Vol. 35 No. 8 SCIENCE IN CHINA (Series A) August 1992

q-DEFORMED BOSON REPRESENTATIONS OF UNIVERSAL ENVELOPING ALGEBRAS IN THE CASE WHERE q IS A ROOT OF UNITY*

SUN CHANG-PU(孙昌璞)

(*Physics Department, Northeast Normal University, Changchun* 130024, *and Nankai Institute of Mathematics, Tianjin* 300071, *PRC*)

AND GE MO-LIN(葛墨林)

(*Nankai Institute of Mathematics, Tianjin* 300071, *PRC*)

Received November 14, 1990.

ABSTRACT

By further improving the method of q-deformed boson realization, the standard basis is built for the typical subalgebra chains. When q is a root of unity, the irreducible and indecomposable representations of quantum universal enveloping algebras $(A_{l-1})_q$ and $(C_l)_q$ are constructed and their reduction structure and decompositions are analyzed.

Keywords: quantum group, quantum universal enveloping algebra, indecomposable representation, subalgebra chain.

I. INTRODUCTION

The quantum group and the quantum universal enveloping algebra (QUEA) originate from the nonlinear physical problems associated with the Yang-Baxter equation (YBE)[1-8]. Recently, their representation theories have attracted physicists and mathematicians' widespread attention[9-15]. The q-deformed boson realization method for QUEA presented by some authors independently[13-15] is a breakthrough in this field. As a quantum correspondence of the boson realization (Jordan-Schwinger mapping) for classical Lie algebras, this method is a powerful tool for constructing the finite-demensional representation of QUEA. At present, it has been generalized and applied to many areas[19-24]. Our study mainly deals with the QUEA $U_q sl(l)=(A_{l-1})_n$. We constructed all the symmetrized representations for the first time for the case where q is not a root of unity (i.e. there is no integer p satisfying $q^p=1$). Our work can be regarded as a natural development of the previous systematic work about Lie (super-) algebras[16-18].

However, all the studies on the q-deformed boson realization failed to deal with the case where q is a root of unity ($q^p=1$, $p=3,4,\cdots$). That is a case

* Project supported in part by the National Natural Science Foundation of China.

significant in physics[25-27], possessing the characteristics specific for QUEA and is quite unlike the case where q is not a root of unity. For the latter case all the discussions can be basically regarded as a q-deformation of the theories of the classical Lie algebra. In this paper we try to develop a systematic method and a fundamental theory for the case where q is a root of unity. For convenience, we assume that $U_q(\mathscr{L})=(\mathscr{L})_q$ is a QUEA of a Lie algebra $\mathscr{L}$, $\mathbb{C}$ the complex number field and End(V) the set of all the linear transformation matrices on the linear space V; $\mathbb{Z}=\{0,\pm1,\pm2,\cdots\}$; $\mathbb{Z}^+=\{0,1,2,\cdots\}$. Following Lusztig[11], we suppose p is an odd integer $\geqslant 3$ without loss of generality.

II. The q-Deformed Boson Realization of the Typical Subalgebra Chains

The q-deformed boson operators $\hat{a}_i=\hat{a}_i^-$, $\hat{a}_i^{-1}$ and $\hat{N}_i(i=1,2,\cdots l)$ which were first introduced in Refs. [13—15] generate an associative algebra $\mathscr{B}_q(l)$ over $\mathbb{C}$ with unit and satisfy

$$\hat{a}_i\hat{a}_j^+=\delta_{ij}q^{\hat{N}_i}+q^{-\delta_{ij}}\hat{a}_j^+\hat{a}_i,$$
$$\hat{a}_i^\pm\hat{a}_j^\pm=\hat{a}_j^\pm\hat{a}_i^\pm,\ [\hat{N}_i,\hat{a}_j^\pm]=\pm\delta_{ij}\hat{a}_j^\pm, \tag{1}$$

where $q\in\mathbb{C}$. This algebra is called q-deformed boson algebra; its representation theory has been systematically studied recently[28]. Corresponding to $\mathscr{B}_q(l)$, the q-deformed Fock space $\mathscr{F}_q(l)$ is defined as

$$\{|\boldsymbol{m}\rangle=|m_1,m_2,\cdots,m_l\rangle=\hat{a}_1^{+m_1}\hat{a}_2^{+m_2}\cdots\hat{a}_l^{+m_l}|0\rangle|\hat{a}_i|0\rangle=\hat{N}_i|0\rangle$$
$$=0,i=1,2,\cdots,l\},$$

where $\boldsymbol{m}=(m_1,m_2,\cdots,m_l)$ is a point in the lattice point set $\mathbb{Z}^{+l}$: $\{(m_1,m_2,\cdots,m_l)=\boldsymbol{m}|m_1,m_2,\cdots,m_l\in\mathbb{Z}^+\}$. Denoting the unit vectors in $\mathbb{Z}^{+l}$ by

$$\boldsymbol{e}_1(l)=(1,0,0,\cdots,0),\ \boldsymbol{e}_2(l)=(0,1,0,\cdots,0),\cdots,$$
$$\boldsymbol{e}_l(l)=(0,0,0,\cdots,1),$$

we have

$$\hat{a}_j^\pm|\boldsymbol{m}\rangle=\left[\frac{1}{2}(n_j\mp n_j+1\pm1)\right]|\boldsymbol{m}\pm\boldsymbol{e}_j(l)\rangle,$$
$$\hat{N}_j|\boldsymbol{m}\rangle=m_j|\boldsymbol{m}\rangle. \tag{2}$$

The above equation defines a natural representation ρ_1: $\mathscr{B}_q(l)\rightarrow$ End $(\mathscr{F}_q(l))$ where $[f]=(q^f-q^{-f})/(q-q^{-1})$ for any operator or number f; $[f]\xrightarrow{q\to1}f$. It is worth while to point out that the representation ρ_1 is irreducible when $q^p\neq1$ for any $p(\neq0)\in\mathbb{Z}$; otherwise the representation ρ_1 is indecomposable (reducible, but not completely reducible) when $q^p=1$.

In the sense of (2), regarding $\mathscr{B}_q(l)$ as a linear operator algebra on $\mathscr{F}_q(l)$, we define a q-deformed boson realization $B((\mathscr{L})_q)$: $\{x=B(\hat{x})|\hat{x}\in(\mathscr{L})_q\}$ by a 1—1 homomorphic mapping B from the QUEA $(\mathscr{L})_q$ to $B_q(\mathscr{L})$. Considering the q-deformed boson realization of $(A_{l-1})_q$ given in Ref. [14], we immediately get a q-deformed boson realization for $(C_l)_q$:

$$\begin{cases} H_i = \hat{N}_i - \hat{N}_{i+1}, & \text{(3a)} \\ E_i = \hat{a}_i^+ \hat{a}_{i+1}^-, \ \ F_i = \hat{a}_{i+1}\hat{a}_i, \ \ i = 1, 2, \cdots, l-1, & \text{(3b)} \\ E_l = [2]^{-1}\hat{a}_l^{+2}, \ \ F_l = -[2]^{-1}\hat{a}_l^2. & \text{(3c)} \end{cases}$$

$$H_l = \hat{N}_l + 1/2.$$

Using (2), it is verified by a direct calculation that H_j, E_j and $F_j (j = 1, 2, \cdots)$ defined by (3) indeed satisfy the basic relations of $(C_l)_q$, namely, the q-deformed commutation relations

$$\begin{aligned} &[H_i, E_j] = \alpha_{ij}E_j, \ \ [H_i, F_j] = -\alpha_{ij}F_j, \\ &[H_i, H_j] = 0, \\ &[E_i, F_j] = \delta_{ij}\frac{q_i^{H_i} - q_i^{-H_i}}{q_i - q_i^{-1}}, \ q_i = q(i \leqslant l-1), \ \ q_l = q^2 \end{aligned} \tag{4}$$

and the Serre relations

$$\begin{aligned} &G_j^2 G_{j\pm1} - (q + q - 1)G_j G_{j\pm1} G_j + G_{j\pm1} G_j^2 = 0, j \leqslant l-1, \\ &\sum_{m=0}^{3} (-1)^m \left(\frac{[3]_{q^2}!}{[m]_{q^2}![3-m]_{q^2}!}\right)^{-1} G_{l-1}^{3-m} G_l G_{l-1}^m = 0, \ \ G = F, \ E, \end{aligned} \tag{5}$$

where

$$\begin{aligned} &[m]_t! = [m]_t[m-1]_t \cdots [2]_t[1]_t; \ \ [m]_t = (t^m - t^{-m})/(t - t^{-1}); \\ &\alpha_{l,l} = 2, \ \ \alpha_{ij} = 2\delta_{ij} - \delta_{i,j+1} - \delta_{ij-1}, \\ &\alpha_{i,l} = -2\delta_{il-1}, \ \ \alpha_{l,i} = -\delta_{il-1}, i, \ \ j = 1, 2, \cdots, l-1. \end{aligned}$$

It is observed from Eqs. (3) and (4) that E_i, F_i and $H_i (i = 1, 2, \cdots, k \leqslant l-1)$ are closed under the q-deformed relation, i.e. they generate a subalgebra $(A_k)_q (k \leqslant l-1)$. Thus, Eqs. (3) give a realization of the subalgebra chain I:

$$(C_l)_q \supset (A_{l-1})_q \supset (A_{l-2})_q \supset \cdots \supset (A_2)_q \supset (A_1)_q.$$

In fact, since E_j, F_j and $H_j (j = k, k+1, \cdots, l-1)$ generate subalgebra $(C_{l-k+1})_q$, there is another subalgebra chain II:

$$(C_l)_q \supset (C_{l-1})_q \supset (C_{l-2})_q \supset \cdots \supset (C_3)_q \supset (C_2)_q.$$

The two subalgebra chains are fundamental from which one can determine other subalgebra chains, such as the subalgebra chain III with branches:

$$\begin{array}{l} (A_{l-1})_q \rightarrow (A_{l-2})_q \rightarrow \cdots \rightarrow (A_1)_q \\ \nearrow \qquad\qquad (A'_{l-2})_q \rightarrow (A'_{l-3})_q \rightarrow \cdots \rightarrow (A'_1)_q \\ \qquad\qquad\quad \nearrow \\ (C_l)_q \rightarrow (C_{l-1})_q \rightarrow \cdots \rightarrow (C_3)_q \rightarrow (C_2)_q \\ \qquad\qquad\qquad\qquad\qquad\qquad\quad \searrow \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad (A''_2)_q \rightarrow (A''_1)_q, \end{array}$$

where $(A'_k)_q, (A''_k)_q, \cdots$ denote the different "embeddings" of $(A_k)_q$ in $(C_l)_q$, e.g. $(A_1)_q$, $(A'_1)_q$ and $(A''_1)_q$ possess different sets of generators: $\{H_1, E_1, F_1\}$, $\{H_2, E_2, F_2\}$ and $\{H_{l-2}, E_{l-2}, F_{l-2}\}$; "$\rightarrow$" in $A \rightarrow B$ represents "$\supset$" in $A \supset B$.

Because there is no obvious finite dimensional invariant subspace in the representation space $\mathscr{F}_q(l)$ of $(C_l)_q$ under the action of $(C_k)_q$ $(k=2,3,\cdots)$, we will not take the subalgebra chain II into account. For the basic subalgebra chain I, we define the standard basis for $\mathscr{F}_q(l)$, which is denoted by number set $(\lambda_l, \lambda_{l-1}, \cdots, \lambda_1)$, so that the irreducible representations of $(A_{l-1})_q$, $(A_{l-2})_q$, $\cdots$, $(A_1)_q$ are respectively denoted by $\lambda_l, \lambda_{l-1}, \cdots, \lambda_2$. Then, we can construct the finite dimensional representations for each subalgebra in this subalgebra chain in a universal framework, so that the representation of $(A_k)_q$ constrained on $(A_{k-1})_q$ is automatically decomposed.

III. Representations of Subalgebra Chain for $(C_l)_q$

From Eqs. (2) and (3), we obtain

$$\begin{cases} H_i|m\rangle = (m_i - m_{i+1})|\boldsymbol{m}\rangle, \\ E_i|m\rangle = [m_{i+1}]|\boldsymbol{m} + \boldsymbol{e}_i(l) - \boldsymbol{e}_{i+1}(l)\rangle, \\ F_i|m\rangle = [m_i]|\boldsymbol{m} + \boldsymbol{e}_{i+1}(l) - \boldsymbol{e}_i(l)\rangle, \quad i = 1,2,\cdots l-1, \\ H_l|\boldsymbol{m}\rangle = (m_l + 1/2)|\boldsymbol{m}\rangle, \\ E_l|\boldsymbol{m}\rangle = [2]^{-1}|\boldsymbol{m} + 2\boldsymbol{e}_l(l)\rangle, \\ F_l|\boldsymbol{m}\rangle = -[2]^{-1}[m_l - 1][m_l]|\boldsymbol{m} - 2\boldsymbol{e}_l(l)\rangle, \end{cases} \tag{6}$$

which defines an infinite dimensional representation Γ_l: $(C_l)_q \to \mathrm{End}\,(\mathscr{F}_q(l))$. Because $(-1)^{\sum_{i=1}^{l} m_l}$ is invariant under the action of this representation, the representation is reduced to $\Gamma_l = \Gamma_l^+ \oplus \Gamma_l^-$ and the space decomposition is $\mathscr{F}_q(l) = \mathscr{F}_q^+(l) \oplus \mathscr{F}_q^-(l)$ correspondingly:

$$\mathscr{F}_q^+(l) = \{|\boldsymbol{m}\rangle \in \mathscr{F}_q(l) | (-1)^{\sum_{i=1}^{l} m_i} = 1\},$$

$$\mathscr{F}_q^-(l) = \{|\boldsymbol{m}\rangle \in \mathscr{F}_q(l) | (-1)^{\sum_{i=1}^{l} m_i} = -1\}.$$

Because the discussion about $\mathscr{F}_q^-(l)$ completely parallels that about $\mathscr{F}_q^+(l)$, we will only discuss $\mathscr{F}_q^+(l)$. Define the standard basis for $\mathscr{F}_q^+(l)$:

$$f(\boldsymbol{\lambda}|J) = |\lambda_1,\ \lambda_2 - \lambda_1, \cdots, \lambda_{l-1} - \lambda_{l-2},\ 2J - \lambda_{l-1}\rangle,\ J = 0,1,2,\cdots,$$

where $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \cdots, \lambda_{l-1}) \in Z^{+l-1}$ and $\lambda_{k-1} = 0, 1, 2, \cdots, \lambda_k$ for a given $\lambda_k (k = 2, 3, \cdots,)$ and we denote $\lambda_l = 2J$, $\lambda_0 = 0$. Then, we have

$$E_i f(\boldsymbol{\lambda}|J) = [\lambda_{i+1} - \lambda_i] f(\boldsymbol{\lambda} + \boldsymbol{e}_i(l-1)|J), \tag{7a}$$

$$F_i f(\boldsymbol{\lambda}|J) = [\lambda_i - \lambda_{i-1}] f(\boldsymbol{\lambda} - \boldsymbol{e}_i(l-1)|J), \tag{7b}$$

$$H_i f(\boldsymbol{\lambda}|J) = (2\lambda_i - \lambda_{i+1} - \lambda_{i-1}) f(\boldsymbol{\lambda}|J), \quad i = 1,2,\cdots,l-1, \tag{7c}$$

$$E_l f(\boldsymbol{\lambda}|J) = [2]^{-1} f(\boldsymbol{\lambda}|J+1),\ H_l f(\boldsymbol{\lambda}|J) = \left(2J - \lambda_{l-1} + \frac{1}{2}\right) f(\boldsymbol{\lambda}|J), \tag{7d}$$

$$F_l f(\boldsymbol{\lambda}|J) = -[2]^{-1}[2J - \lambda_{l-1}][2J - \lambda_{l-1} - 1] f(\boldsymbol{\lambda}|J-1).$$

In order to explain the character of the standard basis, we first discuss the case of $q^p \neq 1 (p(\neq 0) \in \mathbf{Z})$, in which the representation $\Gamma_l^+ : (C_l)_q \to \mathrm{End}(\mathscr{F}_q^+(l))$

of $(C_l)_q$ defined by Eqs. (7a, b, c, d) is irreducible. Because J is invariant under the action of $\Gamma_l^+((A_{l-1})_q)$, $V_l^{[J]}$:

$$\{f(\boldsymbol{\lambda}|J') \in \mathscr{F}_q^+(l) | J' = \text{fixed} J\} \cong \left\{|m_1, m_2, \cdots, m_l\rangle | \sum_{k=1}^{l} m_k = 2J\right\}$$

is an $(A_{l-1})_q$-invariant subspace, with an irreducible representation $\Gamma_l^{[2J]}$[14] on it. By constraining Γ_l^+ on the subalgebra $(A_{l-1})_q$ of $(C_l)_q$, the subalgebra decomposition is automatically realized:

$$\Gamma_l^+|(A_{l-1})_q = \sum_{J=0}^{\infty} \Gamma_l^{(2J)}, \quad \mathscr{F}_l^+(q) = \bigoplus_{J=0}^{\infty} V_l^{[J]}.$$

In the same way, we may regard λ_k as the label for the irreducible representation $\Gamma_k^{[\lambda_k]}$ of $(A_{k-1})_q$ and

$$V_k^{[\lambda_k]}: \{f(\boldsymbol{\lambda}|J) | \lambda_k, \lambda_{k+1}, \cdots, \lambda_l \text{ are fixed}\}$$
$$\cong \Big\{f_{\lambda_k}(\boldsymbol{\lambda}) = f_{\lambda_k}(\lambda_1, \cdots, \lambda_{k-1}) \equiv |m_1, m_2, \cdots, m_{k-1}\rangle$$
$$= \hat{a}_1^{+\hat{m}_1} \hat{a}_2^{+m_2} \cdots \hat{a}_{k-1}^{+m_{k-1}} |0\rangle \,\Big|\, \sum_{i=1}^{k-1} m_i = \lambda_k \text{ or } m_i = \lambda_i - \lambda_{i-1}, i = 1, 2, \cdots, k-1\Big\}$$

is the corresponding invariant subspace. Then, we have the automatic decomposition of representation:

$$\Gamma_{k+1}^{[\lambda_{k+1}]}|_{(A_{k-1})_q} = \bigoplus_{\lambda_k=0}^{\lambda_{k+1}} \Gamma_k^{[\lambda_k]},$$

$$V_{k+1}^{[\lambda_{k+1}]}|_{(A_{k-1})_q} = \bigoplus_{\lambda_k=0}^{\lambda_{k+1}} V_k^{[\lambda_k]}.$$

Now, we turn to the discussion on the case in which q is a root of unity. In this case, though the above results about the reduction and decomposition still hold, the invariant subspaces are no longer invariant while the irreducible representations are no longer irreducible. This is because there exists an extreme vector $f(\boldsymbol{\lambda}|J)$ such that

$$E_i f(\boldsymbol{\lambda}|J) = 0, \quad F_{i+1} f(\boldsymbol{\lambda}|J) = 0, \quad \boldsymbol{\lambda}_{i+1} - \boldsymbol{\lambda}_i = \alpha_i p, \quad \alpha_i \in \mathbb{Z}$$

are satisfied for $q^p = 1$, i.e. $[\alpha p] = 0 (\alpha \in \mathbb{Z})$: About the reduction and decomposition of subspaces, we have

Theorem 1. *There exists a $(C_l)_q$-invariant subspace $S_l(j, \alpha)$:*

$$\{f(\boldsymbol{\lambda}|J) | \lambda_{j+1} - \lambda_j \geqslant \alpha p, \ J \in \mathbb{Z}^+\} \ (j = 1, 2, \cdots, l-1)$$

in $\mathscr{F}_q^+(l)$. When $l \geqslant 2$, there is not any invariant complementary subspace to $S_l(j, \alpha)$, namely, Γ_l^+ is indecomposable (reducible, but not completely reducible).

Proof. 1) We first prove that $S_l(j, \alpha)$ is an invariant subspace. To this end, we define subspace $W(j, k)$: $\{f(\boldsymbol{\lambda}|J) \in \mathscr{F}_q^+(l) | \lambda_{j+1} - \lambda_j = k\}$ for $k \in \mathbb{Z}^+$, then $S_l(j, \alpha) =$

$\sum_{k=\alpha p}^{\infty}\oplus W(j,k)$. It follows from (7a, b) that

$$E_{j+1}f(\boldsymbol{\lambda}|J)=[\lambda_{j+2}-\lambda_{j+1}]f(\boldsymbol{\lambda}+\boldsymbol{e}_{j+1}(l-1)|J),$$
$$F_{j+1}f(\boldsymbol{\lambda}|J)=[\lambda_{j+1}-\lambda_{j}]f(\boldsymbol{\lambda}-\boldsymbol{e}_{j+1}(l-1)|J). \tag{8}$$

We observe from Eqs. (7a,b) and (8) that for any $f(\boldsymbol{\lambda}|J)\in W(j,k)\supset S_l(j,\alpha)$

$$E_{j+1}f(\boldsymbol{\lambda}|J)\in W(j,k+1)\subset S_l(j,\alpha),$$
$$F_jf(\boldsymbol{\lambda}|J)\in W(j,k+1)\subset S_l(j,\alpha);$$
$$E_if(\boldsymbol{\lambda}|J)\in W(j,k)\subset S_l(j,\alpha),$$
$$F_if(\boldsymbol{\lambda}|J)\in W(j,k)\subset S_l(j,\alpha),\ i\neq j,\ j+1.$$

Because $[\underline{\lambda}_{j+1}-\underline{\lambda}_j]=[\alpha p]=0$ for $f(\boldsymbol{\lambda}|\underline{J})\in W(j,\alpha p)$,

$$E_jf(\boldsymbol{\lambda}|\underline{J})=F_{j+1}f(\boldsymbol{\lambda}|\underline{J})=0.$$

Moreover, when $k=\alpha p+1,\ \alpha p+2,\ \alpha p+3\cdots$ and $f(\boldsymbol{\lambda}|J)\in W(j,k)$,

$$E_jf(\boldsymbol{\lambda}|J)\subset W(j,k-1)\subset S_l(j,\alpha),$$
$$F_{j+1}f(\boldsymbol{\lambda}|J)\subset W(j,k-1)\subset S_l(j,\alpha).$$

In conclusion, $\forall f(\boldsymbol{\lambda}|J)\in S_l(j,\alpha)$, $E_if(\boldsymbol{\lambda}|J)$, $F_if(\boldsymbol{\lambda}|J)\in S_l(j,\alpha)$.

2) We prove that there is not any invariant complementary subspace to $S_l(j,\alpha)$. In fact, if we assume that there exists an invariant complementary subspace $\bar{S}_l(j,\alpha)$, then there exists a vector

$$v=\sum_{\lambda_{j+1}-\lambda_j\geqslant\alpha p}c_{\boldsymbol{\lambda}}f(\boldsymbol{\lambda}|J)+\sum_{\lambda_{j+1}-\lambda_j<\alpha p}b_{\boldsymbol{\lambda}}f(\boldsymbol{\lambda}|J)$$

in $\bar{S}_l(j,\alpha)$ so that one of $c'_{\lambda}s(\in\mathbb{C})$ and one of $b'_{\lambda}s(\in\mathbb{C})$ are not zero at least. Suppose that $b_{\underline{\lambda}}$ corresponds to a vector $f(\boldsymbol{\lambda}|\underline{J})\in W(j,\underline{k})$ such that $\underline{\lambda}_{j+1}-\underline{\lambda}_j=\underline{k}$ is minimum. Then,

$$(E_{j+1})^{\alpha p-\underline{k}}\cdot v(\neq 0)\subset S_l(j,\alpha).$$

Because $\bar{S}_l(j,\alpha)$ is $(C_l)_q$-invariant, $(E_{j+1})^{\alpha p-\underline{k}}v\in\bar{S}_l(j,\alpha)$. Thus,

$$S_l(j,\alpha)\cap\bar{S}_l(j,\alpha)\neq\{0\}.$$

This conclusion and $S_l(j,\alpha)\oplus\bar{S}_l(j,\alpha)=\mathscr{F}_q^+(l)$ implied by presumption are contradictory. Then, Theorem 1 holds. Q. E. D.

A corollary about the properties of representation for $(A_k)_q(k=1,2,\cdots,l-1)$ immediately follows from Theorem 1.

Corollary 1. *When $q^p=1$, there exists an $(A_{l-1})_q$-invariant subspace*

$$S_l^{[J]}(j,\alpha):\{f(\boldsymbol{\lambda}|J')\,|\,\lambda_{j+1}-\lambda_j^{\cdot}\geqslant\alpha p,\ J'=J\}$$

in $V_l^{[J]}$ for fixed J. When $l\geqslant 2$, there does not exist any invariant complementary space to $S_l^{[J]}(j,\alpha)$; the representation $\Gamma_l^{(25)}$ is indecomposable.

IV. Finite Dimensional Representations of $(C_2)_q$

According to the general theorem and corollary mentioned above, we will analyse the representation Γ_2^+

$$\begin{cases} E_1 f(n|J) = [2J-n] f(n+1|J), \quad F_1 f(n|J) = [n] f(n-1|J), \\ H_1 f(n|J) = 2(n-J) f(n|J), \\ H_2 f(n|J) = \left(2J - n + \dfrac{1}{2}\right) f(n|J), \\ E_2 f(n|J) = [2]^{-1} f(n|J+1), \\ F_2 f(n|J) = -[2J-n][2J-n][2]^{-1} f(n|J-1) \end{cases} \tag{9}$$

defined by Eqs. (7) in detail where $f(n|J) = a_1^{+n} a_2^{+2J-n}|0\rangle \in \mathscr{F}_q^+(2)(n = 0, 1, 2, \cdots, 2J$ for a given $J)$. Noticing the two $(C_2)_q$-invariant subspaces $S_2(1, \beta)$: $\{f(n|J) \in \mathscr{F}_q^+(2) | n \geqslant \beta p\}$, $S_2(2, \alpha)$ $\{f(n|J) \in \mathscr{F}_q^+(2) | 2J - n \geqslant \alpha p\}$ and their sum $S_{12}(\alpha\beta) = S_2(1, \beta) + S_2(2, \alpha)$ which is still a $(C_2)_q$-invariant subspace, we have a theorem about the finite dimensional representations of $(C_2)_q$.

Theorem 2. *The quotient space $Q_{12}(\alpha, \beta) = \mathscr{F}_q^+(2)/S_{12}(\alpha, \beta)$ is finite dimensional and its dimension is*

$$\dim Q_{12}(\alpha, \beta) = \frac{1}{2}(\alpha\beta p^2 + \sigma(\alpha p) \cdot \sigma(\beta p)), \tag{10}$$

where $\sigma(x) = 0$ for even number x and $\sigma(x) = 1$ for odd number x, i. e., $\sigma(x) = \frac{1}{2}(1 - (-1)^x)$.

Proof. The basis for the quotient space $Q_{12}(\alpha, \beta)$ can be chosen as

$$\bar{f}(n|J) = f(n|J) \operatorname{Mod} S_{12}(\alpha, \beta), \quad 0 \leqslant 2J - n \leqslant \alpha p - 1, \quad 0 \leqslant n \leqslant \beta p - 1.$$

For given βp, n takes βp values: $0, 1, 2, \cdots, \beta p - 1$; for each given n, $n/2 \leqslant J \leqslant \frac{1}{2}(\alpha p + n - 1)$. Since J is a positive integer, J takes $\frac{1}{2}(\alpha p + \sigma(\alpha p))$ values for even number n and J takes $\frac{1}{2}(\alpha p - \sigma(\alpha p))$ values for odd number n. Because there are $\frac{1}{2}(\beta p + \sigma(\beta p))$ even numbers and $\frac{1}{2}(\beta p - \sigma(\beta p))$ odd numbers in the series of number $0, 1, 2, \cdots, \beta p - 1$, the number pair takes

$$\frac{1}{2}(\beta p + \sigma(\beta p)) \frac{1}{2}(\alpha p + \sigma(\alpha p)) + \frac{1}{2}(\beta p - \sigma(\beta p)) \frac{1}{2}(\alpha p - \sigma(\alpha p))$$
$$= \frac{1}{2}(\alpha\beta p^2 + \sigma(\alpha p)\sigma(\beta p)) = \dim Q_{12}(\alpha, \beta)$$

values. Q.E.D.

Representation (9), as the linear transformations on the space $\mathscr{F}_q^+(2)$, can induce a finite dimensional representation $\bar{\Gamma}_2^+ : (C_2)_q \to E_n d Q_{12}(\alpha, \beta)$:

$$g\bar{f}(n|J) \equiv \bar{\Gamma}_2^+(g)\bar{f}(n|J) = \overline{gf(n|J)}, \quad \forall g \in (C_2)_q$$

on $Q_{12}(\alpha, \beta)$. It shows that the actions of $(C_2)_q$ on $Q_{12}(\alpha, \beta)$ possess the same forms as that for Eqs. (9) except for the actions on the extreme vectors $f(\beta p - 1|J)$ and $f(2J + 1 - \alpha p|J)$ $(\alpha, \beta \in \mathrm{Z}^+)$. For example, $E_1\bar{f}(\beta p - 1|J) = 0$, $E_2\bar{f}(2J + 1 -$

$\alpha p|J)=0$. When $p=3$ and $\alpha=\beta=1$, we have a 5-dimensional representation

$$\begin{cases} E_1=E_{4,3}+[2]E_{3,2}, \quad E_2=[2]^{-1}(E_{2,1}+E_{5,4}), \\ F_1=E_{2,3}+[2]E_{3,4}, \quad F_2=-(E_{1,2}+E_{4,5}), \\ H_1=2(E_{4,4}-E_{2,2}), \\ H_2=\dfrac{1}{2}(E_{1,1}+5E_{2,2}+3E_{3,3}+E_{4,4}+5E_{5,5}), \end{cases} \tag{11}$$

where E_{ij} are such matrix units that $(E_{ij})_{kl}=\delta_{ik}\delta_{jl}$, $i,j=1,2,\cdots,s$.

V. Indecomposable Representations of $(A_2)_q$

According to Corollary 1, on the space $V_3^{[\lambda]}:\{f_\lambda(\lambda_1,\lambda_2)=\bar{a}_1^{+\lambda_1}\bar{a}_2^{+\lambda_2-\lambda_1}\bar{a}_3^{+\lambda-\lambda_2}|0\rangle|\lambda_2=0,1,\cdots,\lambda;\lambda_1=0,1,\cdots,\lambda_2$ for given $\lambda_2\}$ we analyse the representation $\Gamma_3^{[\lambda]}$ of $(A_2)_q$:

$$\begin{cases} E_1f_\lambda(\lambda_1,\lambda_2)=[\lambda_2-\lambda_1]f_\lambda(\lambda_1+1,\lambda_2), \quad E_2f_\lambda(\lambda_1,\lambda_2)=[\lambda-\lambda_2]f_\lambda(\lambda_1,\lambda_2+1), \\ F_1f_\lambda(\lambda_1,\lambda_2)=[\lambda_1]f_\lambda(\lambda_1-1,\lambda_2), \quad F_2f_\lambda(\lambda_1,\lambda_2)=[\lambda_2-\lambda_1]f_\lambda(\lambda_1,\lambda_2-1), \\ H_1f_\lambda(\lambda_1,\lambda_2)=(2\lambda_1-\lambda_2)f_\lambda(\lambda_1,\lambda_2), \quad H_2f_\lambda(\lambda_1,\lambda_2)=(2\lambda_2-\lambda_1-\lambda)f_\lambda(\lambda_1,\lambda_2). \end{cases} \tag{12}$$

Now, we introduce the lattice diagram (Fig. 1) to describe the representation. Here, the lattice point (λ_1,λ_2) in the right triangle OAB denotes a weight vector $f_\lambda(\lambda_1,\lambda_2)$; the up-, down-, right- and left-arrows denote the actions of E_1, F_1, E_2 and F_2, respectively. The three shadowed domains (Fig. 2) cut out by three lines $l_1:\lambda_2-\lambda_1=\alpha p$, $l_2:\lambda-\lambda_2=\beta p$ and $l_3:\lambda_1=\gamma p(\alpha,\beta,\gamma\in Z^+)$ correspond to three invariant subspaces $S_3(1,\alpha):\{f_\lambda(\lambda_1,\lambda_2)|\lambda_2-\lambda_1\geqslant\alpha p\}$, $S_3(2,\alpha):\{f_\lambda(\lambda_1,\lambda_2)|\lambda-\lambda_2\geqslant\beta p\}$ and $S_3(0,\gamma):\{f_\lambda(\lambda_1,\lambda_2)|\lambda_1\geqslant\gamma p\}$, respectively. Their dimensions are $D_\delta=\frac{1}{2}(\lambda-\delta p+1)(\lambda-\delta p+2)(\delta=\alpha,\beta,\gamma)$.

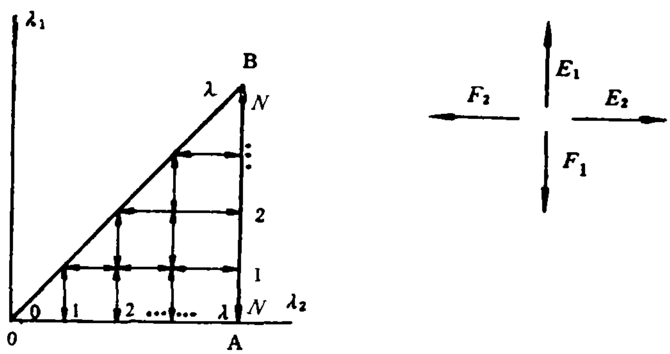

Fig. 1

Becuase the cross of the invariant subspaces is still an invariant subspace, we can define the following subspaces

$$Q_1=S_3(1,\alpha)\cap S_3(2,\beta)\cap S_3(0,\nu),$$
$$Q_2=S_3(1,\alpha)\cap S_3(2,\beta), \quad Q_3=S_3(2,\beta)\cap S_3(0,\nu),$$
$$Q_4=S_3(0,\nu)\cap S_3(1,\alpha).$$

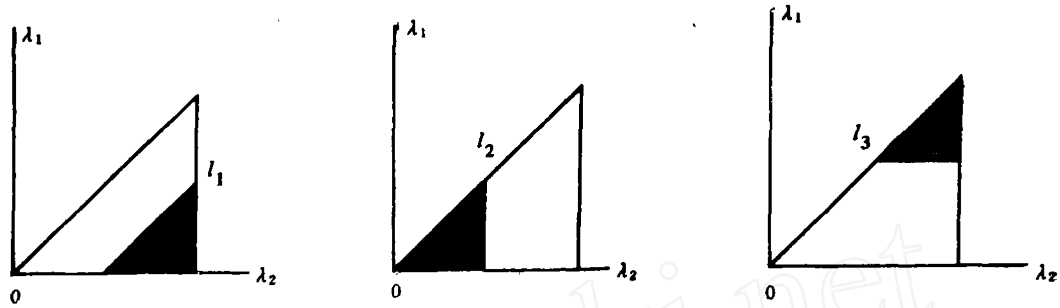

Fig. 2

According to whether Q_i is $\{0\}(i=1,2,3,4)$, the invariant subspaces are classified into six types corresponding to the following diagrams (Fig. 3).

When $p=3$ and $N=4$, the 15-dimensional representation, as an example, is written in an explicit form

$$\begin{cases} E_1 = E_{6,2} + E_{9,5} + E_{10,7} + E_{13,11} + [2](E_{7,3} + E_{11,8} + E_{14,12}) + E_{15,14}, \\ F_1 = E_{2,6} + E_{3,7} + E_{5,9} + E_{4,8} + [2](E_{7,10} + E_{8,11} + E_{9,12}) + E_{14,15}, \\ E_2 = E_{2,1} + E_{9,8} + E_{12,11} + E_{14,13} + E_{5,4} + [2](E_{4,3} + E_{8,7} + E_{11,10}), \\ F_2 = E_{1,2} + E_{4,5} + E_{6,7} + E_{10,11} + E_{13,14} + [2](E_{2,3} + E_{7,8} + E_{10,11}), \\ H_1 = -E_{2,2} - 2E_{3,3} - 3E_{4,4} - 4E_{5,5} + E_{6,6} - E_{8,8} - 2E_{9,9} + 2E_{10,10} \\ \qquad + E_{11,11} + 3E_{13,13} + 2E_{14,14} + E_{15,15}, \\ H_2 = -4E_{1,1} - 2E_{2,2} + 2E_{4,4} + 4E_{5,5} - 3E_{6,6} - E_{7,7} + E_{8,8} + 3E_{9,9} \\ \qquad - 2E_{10,10} + 2E_{12,12} - E_{13,13} + E_{14,14}. \end{cases} \tag{13}$$

It can be seen from its diagram representation that under the representation (13) there are three invariant subspaces $U_1:\{f_4(0,0), f_4(0,1), f_4(1,1)\}$, $U_2:\{f_4(0,3), f_4(0,4), f_4(1,4)\}$ and $U_3:\{f_4(3,3), f_4(3,4), f_4(4,4)\}$.

VI. Representations of $(A_1)_q$ and Their Lusztig's Extensions

Using the q-deformed boson realization of $(A_1)_q$:

$$J_+ \equiv E_1 = \hat{a}_1^+\hat{a}_2,\ J_- \equiv F_1 = \hat{a}_2^+\hat{a}_1,\ J_3 = H_1 = \hat{N}_1 - \hat{N}_2,$$

we obtain a representation

$$\begin{cases} J_\pm f_\lambda(n) = \dfrac{1}{2}\{(1\pm1)[\lambda-n] + (1\mp1)[n]\}f_\lambda(n\pm1), \\ J_3 f_\lambda(n) = (2n-\lambda)f_\lambda(n), \end{cases} \tag{14}$$

on the invariant subspace $V^{[\lambda]}:\{f_\lambda(n) = a_1^{+n}a_2^{+\lambda-n}|0\rangle \mid n=1,2,\cdots,\lambda\}$. Because there exist extreme vectors $f_\lambda(\alpha p)$ and $f_\lambda(\lambda-\beta p)$ (α, $\beta\in Z^+$ and $\beta\leqslant\lambda/p$) such that $J_+f_\lambda(\lambda-\beta p) = J_-f_\lambda(\alpha p) = 0$, there are two kinds of invariant subspaces $V_\alpha^{[\lambda]}$: $\{f_\lambda(\alpha p+n) \mid n=0,1,2,\cdots\}$ and $W_\beta^{[\lambda]}:\{f_\lambda(\lambda-\beta p-k) \mid k=0,1,2,\cdots,\lambda-\beta p\}$. According to the sense of $V_\alpha^{[\lambda]}\cap W_\beta^{[\lambda]}$, we can give a complete classification for the representations of $(A_1)_q$ at $q^p=1$.

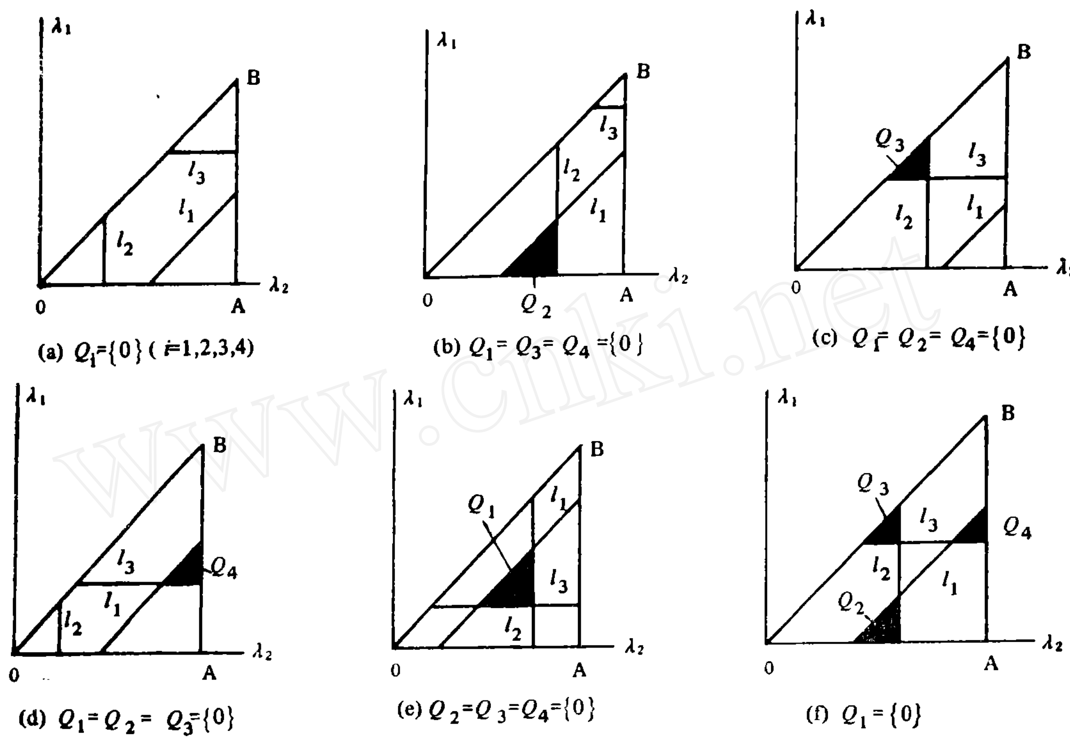

Fig. 3

Theorem 3. *The representation* (13) *of* $(A_1)_q$ *has three types of reduced decompositions at* $q^p=1$. *Type* I: *When* $\alpha p-1>\lambda-\beta p$, *representation is indecomposable for invariant subspaces* $V_\alpha^{[\lambda]}$ and $W_\beta^{[\lambda]}$; *Type* II: *When* $\alpha p-1=\lambda-\beta p$, *the representation is completely reducible*; *Type* III: *When* $\alpha p-1<\lambda-\beta p$, *the representation is indecomposable for the invariant subspace* $V_\alpha^{[\lambda]}\cap W_\beta^{[\lambda]}$.

Proof: (i) For type I, due to $V_\alpha^{[\lambda]}\cap W_\beta^{[\lambda]}=\{0\}$, the proof of Theorem 1 gives the result in this theorem.

(ii) For type II, due to $f_\lambda(\alpha p-1)=f_\lambda(\lambda-\beta p)$, we have $J_+f_\lambda(\alpha p-1)=J_+f_\lambda(\lambda-\beta p)=0$. Because $J_-f_\lambda(\alpha p)=0$, the chain of weight vectors $f_\lambda(0)$, $f_\lambda(1)$, $\cdots, f_\lambda(\alpha p-1), f_\lambda(\alpha p), \cdots$, $f_\lambda(\lambda)$ is broken down between $f_\lambda(\alpha p)$ and $f_\lambda(\alpha p-1)$, namely, $V^{[\lambda]}=V_\alpha^{[\lambda]}+W_\beta^{[\lambda]}$ and $V_\alpha^{[\lambda]}\cap W_\alpha^{[\lambda]}=\{0\}$ or $V^{[\lambda]}=V_\alpha^{[\lambda]}\oplus W_\beta^{[\lambda]}$. Accordingly, the representation (13) is completely reduced.

(iii) For type III, there exists an invariant subspace $V_\alpha^{[\lambda]}\cap W_\beta^{[\lambda]}=\{f_\lambda(\alpha, p), f_\lambda(\alpha p+1), \cdots, f_\lambda(\lambda-\beta p)\}$. A proof similar to that for Theorem 1 gives the conclusion in this theorem. Q.E.D.

According to the above theorem, when $p=3$, we analyse some typical lower dimensional representations of $(A_1)_q$:

1)
$$\lambda=3:\begin{cases}J_+=[2]E_{3,2}+E_{4,3},\ J_-=E_{1,2}+[2]E_{2,3},\\ J_3=-\frac{3}{2}E_{1,1}-\frac{1}{2}E_{2,2}+\frac{1}{2}E_{3,3}+\frac{3}{2}E_{4,4};\end{cases}\tag{15}$$

2) $\lambda=4$: $\begin{cases} J_+ = E_{2,1} + [2]E_{4,3} + E_{5,4}, \quad J_- = E_{1,2} + [2]E_{2,3} + E_{4,5}, \\ J_3 = -2E_{1,1} - E_{2,2} + E_{4,4} + 2E_{5,5}; \end{cases}$ (16)

3) $\lambda=5$: $\begin{cases} J_+ = [2]E_{2,1} + E_{3,4} + [2]E_{5,4} + E_{6,5}, \\ J_- = E_{1,2} + [2]E_{2,3} + E_{4,5} + [2]E_{5,6}, \\ J_3 = -\frac{5}{2}E_{1,1} - \frac{3}{2}E_{2,2} - \frac{1}{2}E_{3,3} + \frac{1}{2}E_{4,4} + \frac{3}{2}E_{5,5} + \frac{5}{2}E_{6,6}; \end{cases}$ (17)

4) $\lambda=6$: $\begin{cases} J_+ = [2]E_{3,2} + E_{4,3} + [2]E_{6,5} + E_{7,6}, \\ J_- = E_{1,2} + [2]E_{2,3} + [2]E_{4,5} + E_{5,6}, \\ J_3 = -3E_{1,1} - 2E_{2,2} - E_{3,3} + E_{5,5} + 2E_{6,6} + 3E_{7,7}. \end{cases}$ (18)

The above representations are illustrated as the following Fig. 4 where the up- and down-arrows denote respectively the actions of J_+ and J_- on the weight vectors $f_\lambda(n)$. It can be seen from Fig. 4 that the representations (15) and (16) belong to Type I and representations (17) and (18) to the Types II and III respectively.

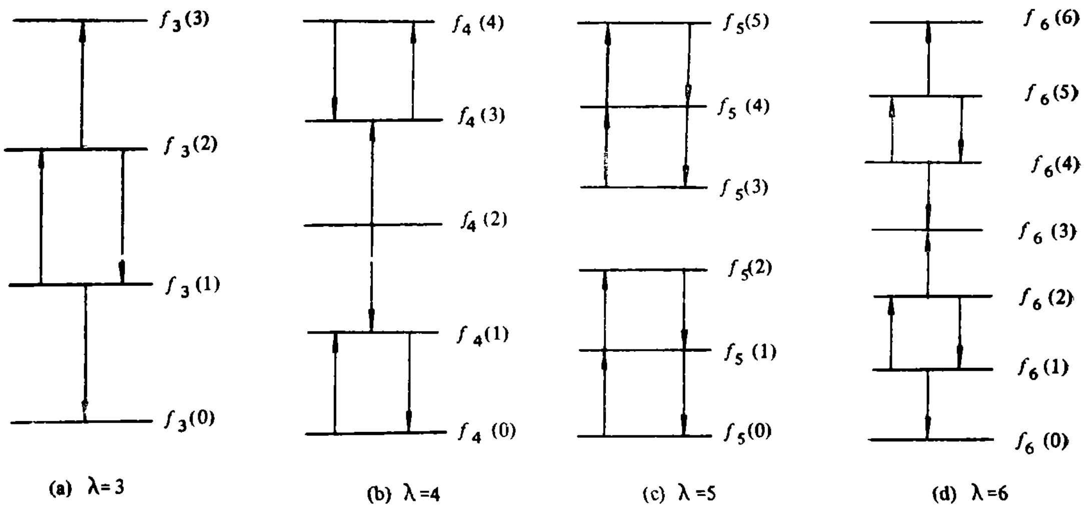

Fig. 4

Now, we study the Lusztig's extention of $(A_1)_q$. According to the PBW theorem[10] for $(A_1)_q$, $\forall x \in (A_1)_q$, $x = \sum_{m,n,r=0}^{\infty} C_{m,n,r} J_+^m J_-^n J_3^r$ where the coefficients C_{mnr} (m, n, $r \in Z^+$) are zero or complex numbers with finite norms. At present, we regard the base $J_+^m J_-^n J_3^r$ for $(A_1)_q$ as an operator on the representation space V of $(A_1)_q$ and then extend $(A_1)_q$ to include x such that norms of $C_{m,n,r}$ are infinite but x has finite limit on V. The extended $(A_1)_q$ is called Lusztig's extention denoted by $(\hat{A}_1)_q$. About $(\hat{A}_1)_q$ and its representations we have

Theorem 4. *On the representation space* $V^{[\lambda]}$ of $(A_1)_q$, *the Lusztig operators* $L_\pm = \lim_{q^p \to 1} \{(1/[p]!)J_\pm^p\}([p]! = [p][p-1]\cdots[2][1])$ have limits

$$L_-f_\lambda(n)=\begin{cases}0, & for\ n<p,\\ \alpha f_\lambda(n-p), & for\ n=\alpha p+n',\ \alpha(\in Z^+)\geqslant 1,\ 0\leqslant n'(\in Z^+)\leqslant p-1;\end{cases} \tag{19}$$

$$L_+f_\lambda(n)=\begin{cases}0, & for\ n>\lambda-p,\\ \beta f_\lambda(n+p), & for\ \lambda-n=\beta p+m,\ \beta(\in Z^+)\geqslant 1,\ 0\leqslant m'(\in Z^+)\leqslant p-1.\end{cases} \tag{20}$$

The Lusztig's extention $(\hat{A}_1)_q$ is generated by $J_\pm$, J_3 and $L_\pm$ and representation (14) of $(A_1)_q$ is naturally extended to a representation of $(\hat{A})_q$.

Proof. Using $[n]=[\alpha p+n']=[n']$ and $\lim\limits_{q^p\to 1}([\alpha p]/[p])=\alpha$, and starting from Eqs. (14), we can verify Eqs. (19) by direct calculation. Using

$$\rho^{[\lambda]}(L_\pm)=\lim_{q^p\to 1}\{(1/[p]!)(\rho^{[\lambda]}(J_\pm))^p\},$$

we can extend the representation of $(A_1)_q$ to that for $(\hat{A}_1)_q$. Q. E. D.

It is necessary to point out that representation (14) as a representation of $(\hat{A}_1)_q$ possesses reduction structure different from that for representation (14), a representation of $(A_1)_q$. This is because $L_\pm$ can mix the weight vectors $f_\lambda(n)$ and $f_\lambda(n\pm p)$ respectively in different $(A_1)_q$-invariant subspaces. For example, in representation (15), $L_+f_3(0)=f_3(3)$. Thus, L_+ mixes two one-dimensional invariant subspaces $\{f_3(0)\}$ and $\{f_3(0)\}$ so that only $\{f_3(0),\ f_3(3)\}$ is $(\hat{A}_1)$ q-invariant. For the same reason, (16)—(18) as representations of $(\hat{A}_1)_q$ have respectively the invariant mixed invariant subspaces $\{f_4(0),\ f_4(1),\ f_4(3),\ f_4(4)\}$, $\{f_5(0),\ f_5(1),\ f_5(2),\ \cdots, f_5(5)\}$ (the whole representation space for (17)) and $\{f_6(0),\ f_6(3), f_6(6)\}$.

The general Lusztig's extensions for $(C_l)_q$ and $(A_l)_q$ are quite complicated and we will write another paper on them.

We are grateful to Prof. Yang C. N., who instructed us to enter the active research field of the Yang-Baxter integrable systems and the relevant problems. We also thank Prof. Takhtajian L. A. and Smirnov F. for many useful discussions.

References

[1] Drinfeld, V. G., *Proc. IMC. Berkeley*, 1986, p. 798.
[2] Jimbo, M., *Lett. Math. Phys.*, **10**(1985), 63; **11**(1986), 247; *Commu. Math. Phys.*, **102**(1986), 537.
[3] Faddeev, L. D. et al., Quantization of Lie group and Lie algebra, *LOMI preprint*, 1987.
[4] Reshetikhin, N. Yu., LOMI preprint E-4 and E-11, 1987; Kirillov, A. N. & Reshetikhin, N. Yu., *LOMI preprint*, 1988.
[5] Takhtajian, L. A., Lecture on Quantum Group, in *Nankai Lectures on Mathematical Physics* (Ed. M. L. Ge), World Scientific, to appear.
[6] Majid, S., *Inter. J. Mod. Phys.*, **A5**(1990), 1.
[7] Yang, C. N., *Phys. Rev. Lett.*, **19**(1967), 1312; Baxter, R. J., *Exactly Solvable Models in Statistical Mechanics*, Academic, 1982.
[8] Yang, C. N., Ge, M. L. (Ed.), *Braid Group, Knot Theory and Statistical Mechanics*, World Scientific, 1989.
[9] Lusztig, G., *Adv. Math.*, **70**(1988), 237.

[10] Rosso, M., *Commu. Math. Phys.*, **117**(1988), 581; **124**(1989), 307.
[11] Lusztig, G., *Contemp. Math.*, **82**(1989), 59.
[12] Roche, P. & Arnaudon, D., *Lett., Math. Phys.*, **17**(1989), 295.
[13] Beidenharn, L. G., *J. Phys.*, **A22**(1989), L873.
[14] Sun, C. P. & Fu, H. C., *ibid.*, A22(1989), L983.
[15] Macfarlane, A. J., *ibid.*, **A22**(1989), 4551.
[16] Sun, C. P., *ibid.*, **A20**(1987), 4551, 5823, L1157.
[17] Sun, C. P. & Fu, H. C., *Il. Nuovo Cimento*, **B115**(1990), 1.
[18] Fu, H. C. & Sun, C. P., *J. Math. Phys.*, **31**(1990), 217.
[19] Hayashi, T., *Commu. Math. Phys.*, **127**(1990), 129.
[20] Sun, C. P. & Fu, H. C., *Commu. Theor. Phys.*, **19**(1990), 217.
[21] Chaichian, M. & Kulish, P., *Phys. Lett.*, **B234**(1990), 291; *ibid.*, **B234**(1990), 401.
[22] Floreanini, R. et al., *ibid.*, **B242**(1990), 383.
[23] Song, X. C., *CCAST preprint*, 90-7, 1990.
[24] Chang, Z., Chen, W. et al., *ASITP* Preprint, 90-16,21,22,23,33, 1990.
[25] Pasguier, V. & Saleur, H., *Nucl. Phys.*, **B330**(1990), 523.
[26] Reshetikhin, N. Yu. & Smirnov, F., *Harvard Univ. MP*, Preprint, 1990.
[27] Smirnov, F. & Takhtajian, L. A., *Colorado Univ. ANP Program*, Preprint, 1990.
[28] Sun, C. P. & Ge, M. L., *J. Math. Phys.*, **32**(1991), 595.
[29] Sun, C. P., Xue, K. et al., *J. Phys.*, **A24**(1991), L545.