. *$\mathscr{A}$ is a Hopf algebra with the coproduct Δ, the counit ε and the antipode S defined above.*

To establish the dual Hopf algebra $\mathscr{A}^0$ of $\mathscr{A}$, we introduce the new generators

$$E_i = X_i^+ \exp\left(-\frac{h}{4}H_i\right),\ i = 1,2,$$

which possess the following properties:

$$\Delta E_i = E_i \otimes 1 + \exp\left(-\frac{h}{2}H_i\right)\otimes E_i,$$
$$S(E_i) = -\exp\left(\frac{h}{2}H_i\right)E_i, \tag{2.5}$$
$$E_i^2 = 0.$$

From the proof of the PBW theorem concerning $sl_q(n+1)$[9], we observe that

$$\{E_2^{r_2}E_3^{r_3}E_1^{r_1}H_1^{S_1}H_2^{S_2} \mid r_1, r_2 = 0,1;\ r_3, S_1, S_2 \in Z^+ = \{0,1,2,\cdots\}\}$$

is a basis of $\mathscr{A}$, where $E_3 = E_1E_2 - e^{\frac{-h}{2}\lambda}E_2E_1$ satisfies

$$E_3E_1 = -e^{\frac{h}{2}\lambda}E_1E_3,\ E_3E_2 = -e^{-\frac{h}{2}\lambda}E_2E_3,$$
$$[H_1, E_3] = [H_2, E_3] = (2+\lambda)E_3,$$
$$\Delta E_3 = E_3 \otimes I + e^{-\frac{h}{2}(H_1+H_2)}\otimes E_3 + (e^{\frac{h}{2}\lambda} - e^{-\frac{h}{2}\lambda})e^{-\frac{h}{2}H_2}E_1\otimes E_2, \tag{2.6}$$
$$S(E_3) = e^{\frac{h}{2}(H_1+H_2)}e^{-\frac{h}{2}\lambda}(E_2E_1 - e^{-\frac{h}{2}\lambda}E_1E_2).$$

III. Construction of a New Quantum Double

Now let us set to investigate the dual Hopf algebra $\mathscr{A}^0$ and construct the corresponding quantum double $\mathscr{D} = D(\mathscr{A})$. It follows from the general quantum double theory that

(i) $\mathscr{A}$ and $\mathscr{A}^0$ are subalgebras of $\mathscr{D}$,

(ii) the linear mapping $\mathscr{A}\otimes\mathscr{A}^0 \to \mathscr{D}$ is bijective,

(iii) $\forall a \in \mathscr{A},\ b \in \mathscr{A}^0$,

$$ba = \sum_{i,j}\langle a^i_{(1)}, Sb^j_{(1)}\rangle\langle a^i_{(3)}, b^j_{(3)}\rangle a^i_{(2)}b^j_{(2)}, \tag{3.1}$$

where for $C = a$ or b

$$(\Delta\otimes id)\cdot\Delta(C) = (id\otimes\Delta)\circ\Delta(C) = \sum_i C^i_{(1)}\otimes C^i_{(2)}\otimes C^i_{(3)},$$

and the bilinear form $\langle,\rangle$: $\mathscr{A}\otimes\mathscr{A}^0 \longmapsto \mathscr{C}$ satisfies

$$\langle a, b_1b_2\rangle = \langle\Delta a, b_1\otimes b_2\rangle, \forall a \in \mathscr{A}, b_1, b_2 \in \mathscr{A}^0,$$

$$\langle a_1a_2, b\rangle = \langle a_2\otimes a_1, \Delta b\rangle, \forall a_1, a_2 \in \mathscr{A}, b \in \mathscr{A}^0, \tag{3.2}$$
$$\langle 1, b\rangle = \varepsilon(b), \langle a, 1\rangle = \varepsilon(a), \forall a \in \mathscr{A}, b \in \mathscr{A}^0$$
$$\langle S(a), S(b)\rangle = \langle a, b\rangle.$$

In order to deduce the Hopf algebra structure of $\mathscr{A}^0$ we first define the basic elements ξ_1, ξ_2, η_1 and η_2 of $\mathscr{A}^0$ as follows:

$$\xi_i(H_i) = \langle H_i, \xi_i\rangle = 1, \ (i = 1,2)$$
$$\xi_i(X) = \langle X, \xi_i\rangle = 0, \text{ when } X \text{ is a basis element of } \mathscr{A} \text{ other than } H_i, \tag{3.3}$$
$$\eta_i(E_i) = \langle E_i, \eta_i\rangle = 1, \ (i = 1,2)$$
$$\eta_i(Y) = \langle Y, \eta_i\rangle = 0, \text{ when } Y \text{ is a basis element of } \mathscr{A} \text{ other than } E_i.$$

Then, from (3.2) and (3.3) by induction we can prove

Lemma 2.

(i) $\eta_1^2 = \eta_2^2 = 0$, (*to guarantee* $\langle , \rangle$ *to be non-degenerate*)

(ii) $\langle H_i^n, \xi_i^{n'}\rangle = n!\,\delta_{nn'}$, (3.4)

(iii) $\langle E_3, \eta_1\eta_2 - e^{-\frac{h\lambda}{2}}\eta_2\eta_1\rangle \equiv \langle E_3, \eta_3\rangle = 1 - e^{-h\lambda}$,

(iv) $\langle E_3^n, \eta_3^{n'}\rangle = \delta_{nn'}\prod_{k=1}^{n}(1 - e^{-h\lambda k})$,

(v) $\langle E_2^{r_2}E_3^{r_3}E_1^{r_1}H_1^{S_1}H_2^{S_2}, \ \eta_2^{r_2'}\eta_3^{r_3'}\eta_1^{r_1'}\xi_1^{s_1'}\xi_2^{s_2'}\rangle$

$$= \left\{\prod_{i=1}^{3}\delta_{r_ir_i'}\prod_{j=1}^{2}\delta_{s_js_j'}\right\}S_1!\,S_2!\prod_{k=1}^{r_3}(1 - e^{-h\lambda k}).$$

$(r_1, r_2 = 0,1, \ r_3, S_1, S_2 \in Z^+)$.

For convenience in the subsequent discussion, we introduce

$$\mathscr{H}_1 = \xi_1 + \frac{\lambda}{2}\xi_2, \quad \mathscr{H}_2 = \xi_2 + \frac{\lambda}{2}\xi_2.$$

By calculation we have

Lemma 3.

$$\Delta\mathscr{H}_i = \mathscr{H}_i\otimes I + I\otimes\mathscr{H}_i, \ (i = 1,2)$$
$$\Delta\eta_i = 1\otimes\eta_i + \eta_i\otimes\exp(2\mathscr{H}_i),$$
$$[\mathscr{H}_1, \mathscr{H}_2] = 0, \ [\mathscr{H}_i, \eta_i] = -\frac{h}{2}\eta_i,$$
$$[\mathscr{H}_1, \eta_2] = -\frac{h}{4}\lambda\eta_2, \ [\mathscr{H}_2, \eta_1] = -\frac{h}{4}\lambda\eta_1,$$
$$[H_1, \eta_2] = -\lambda\eta_2, \ [H_2, \eta_1] = -\lambda\eta_1 \tag{3.5}$$
$$[H_i, \eta_i] = -2\eta_i, \ [\mathscr{H}_i, E_i] = \frac{h}{2}E_i, \ (i = 1,2)$$
$$[\mathscr{H}_1, E_2] = \frac{1}{4}\lambda hE_2, \ [\mathscr{H}_2, E_1] = \frac{1}{4}\lambda hE_1,$$

$$[\mathscr{H}_i, H_j] = 0, \quad i,j = 1,2$$
$$[E_i, \eta_j] = \delta_{ij}(e^{2\mathscr{H}_i} - e^{-\frac{h}{2}1+i}).$$

Now from Lemma 2 and Lemma 3 we can easily get two bases of $\mathscr{A}$ and $\mathscr{A}^0$ dual to each other and determine the quantum double.

Theorem 2. *The basis* $\{E_2^{r_2}E_3^{r_3}E_1^{r_1}H_1^{S_1}H_2^{S_2} \mid r_1, r_2 = 0,1;\ r_3, S_1, S_2 \in \mathbb{Z}^+\}$ *of* $\mathscr{A}$ *and the basis* $\left\{\eta_2^{r_2}\eta_3^{r_3}\eta_1^{r_1}\xi_1^{S_1}\xi_2^{S_2} \Big/ \left\{S_1!S_2!\prod_{k=1}^{r_3}(1-e^{-\lambda hk})\right\} \Big| r_1, r_2 = 0, 1;\ r_3, S_1, S_2 \in \mathbb{Z}^+\right\}$ *of* $\mathscr{A}^0$ *are dual to each other. The quantum double* $\mathscr{D} = D(\mathscr{A})$ *is determined by* (2.1) *and* (3.5).

IV. Universal R-matrix

From the above new realization of quantum double and the explicit dual bases we can write down a new universal R-matrix:

$$\mathscr{R} = \sum_{\substack{r_1,r_2=0,1 \\ r_3,S_1,S_2\in Z^+}} \frac{E_2^{r_2}E_3^{r_3}E_1^{r_1}H_1^{S_1}H_2^{S_2} \otimes \eta_2^{r_2}\eta_3^{r_3}\eta_1^{r_1}\xi_1^{S_1}\xi_2^{S_2}}{S_1!S_2!\prod_{k=1}^{r_3}(1-e^{-h\lambda k})}$$
$$= \sum_{\substack{r_1,r_2=0,1 \\ r_3\in Z^+}} \frac{E_2^{r_2}E_3^{r_3}E_1^{r_1} \otimes \eta_2^{r_2}\eta_3^{r_3}\eta_1^{r_1}}{\prod_{k=1}^{r_3}(1-e^{-\lambda hk})} \exp\left\{\sum_{i=1}^{2} H_i \otimes \xi_i\right\}. \tag{4.1}$$

Now, consider the central extension $\mathscr{D}_c$ of the quantum double $\mathscr{D}$. It is a $\mathscr{C}[h]$-algebra generated by the elements $\hat{H}_i, \hat{E}_i, \hat{F}_i$ and $\hat{C}_i$ $(i = 1,2)$ that satisfy the relations

$$\begin{aligned}
&[\hat{H}_i, \hat{E}_i] = 2\hat{E}_i, \ [\hat{H}_i, \hat{F}_i] = -2\hat{F}_i, \ (i = 1,2) \\
&[\hat{H}_1, \hat{H}_2] = 0, \ \hat{E}_i^2 = \hat{F}_i^2 = 0, \\
&[\hat{H}_1, \hat{E}_2] = \lambda\hat{E}_2, \ [\hat{H}_1, \hat{F}_2] = -\lambda\hat{F}_2, \\
&[\hat{H}_2, \hat{E}_1] = \lambda\hat{E}_1, \ [\hat{H}_2, \hat{F}_1] = -\lambda\hat{F}_1, \\
&[\hat{E}_i, \hat{F}_j] = \delta_{ij}e^{h/2}sh\left(\frac{h}{2}\hat{H}_i\right)\Big/sh(h/2), \\
&[\hat{C}_i, X] = 0, \ X \in \{\hat{H}_i, \hat{F}_i, \hat{E}_i \mid i = 1,2\}.
\end{aligned} \tag{4.2}$$

Then we can prove

Lemma 4. *The mapping* ϕ: $\mathscr{D} \longmapsto \mathscr{D}_c$:

$$\begin{aligned}
&E_i \longmapsto \hat{E}_i, \\
&\eta_i \longmapsto (1-e^{-h})\hat{F}_i \exp\left(\frac{h}{2}\hat{C}_i\right), \\
&H_i \longmapsto \hat{H}_i - \hat{C}_i, \\
&\mathscr{H}_i \longmapsto \frac{h}{4}(\hat{H}_i + \hat{C}_i)
\end{aligned} \tag{4.3}$$

is a homomorphism.

Using the mapping ϕ, from (4.1) one obtains a universal R-matrix connected with $\mathscr{D}_c$:

$$\mathscr{R}=\sum_{\substack{r_1,r_2=0,1\\ r_3,S_1,S_2\in Z^+}}\left\{\hat{E}_2^{r_2}\hat{E}_3^{r_3}\hat{E}_1^{r_1}\otimes\hat{F}_2^{r_2}\hat{F}_3^{r_3}\hat{F}_1^{r_1}\exp\left[\frac{h}{2}(\hat{C}_1r_3+\hat{C}_2r_2+\hat{C}_3r_3+\hat{C}_1r_1)\right]\right.$$

$$\left.\times(1-e^{-h})^{r_1+r_2+2r_3}\right\}\times\left[\prod_{k=1}^{r_3}(1-e^{-\lambda bk})^{-1}\right]$$

$$\times\exp\left\{\frac{h}{4}\left(1-\frac{\lambda^2}{4}\right)^{-1}\cdot\left[(\hat{H}_1-\hat{C}_1)\otimes\left((\hat{H}_1+\hat{C}_1)-\frac{\lambda}{2}(\hat{H}_2+\hat{C}_2)\right)\right.\right.$$

$$\left.\left.+(\hat{H}_2-\hat{C}_2)\otimes\left((\hat{H}_2+\hat{C}_2)-\frac{\lambda}{2}(\hat{H}_1+\hat{C}_1)\right)\right]\right\}. \tag{4.4}$$

Under different representations of $\mathscr{D}_c$ it will produce various R-matrices.

V. Quantum R-Matrix-non-Standard Solution and Colored Solutions

It can be directly verified that

$$\Pi(\hat{H}_1)=\begin{bmatrix}\lambda+2&&\\&\lambda&\\&&0\end{bmatrix},\quad \Pi(\hat{H}_2)=\begin{bmatrix}0&&\\&-\lambda&\\&&-(\lambda+2)\end{bmatrix},$$

$$\Pi(\hat{E}_1)=\begin{bmatrix}0&-[\lambda]_q&0\\0&0&0\\0&0&0\end{bmatrix},\quad \Pi(\hat{F}_1)=\begin{bmatrix}0&0&0\\e^{h/2}&0&0\\0&0&0\end{bmatrix},$$

$$\Pi(\hat{E}_2)=\begin{bmatrix}0&0&0\\0&0&-[\lambda]_q\\0&0&0\end{bmatrix},\quad \Pi(\hat{F}_2)=\begin{bmatrix}0&0&0\\0&0&0\\0&e^{h/2}&0\end{bmatrix},$$

$$\Pi(\hat{C}_i)=C_iI,\ i=1,2,\quad [\lambda]_q=sh\left(\frac{h}{2}\lambda\right)\Big/sh\left(\frac{h}{2}\right),\ C_i\in\mathbb{C}.$$

Define a matrix representation Π of $\mathscr{D}_c$. For convenience, we denote $\Pi_c=\Pi_{(C_1,C_2)}=\Pi$.

(i) When $C_i=0$ $(i=1,2)$, from the above representation we get the following R-matrix.

$$R=\Pi_C\otimes\Pi_C(\mathscr{R})=\exp\left[\frac{\lambda h(2+\lambda)}{2(2-\lambda)}\right]\cdot\text{block diag }(\bar{A}_1,\bar{A}_2,\bar{A}_3):$$

$$\bar{A}_1=\begin{bmatrix}t&0&0\\0&1&t-t^{-1}\\0&0&1\end{bmatrix},\quad \bar{A}_2=\begin{bmatrix}1&0&t-t^{-1}\\0&-t^{-1}&0\\0&0&1\end{bmatrix},\quad \bar{A}_3=\begin{bmatrix}1&t-t^{-1}&0\\0&1&0\\0&0&t\end{bmatrix},$$

where $t=e^{\frac{h}{2}(2+\lambda)}=-e^{\frac{1}{2}\lambda h}$ is an arbitrary parameter. This R-matrix is exactly the non-standard solution obtained by means of the extended Kauffmann's diagrammatic technique[4].

(ii) When $C_1 = C_2 = C$ and $C'_1 = C'_2 = C'_3$, we obtain a completely new solution——an R-matrix colored by a scalar:

$$R(C,C') = \Pi_C \otimes \Pi_{C'}(\mathscr{R}) = \exp\left[\frac{(2+\lambda)\lambda h}{2(2-\lambda)} - \frac{h}{2+\lambda} CC'\right]$$

$$\cdot \text{ block diag}(\bar{B}_1, \bar{B}_2, \bar{B}_3):$$

$$\bar{B}_1 = \begin{bmatrix} u^{-1}u't & 0 & 0 \\ 0 & u' & t-t^{-1} \\ 0 & 0 & u^{-1} \end{bmatrix}, \quad \bar{B}_2 = \begin{bmatrix} uu' & 0 & t-t^{-1} \\ 0 & -t^{-1} & 0 \\ 0 & 0 & u^{-1}u'^{-1} \end{bmatrix},$$

$$\bar{B}_3 = \begin{bmatrix} u & t-t^{-1} & 0 \\ 0 & u'^{-1} & 0 \\ 0 & 0 & uu'^{-1}t \end{bmatrix},$$

where $u = e^{\frac{h}{2}C}$, $u' = e^{\frac{h}{2}C'}$ and the colors C and C' distinguish between the same dimensional representations of $\mathscr{D}_e$. Obviously, the R-matrix $R(C,C')$, which satisfies

$$R_{12}(C,C')R_{13}(C,C'')R_{23}(C',C'') = R_{23}(C',C'')R_{13}(C,C'')R_{12}(C,C'),$$

is a generalization of the colored R-matrices associated with $sl_q(2)$ to a higher rank algebra case.

(iii) In the most general case, C_i and C'_i $(i=1,2)$ are arbitrary, so we denote $\boldsymbol{C} = (C_1, C_2)$, $\boldsymbol{C}' = (C'_1, C'_2)$. Then we have an R-matrix colored by a vector:

$$R(\boldsymbol{C},\boldsymbol{C}') = \Pi_C \otimes \Pi_{C'}(\mathscr{R}) = \exp\left[\frac{\lambda h(2+\lambda)}{2(2-\lambda)} + \frac{h}{(4-\lambda^2)}\left(\boldsymbol{C}\cdot\sigma_x\cdot\boldsymbol{C}'^T \cdot \frac{\lambda}{2} = \boldsymbol{C}\cdot\boldsymbol{C}'^T\right)\right] \times \text{block diag }(Q_1, Q_2, Q_3):$$

$$Q_1 = \begin{bmatrix} u_1^{-1}u'_1 t & 0 & 0 \\ 0 & u'_1 & t-t^{-1} \\ 0 & 0 & u_1^{-1} \end{bmatrix}, \quad Q_2 = \begin{bmatrix} u'_1 u_2 & 0 & t-t^{-1} \\ 0 & -t^{-1} & 0 \\ 0 & 0 & u_1^{-1}u_2'^{-1} \end{bmatrix},$$

$$Q_3 = \begin{bmatrix} u_2 & t-t^{-1} & 0 \\ 0 & u_2'^{-1} & 0 \\ 0 & 0 & u_2 u_2'^{-1} t \end{bmatrix}, \quad u_i = e^{\frac{h}{2}C_i}, \; u'_i = e^{\frac{h}{2}C'_i} \quad (i=1,2),$$

where σ_x is a Pauli matrix.

IV. Discussion and Conclusion

(1) In the new universal R-matrix $\mathscr{R}$, although the integers that r_1 and r_2 can take are finite, r_3 are allowed to take any non-negative integer, so as usual, $\mathscr{R}$ is an infinite series as an abstract operator. However, because there exists a close similarity between the structure of $\mathscr{A}_2(\lambda)$ and that of the Borel subalgebra $\{\bar{X}_i^+, \bar{H}_i | i=1,2\}$ of the simple Lie algebra A_2, we can expect that in a finite dimen-

sional representation of $\mathscr{D}$ E_i and η_i $(i=1,2)$ can be written as uppertriangular and lowertriangular matrices, respectively. In this case, E_3 and η_3 are also uppertriangular and lowertriangular matrices respectively, so we have $E_3^l = \eta_3^l = 0$ for some non-negative integer l and the formal infinite series will be cut off at some term. Consequently, we can get new finite dimensional R-matrices without dealing with the summation of an infinite series.

(2). We have shown that it is possible to construct new realizations of the quantum double other than quantum (super) algebras and obtain new R-matrices. The classification of the possible new quantum doubles remains to be studied further.

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