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q-DEFORMED VERMA REPRESENTATIONS OF QUANTUM ALGEBRA $sl_q(3)$ AND NEW TYPE YANG-BAXTER MATRICES*

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ABSTRACT

The q-deformation of Verma theory for the Lie algebra is studied in this paper. The indecomposable representations and the induced representations of quantum universal enveloping algebra $sl_q(3)$ are constructed on the q-deformed Verma space and the quotient spaces respectively. We put stress on the discussion of the case in which q is a root of unity. Using the new representation constrained in the subalgebra $sl_q(2)$, we systematically construct the new series of solutions (Yang-Baxter matrices) for Yang-Baxter equation without spectral parameter.

Keywords: quantum universal enveloping algebra, Verma module, indecomposable representation, Yang-Baxter equation.

I. INTRODUCTION

The Yang-Baxter equation (YBE) plays a crucial role in the problems of nonlinear integrable systems and its studies have become one of the most active research fields in mathematics and physics[1]. The works of Drinfeid and Jimbo et al. show that the solutions of YBE are associated with the representation theory of quantum universal enveloping algebras (also called quantum algebra for short). In this way the standard solutions have been obtained by making use of the finite-dimensional irreducible representations of quantum algebra[2]. Other class of solutions different from the standard ones were obtained from the generalized Kauffman diagrame technique[3]. They are called the non-standard solutions and are still related to the representation of quantum algebras[4]. So it is necessary to further develop the representation theory of quantum algebras.

The purpose of this paper is to obtain new type representations of quantum algebra $sl_q(3)$ through the quantum generalization of the Verma theory for the Lie algebra. The representations obtained here are different from various irreducible representations given in Refs. [6—10]. By using them and the universal R-matrix, a new

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series of solutions for the YBE without the spectral parameter (braid group relations) is constructed. It is worth pointing out that the method for constructing representations of $sl_q(3)$ and $sl_q(2)$ explicitly in this paper can be generalized to other quantum algebras.

In this paper we put stress on the discussion of the case where q is a root of unity, i.e. $q^p = 1$. For convenience, we assume that $p = 3, 5, 7, \cdots$[6]. We denote the associative algebra (C-algebra) over the complex number field C generated by $x_i, y_i, \cdots (i = 1, 2, 3 \cdots N)$ by $\mathbb{C}\langle x_i, y_i \cdots | i = 1, 2, \cdots N\rangle$, and $\mathbb{Z} = \{0, \pm 1, \pm 2, \cdots\}$, $\mathbb{Z}^+ = \{0, 1, 2, \cdots\}$, $\mathring{\mathbb{Z}}^+ = \{1, 2, \cdots\}$.

II. Quantum Verma Module for $sl_q(3)$

The generators $E_i^\pm$ and H_i $(i = 1, 2)$ of quantum algebra $sl_q(3) = \mathbb{C}\langle E_i = E_i^+, F_i = E_i^-, H_i | i = 1, 2\rangle$ satisfy the q-deformed commutation relations

$$\begin{cases} [H_i, E_i^\pm] = \pm 2E_i^\pm, \ [H_i, E_j^\pm] = \mp E_j^\pm, \ (i \neq j), \\ [H_i, H_k] = 0, \ [E_k, F_k] = \delta_{ik}[H_i], \ i, j, k = 1, 2; \end{cases} \tag{1}$$

and the Serra relations

$$(E_i^\pm)^2 E_j^\pm - (q + q^{-1}) E_i^\pm E_j^\pm E_i^\pm + E_j^\pm (E_i^\pm)^2 = 0, i \neq j, i, j = 1, 2, \tag{2}$$

where we have defined $[f] = (q^f - q^{-f})/(q - q^{-1})$.

Considering that $E_1^\pm$ and $E_2^\pm$ respectively correspond to the simple root α_1 and α_2 of the classical Lie algebra A_2 under the classical limit $q \to 1$, we define the elements of $sl_q(3)$ corresponding to the third positive root $\alpha_1 + \alpha_2$:[7]

$$E_3^\pm = E_1^\pm E_2^\pm - q E_2^\pm E_1^\pm, \ E_3^+ = E_3, \ E_3^- = F_3.$$

Then, it follows from the q-deformed PBW theorem[7] that the basis for $sl_q(3)$ is determined by

$$\{E_1^{m_1} E_2^{m_2} E_3^{m_3} H_1^{s_1} H_2^{s_2} F_1^{n_1} F_2^{n_2} F_3^{n_3} | n_i, m_i, s_j \in \mathbb{Z}^+, m_i, n_i (i = 1, 2) = 1, 2, 3, s_1, s_2 = 1, 2\}.$$

Let $|\lambda\rangle$ be an extreme vector (the lowest weight vector) such that

$$H_i|\lambda\rangle = \lambda_i|\lambda\rangle, \ F_i|\lambda\rangle = 0, \ i = 1, 2.$$

The equations $\lambda(H_i) = \lambda_i \in \mathbb{C} (i = 1, 2)$ define a weight function $\lambda \in \mathscr{H}^*$ over the Cartan subalgebra $\mathscr{H} = \mathbb{C}\langle H_1, H_2\rangle$. Under the action of $sl_q(3)$, $|\lambda\rangle$ generates a linear spaces $V(\lambda) = sl_q(3)|\lambda\rangle$:

$$\mathrm{span}\{f_\lambda(m, n, k) = E_1^m E_2^n E_3^k |\lambda\rangle | m, n, k \in \mathbb{Z}^+\}.$$

This space is called q-(deformed) Verma space, on which the q-(deformed) Verma representation $\rho^{[\lambda]}: sl_q(3) \to \mathrm{end}(V(\lambda)): \rho^{[\lambda]}(g) f_\lambda(m, n, k) = g E_1^m E_2^n E_3^k |\lambda\rangle (\forall g \in sl_q(3))$ is naturally carried. Using the basic relations (1) and (2), we explicitly write these representations:

$$\begin{cases} H_1 f_\lambda(m, n, k) = (2m - n + k + \lambda_1) f_\lambda(m, n, k), \\ E_1 f_\lambda(m, n, k) = f_\lambda(m + 1, n, k), \\ F_1 f_\lambda(m, n, k) = q^{\lambda_1}[k] f_\lambda(m, n + 1, k - 1) \\ \qquad - [m][m - 1 - n + k + \lambda_1] f_\lambda(m - 1, n, k) \end{cases} \tag{3a}$$

$$\begin{cases} H_2 f_\lambda(m, n, k) = (2n - m + k + \lambda_2) f_\lambda(m,n,k), \\ E_2 f_\lambda(m, n, k) = q^{-m} f_\lambda(m, n+1, k) - q^{-n-1}[m] f_\lambda(m-1, n, k+1), \\ F_2 f_\lambda(m, n, k) = q^k [n][1 - \lambda_2 - n] f_\lambda(m, n-1, k) \\ \qquad - q^{1-\lambda_2-n}[k] f_\lambda(m+1, n, k-1). \end{cases} \tag{3b}$$

Due to the representation (3), $V(\lambda)$ can be regarded as an $sl_q(3)$-left module. We call it the quantum Verma module or q-deformed Verma module.

When q is not a root of unity ($q^p \neq 1, \forall p \in \mathring{Z}^+$), all the conclusions about repre sentations (3a),(3b) are quite similar to the classical case ($q = 1$):

(i) If $-\lambda_1 \notin Z^+$ or $-\lambda_2 \notin Z^+$, representations (3a), (3b) are irreducible.

(ii) $-\lambda_i \in Z^+ (i = 1,2)$ and λ_1 or $\lambda_2 \neq 0$, the representations (3a),(3b)are indecomposable. In this case, $\lambda = (\lambda_1, \lambda_2)$ is called dominant integral weight. However, if q is a root of unity, the problems in the cases (i) and (ii) appear very complicated and we will obtain some new solutions for YBE in this sense.

III. Representations on the Invariant Subspaces and Quotient Spaces

In this section we analyse the reduction of the q-Verma space when $q^p \neq 1$ and λ is a dominant integral weight. To this end, we try to find an extreme vector $|M\rangle \in V(\lambda)$ such that

$$F_i|M\rangle = 0, \ H_i|M\rangle = M_i|M\rangle, \ i = 1,2, \tag{4}$$

where we require that $M = (M_1 = M(H_1), M_2 = M(H_2)) \in \mathscr{H}^*$ is a dominant integral linear function "larger" than λ, that is to say, $M_1 - \lambda_1 > 0$ or $M_2 > \lambda_2$ for $M_1 = \lambda_1$. Then the standard cyclic module

$$S_M = sl_q(3)|M\rangle = \text{span } \{E_1^m E_2^k E_3^n |M\rangle | m,n,k \in Z^+\}$$

generated by $|\lambda\rangle$ is a non-trivial invariant subspace.

In the weight space $V[M_1, M_2]$: $\{|x\rangle \in V(\lambda) | H_i|x\rangle = M_i|x\rangle i = 1,2\}$ we make an ansatz:

$$|M\rangle = \sum_{k=0}^{\min(\alpha,\beta)} C_k f_\lambda(\alpha - k, \beta - k, k) \quad C_k \in C, \tag{5}$$

where the non-negative numbers

$$\alpha = \frac{1}{3}(2M_1 + M_2 - 2\lambda_1 - \lambda_2), \ \beta = \frac{1}{3}(2M_2 + M_1 - 2\lambda_2 - \lambda_1)$$

are the parameters of $V[M_1, M_2]$. Substituting (5) into (4) and using (3), we obtain two recurrence relations about C_k. The requirement that the two relations must be identical results in three classes of C_k. Accordingly, we obtain three extreme vectors:

$$|M(1)\rangle = f_\lambda(1 - \lambda_1, 0, 0),$$
$$|M(2)\rangle = f_\lambda(0, 1 - \lambda_{2,0}),$$
$$|M(3)\rangle = \sum_{k=3-\lambda_1,}^{2-\lambda_1-\lambda_2} \left\{ \prod_{i=3-\lambda_1}^{k} ([i]^{-1} q^{-\lambda_1} [3 - \lambda_1 - \lambda_2 - i][i + \lambda_1 - 2]) \right\}$$

$$\times f(2-\lambda_1-\lambda_2-k, 2-\lambda_1-\lambda_2-k, k). \quad (6)$$

The corresponding weights are $M(1) = (2-\lambda_1, \lambda_1+\lambda_2-1)$, $M(2) = (\lambda_1+\lambda_2-1, 2-\lambda_2)$ and $M(3) = (2-\lambda_2, 2-\lambda_1)$ respectively. Because $S_{M(1)}$ and $S_{M(2)}$ are still the invariant subspaces of $V(\lambda)$ and they are the largest normal subspaces generated by $E_1^{1-\lambda_1}$ and $E_2^{1-\lambda_2}$, the quotient space $\underline{V(\lambda)/(S_{M(1)}+S_{M(2)})}$ is finite-dimensional. When $q^p \neq 1$, $\rho(x)$ will induce finite-dimensional irreducible representation on it. Such representations corresponding to the case of Lie algebra can be described by the Gelfand's basis. A detailed discussion on it can be seen in Refs. [10,11].

Now, we study the representation on the quotient space $Q(\lambda) = V(\lambda)/J(\lambda) = \mathrm{span}\{\bar{f}_\lambda(m, n, k) = f_\lambda(m, n, k) \bmod J(\lambda) \mid m, k \in \mathbb{Z}^+, n = 0, 1, 2, \cdots, -\lambda_2\}$ corresponding to invariant subspace $J(\lambda) = \mathrm{span}\{f_\lambda(m, 1-\lambda_2+n, k) \mid m, n, k \in \mathbb{Z}^+\}$. This representation is

$$\begin{cases} H_1\bar{f}_\lambda(m, n, k) = (2m-n+k+\lambda_1)\bar{f}_\lambda(m, n, k), \\ E_1\bar{f}_\lambda(m, n, k) = \bar{f}_\lambda(m+1, n, k), \\ F_1\bar{f}_\lambda(m, n, k) = \theta(-\lambda_2-1-n)q^{\lambda_1}[k]\bar{f}_\lambda(m, n+1, k-1) \\ \qquad -[m][m-1-n+k+\lambda_1]\bar{f}_\lambda(m-1, n, k), \end{cases} \quad (7a)$$

$$\begin{cases} H_2\bar{f}_\lambda(m, n, k) = (2n-m+k+\lambda_2)\bar{f}_\lambda(m, n, k), \\ E_2\bar{f}_\lambda(m, n, k) = \theta(-\lambda_2-1-n)q^{-m}\bar{f}_\lambda(m, n+1, k) \\ \qquad -q^{-n-1}[m]\bar{f}_\lambda(m-1, n, k+1), \\ F_2\bar{f}_\lambda(m, n, k) = q^k[n][1-\lambda_2-n]\bar{f}_\lambda(m, n-1, k) \\ \qquad -q^{1-\lambda_2-n}[k]\bar{f}_\lambda(m+1, n, k-1), \end{cases} \quad (7b)$$

where $\theta(x) = 0 (x < 0)$ and $\theta(x) = 1 \ (x \geqslant 0)$.

It can be seen from (7a) and (7b) that the quotient module $\pi(\lambda_1) = Q(\lambda)|_{\lambda_2=0}/S[\lambda_1] = \mathrm{span}\{F(m, k) = \bar{f}_\lambda(m, 0, k)|_{\lambda_2=0} \bmod S[\lambda] \mid 0 \leqslant m+k \leqslant -\lambda_1\}$ is finite-dimensional and the dimension is

$$\dim\pi(\lambda_1) = \frac{1}{2}(1-\lambda_1)(2-\lambda_2), \quad (8)$$

where the $sl_q(3)$ is the left module. $Q(\lambda)|_{\lambda_2=0}$ can be regarded as an invariant submodule $S[\lambda_1]: \{\bar{f}_\lambda(m, O, k) \mid m+k \geqslant 1-\lambda_1\}$.

The quotient representation on $\pi(\lambda_1)$ is

$$\begin{cases} H_1F(m, k) = (2m+k+\lambda_1)F(m, k), \\ E_1F(m, k) = \theta(-\lambda_1-1-m-k)F(m+1, k), \\ F_1F(m, k) = [m][1-\lambda_1-m-k]F(m, k-1), \end{cases} \quad (9a)$$

$$\begin{cases} H_2F(m, k) = (k-m)F(m, k), \\ E_2F(m, k) = -q^{-1}[m]F(m-1, k+1), \\ F_2F(m, k) = -q[k]F(m+1, k-1). \end{cases} \quad (9b)$$

It can be proved that this representation is irreducible when $-\lambda_1 \in \mathbb{Z}^+$, and $q^p \neq 1$.

IV. Representations of $sl_q(3)$ at Root of Unity

When q is a root of unity, equation $q^p = 1$ leads to $[\alpha p] = 0$ $(\forall \alpha \in Z)$ and there exist extreme vectors $f_\alpha \equiv f(\alpha_1 p, \alpha_2 p, \alpha_3 p)(\alpha_1, \alpha_2, \alpha_3 \in Z^+)$ satisfying

$$\begin{cases} H_1 f_\alpha = \{(2\alpha_1 - \alpha_2 + \alpha_3)p + \lambda_1\} f_\alpha, \\ H_2 f_\alpha = \{(2\alpha_2 - \alpha_1 + \alpha_3)p + \lambda_2\} f_\alpha, \\ F_i f_\alpha = 0, \quad i = 1, 2. \end{cases}$$

Under the action of $sl_q(3)$, each vector f_α generates an invariant subspace $S^\alpha = S(\alpha_1, \alpha_2, \alpha_3) = \text{span}\ \{f_\lambda(m, n, k) | m \geqslant \alpha_1 p,\ n \geqslant \alpha_2 p,\ k \geqslant \alpha_3 p\}$. The corresponding quotient space $\Omega^\alpha = \Omega(\alpha_1, \alpha_2, \alpha_3) = V(\lambda)/S^\alpha = \text{span}\{\omega(m, n, k) = f_\lambda(m, n, k)\text{mod}S^\alpha\}$ is finite-dimensional and its dimension is $\dim\Omega^\alpha = \alpha_1\alpha_2\alpha_3 p$. On Ω^α, the Verma representations (3a),(3b) induce a finite-dimensional representation. The action of this representation on the base $\omega(m, n, k)$ is formally the same as that of Verma representation on the base $f_\lambda(m, n, k)$. Thus, we need not write it here.

Now, we study the representation with two labels at $q^p = 1$. Because $q^p = 1$, $F_1 F(\beta_1 p, k) = E_2 F(\beta_2 p, k) = F_2 F(m, \beta_2 p) = 0(\beta_1, \beta_2, m, k \in Z^+)$ and thus there exist two invariant subspaces:

$$U(\beta_1) = \text{span}\{F(m, k) \in \pi(\lambda_1) | m \geqslant \beta_1 p\},$$
$$U(\beta_2) = \text{span}\{F(m, k) \in \pi(\lambda_2) | k \geqslant \beta_2 p\},$$

when $(\beta_1 + \beta_2)p \leqslant -\lambda_1$, $U(\beta_1) \cap U(\beta_2) \neq \{0\}$, i.e. $U(\beta_1) \cap U(\beta_2)$ is a nontrivial invariant subspaces. On $U(\beta_1)$, $U(\beta_2)$ and $U(\beta_1) \cap U(\beta_2)$, the representations (9a),(9b) will define various subrepresentations.

For example, when $\lambda_1 = 3p$ and $p = 3$, $U(1) \cap U(2)$ is a 10-dimensional invariant subspace. On its ordered basis $\{F(3,3), F(4,3), F(5,3), F(6,3), F(3,4), F(4,4), F(5,4), F(3,5), F(4,5), F(3,6)\}$, we explicitly write this representations as

$$\begin{cases} H_1 = \text{diag}(18, 20, 22, 24, 19, 21, 23, 20, 22, 21), \\ H_2 = \text{diag}(3, 3, 3, 3, 4, 4, 4, 5, 5, 6), \\ E_1 = e_{2,1} + e_{3,2} + e_{4,3} + e_{6,5} + e_{7,6} + e_{9,8}, \\ E_2 = -q^{-1}(e_{5,2} + [2]e_{6,3} + e_{8,6} + [2]e_{9,7} + e_{10,9}), \\ F_1 = -[2]e_{2,3} - e_{5,6} - [2]^2 e_{6,7} - [2]e_{8,9}, \\ F_2 = -q(e_{2,5} + e_{3,6} + e_{7,4} + [2]e_{6,8} + [2]e_{7,9}), \end{cases} \tag{10}$$

where $e_{i,j}$ $(i, j = 1, 2, \cdots, 10)$ are the matrix unit so that $(e_{i,j})_{kl} = \delta_{ik}\delta_{jl}$.

In fact, it can be proved that the representations (3a), (3b), (7a), (7b), are indecomposable when $q^p = 1$,and the representation (g, a, b) is also indecomposable when $-\lambda_1 > p$. All the proofs for the indecomposable properties of representations are given in Appendix.

V. Representations of Subalgebra $sl_q(2)$

Now we study the infinite-dimensional representation $\Gamma^{(\lambda)} = \rho^{[\lambda]} | sl_q(2)$ of $sl_q(2)$

which is obtained by restricting representation $\rho^{[\lambda]}$ of $sl_q(3)$ on its subalgebra $sl_q(2)=\mathbb{C}\langle E_1, F_1, H_1\rangle$. In fact. Eqs. (3a) have given its explicit expression.

Because the sum $n+k$ of the labels n and k for $f_\lambda(m, n, k)$ is invariant under the action of $sl_q(2)$, $W(N)=\text{span}\{f_\lambda(m, n, k)\,|\,n+k=N\}$ for a given $N\in\mathbb{Z}^+$ is an invariant subspace with the basis $\chi_N(m,s)=f_\lambda(m,s,N-s)$ $(s=0,1,2,\cdots,N)$. It follows from (3a) that an infinite-dimensional representation of $sl_q(2)$ is defined by

$$\begin{cases} H_1\chi_N(m, s)=(2m-2s+N+\lambda_1)\chi_N(m, s), \\ E_1\chi_N(m, s)=\chi_N(m+1, s), \\ F_1\chi_N(m, s)=q^{\lambda_1}[N-s]\chi_N(m, s-1) \\ \qquad -[m][m+N-1-2s+\lambda_1]\chi_N(m-1,s). \end{cases} \tag{11}$$

Now we analyse the properties of representation (11). Because $[\nu_i p]=0$ $(i=1, 2, \nu_i\in\mathbb{Z}^+)$, there exist three kinds of invariant subspaces:

$$\begin{cases} W_1(N, \nu_1)=\text{span}\{\chi_N(m, N-\nu_1 p-s)\,|\,s=0,1,\cdots,N-\nu_1 p, m\in\mathbb{Z}^+\}, \\ W_2(N,\nu_2)=\text{span}\{\chi_N(\nu_2 p+m,s)\,|\,m,s\in\mathbb{Z}^+\}, \\ W_3(N, \nu_2)=W_1(N, \nu_1)\cap W_2(N, \nu_2). \end{cases}$$

Because $W_3(N, \nu_1, \nu_2)$ is an invariant subspace of $W_1(N, \nu_1)$, the quotient space $\Omega_N(\nu_1, \nu_2)=W_1(N, \nu_1)/W_3(N, \nu_1, \nu_2)=\text{span}\ \{\bar{\chi}_N(m, s)=\chi_N(m, s) \bmod W_3(N, \nu_1, \nu_2)\,|\,m=0, 1, \cdots, \nu_2 p-1, s=0, 1, 2, \cdots, N-\nu_1\}$ is finite-dimensional and its dimension is $\dim\Omega_N(\nu_1, \nu_2)=(\nu_1 p-1)\ (\nu_2 p-1)$. On $\Omega_N(\nu_1, \nu_2)$, the representation (11) induces a finite- dimensional representation as follows:

$$\begin{cases} H_1\bar{\chi}_N(m, s)=(2m-2s+N+\lambda_1)\bar{\chi}_N(m, s), \\ E_1\bar{\chi}_N(m, s)=\theta(\nu_1 p-2-m)\bar{\chi}_N(m+1, s), \\ F_1\bar{\chi}_N(m, s)=q^{\lambda_1}[N-s]\bar{\chi}_N(m,s-1)-[m][m-1+\lambda_1 \\ \qquad -2s+N]\bar{\chi}_N(m-1,s), \end{cases} \tag{12}$$

when ν_1 or $\nu_2\geqslant 2$, this representation is indecomposable.

We study the representation with single label as follows. Denote the basis for $W(0)$ by $\phi(m)=\chi_0(m, 0)$ and let $\mu=-\lambda_1(\geqslant 0)$. Then we obtain a representation of $sl_q(2)$ on $W(O)$:

$$\begin{cases} H_1\phi(m)=(2m-\mu)\phi(m), \\ E_1\phi(m)=\phi(m+1), \\ F_1\phi(m)=[m][1+\mu-m]\phi(m-1). \end{cases} \tag{13}$$

In fact, the above representation is just the representation of $sl_q(2)$ on its own Verma spaces and its properties have been discussed when μ is an integer[9]. Now, we discuss the case where μ is not an integer.

Because $F_1\phi(\alpha p)=0(\alpha\in\mathbb{Z}^+)$, $\mathscr{H}(\alpha, p)=\text{span}\{\phi(\alpha p+n)\,|\,n\in\mathbb{Z}^+\}$ is an $sl_q(2)$-invariant subspace. On the corresponding quotient space $Q_p^\alpha=W(O)/\mathscr{H}(\alpha, p)=\text{span}\{\phi_J(M)=\phi(J+M) \bmod \mathscr{H}(\alpha, p)\,|\,M=J, J-1, \cdots, -J\}$ for given $J=1/2(\alpha p-1)$, representation (13) induces a finite-dimensional representation $\mathscr{T}^{[J]}$:

$$\begin{cases} H_1\phi_J(M) = (2(J+M) - \mu)\phi_J(M), \\ E_1\phi_J(M) = \theta(J-1-M)\phi_J(M), \\ F_1\phi_J(M) = [J+M][\mu+1-J-M]\phi_J(M-1). \end{cases} \tag{14}$$

This representation includes an arbitrary complex parameter μ independent of the dimension of the representation. This is the key to the construction of new R-matrix. When $\alpha \geqslant 2$, like representation (13) this representation is indecomposable.

VI. The New Type Yang-Baxter Matrices

According to Drinfeld[3], the R-matrix (Yang-Baxter matrix)

$$\check{R}^{\lambda_1\lambda_2} = P\rho^{[\lambda_1]}\otimes\rho^{[\lambda_2]}(\mathscr{R}) = P \cdot R^{\lambda_1\lambda_2}$$

satisfies the parameter-free Yang-Baxter equation (braid group relation)

$$(\check{R}^{\mu\lambda}\otimes I)(I\otimes\check{R}^{\nu\lambda})(\check{R}^{\nu\mu}\otimes I) = (I\otimes\check{R}^{\nu\mu})(\check{R}^{\nu\lambda}\otimes I)(I\otimes\check{R}^{\mu\lambda}). \tag{15}$$

It is constructed in terms of the universal R-matrix $\mathscr{R} = \sum_a e_a\otimes e^a$ and the representation $\rho^{[\lambda_i]}$ $(\lambda_i = \mu, \nu, \lambda, i = 1, 2)$ of quantum algbra $\bar{u}_q(g)\{e_a, e^a | a = 1, 2, \cdots\infty\}$. P is such a permutation operator that $P(x_1\otimes x_2) = x_2\otimes x_1 (\forall x_i \in V^{[\lambda_i]})$. Because the representations used in the previous works were irreducible, the obtained R-matrices are limited on number, and can be expressed in terms of $q-C-G$ coefficients[2]. We call these R-matrices standard ones.

In the following, we use the new type representation (14) of $sl_q(2)$ and the universal $sl_q(2)$-R-matrix:

$$R = q^{H_1\otimes H_1/2}\sum_{n=0}^{\infty}\frac{(1-q^{-2})^n}{[n]!}(q^{H_1/2}E_1\otimes q^{-H_1/2}F_1)^n q^{n(n-1)/2} \tag{16}$$

to construct new type R-matrices. Here, although q, as a cyclic parameter, is not continuous, these new solutions still contain the continuous parameter similar to q in the standard R-matrices due to the arbitrary parameter λ. The new type R-matrix is defined by

$$R^{J_1J_2} = \mathscr{F}^{[J_1]}\otimes\mathscr{F}^{[J_2]}(\mathscr{R}) \in \mathrm{End}(Q_p^a\otimes Q_p^a), \tag{17}$$

and

$$R^{J_1J_2}(\phi_{J_1}(M_1)\otimes\phi_{J_2}(M_2)) = \sum_{M_1', M_2'}(R^{J_1J_2})_{M_1M_1}^{M_1'M_2'}\phi_{j_1}(M_1)\otimes\phi_{J_2}(M_2)$$

gives an explicit expression for $R^{J_1J_2}$

$$\begin{aligned}(R^{J_1J_2})_{M_1M_2}^{M_1', M_2'} = {} & q^{1/2(2J_1+2M_1-\mu)(2J_2+2M_2-\mu)}\delta_{M_1}^{M_1'}\delta_{M_2}^{M_2'} \\ & + \theta(M_2 - M_2' - 1)q^{1/2}(M_2 - M_2')(3M_2 \\ & - 3M_2' + J_1 + M_1 - J_2 - M_2) \\ \delta_{M_1+M_2}^{M_1'+M_2'} & \prod_{k=0}^{M_2-M_2'-1}\{[J_2 + M_2 - k][\mu + 1 - J_2 - M_2 + k]\}. \end{aligned} \tag{18}$$

The δ-symbol $\delta_{M_1+M_2}^{M_1'+M_2'}$ shows that the new R-matrix (18) still satisfies "weight conservation" (label conservation) and enables R-matrices to possess the diagonal block structure. The symbol $\theta(M_2 - M_2' - 1)$ enables R-matrices to possess the "quasi-triangle" structure. Let $t = q^{1-\mu}$ and $\omega = q - 1$. It is observed that the R-matrices given by (18) include the continuous parameter t and the cyclic parameter ω. Therefore, when μ is not an interger, the new R-matrices given by (18) do not reduce to the standard solutions.

For example, when $p = 3$, $J_1 = J_2 = 1$, we obtain a 9×9 R-matrix: $R^{11} =$ diag. block (A_1, A_2, A_3) as follows.

$$A_1 = \begin{bmatrix} t & 0 & 0 \\ 0 & 1 & t - t^{-1} \\ 0 & 0 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} \omega^2 t^{-2} & \omega t^{-3} t^{-1} & 0 \\ 0 & \omega^2 t^{-1} & 0 \\ 0 & 0 & \omega t^{-3} \end{bmatrix},$$

$$A_3 = \begin{bmatrix} t & \omega t^{-2} - 1 & (t - t^{-1})(1 - \omega t^{-2}) \\ 0 & \omega t^{-1} & \omega^2 t^{-1}(t - t^{-1}) \\ 0 & 0 & t^{-1} \end{bmatrix}. \tag{19}$$

This is a new solution different from the standard ones. Using the extended Kauffmon diagram theory, we can directly check that R^{11} defined by Eq. (19) indeed satisfies the YBE.

The representation theory of quantum algebra at $q^p = 1$ has been well discussed more recently[12-14], but the concrete method constructing the representations in explicit form is built in this paper. This method can be used for $sl_q(n)$ $(n \geqslant 3)$ and the results will be published elsewhere.

Apendix

The Proof for the Indecomposable Properties of Representations

The representations (3), (7a),(7b), (9), (11), (12), (13) and (14) are usually indecomposable. This conclusion can be universally proved by the following proposition.

Let $V = \text{span}\{v(\boldsymbol{m}) = v(m_1, m_2, \cdots m \mid m_i \in \mathbb{Z}^+, i = 1, 2, \cdots, 1\}$ be a representation space for an associative algebra A. The basis $v(\boldsymbol{m})$ is graded by a quasi-positive integral linear function $L(\boldsymbol{m})$ over $\mathbb{Z}^{+l}$. Accordingly, $V = \sum_{N=0}^{M} V^{[N]}$, $V^{[N]} = \text{span}\{v(\boldsymbol{m}) \mid L(\boldsymbol{m}) = N\}$ where M may be infinite. Define $W^{[k']} = \sum_{N=K'}^{M} V^{[N]}$, then we have

Proposition. *For a given $k \in \mathbb{Z}^+$, if $W^{[k]}$ is an A-invariant subspace in V and there exists $b \in A$ such that $bv(\boldsymbol{m}) \in W^{[k+1]}$ for any $v(\boldsymbol{m}) \in v^{[k']}$ and $k' \in \mathbb{Z}^+$, then the representation of A over V is indecomposable (reducible, but not completely reducible).*

Proof. Assume V are completely reducible, then there exist an invariant complementary space $\overline{W}^{[k]}$ such that

$$V = W^{[k]} \oplus \overline{W}^{[k]}. \tag{A1}$$

In this case, there must be a vector $v = v_1 + v_2$ in $W^{[k]}$ so that $v_1(\neq 0) \in W^{[k']}(k' \leqslant k-1)$ and $v_2(\neq 0) \in W^{[k+1]}$. Due to the conditions for the proposition,

$$b^n v^{[k']} \subset W^{[k'+1]}, \quad b^n W^{[k'+1]} \subset W^{[k'+1+n]},$$

that is to say, there exists a non-zero vector $b^{k'-k} v(m)$:

$$b^{k'-k} v(\boldsymbol{m}) \in w^{[k]}. \tag{A2}$$

Because of (A_1), $\overline{W}^{[k]}$ is invariant, i.e.

$$b^{k'-k} v(\boldsymbol{m}) \in \overline{W}^{[k]}. \tag{A3}$$

However, (A1) implies that $W^{[k]} \cap \overline{W}^{[k]} \neq \{0\}$. Because (A1) and (A2) imply that $W^{[k]} \cap \overline{W}^{[k]} \neq \{0\}$, the contradictory appears. Then the proposition is proved.

Now we apply the above proposition to representation (12). The graded structure of $V(\lambda)$ is

$$V(\lambda) = \sum_{n=0}^{\infty} V^{[n]}(\lambda),$$

$$V^{[n]}(\lambda) = \text{span}\{f(m,n,k) \mid m,\ k \in \mathbb{Z}^+\}.$$

It follows from (12) that the subspaces

$$w^{[\lambda_1+1]} = J(\lambda) = \sum_{n=1-\lambda}^{\infty} V^{[n]}(\lambda) = \text{span }\{f(m, \lambda_1 + 1 + n, k) \mid m, n, k \in \mathbb{Z}^+\}$$

is $sl_q(2)$-invariant. Making use of the following properties of

$$W_\lambda^{[k]} = \sum_{n=k}^{\infty} V^{[n]}(\lambda),$$

$$F_1 v^{[k]}(\lambda) \subset W_\lambda^{[k+1]}, \quad E_2 v^{[k]}(\lambda) \subset W_\lambda^{[k+1]},$$

we prove that the representation defined by Eq.(12) is indecomposable.

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References

[1] Yang, C. N.. *Phys. Rev. Lett.*, **19**(1967), 1312; Baxter, R. J., *Ann. Phys.*, **70**(972), 193; **76**(1973), 1,25,48; Yang, C. N. & Ge, M. L. (Ed.) *Braid Group, Knot Theory and Statistical Mechanics*, World Scientific, 1989.

[2] Drinfeld, V. G., *Proc. IMC.*, Berkeley, 1986, p. 798; Jimbo, M., *Lett. Math. Phys.*, **10**(1985),63; **11**(1986), 247; Yu, R., *Reshetikhin, LOMI preprint*, 1987, E-4, E-11.

[3] Ge, M. L. et al., *Inter. J. Mod. Phys.*, **A4**(1989), 3351; *J. Phys.* **A23**(1990), 605, 619; Ge. M. L. & Xue, K., *J. Math. Phys.*, **34**(1991), in press.

[4] Ge, M. L., Sun, C. P. et al., *J. Phys.*, **A23**(1990), L645.

[5] Verma, D. N., *Bull. Amer. Math. Soc.*, **74**(1968), 160.

[6] Lustztig, G., *Adv. Math.*, **70**(1988), 237; *Contemp. Math.*, **82**(1989), 59; Roche, P. R. & Araudo, L., *Lett. Math. Phys.*, **17**(1989), 295.

[7] Rosso, J., *Commun. Math. Phys.*, **117**(1988), 251; **124**(1989), 307.

[8] Biedenharn, L. C., *J. Phys.*, **A22**(1989), L873; Sun, C. P. & Fu, H. C., *J. Phys.*, **A22**(1989), L983, Macfarlane, A. J., *J. Phys.*, **A22**(1989), 4551; Sun, C. P. & Fu, H. C., *Commun. Theor.*

Phys., **13**(1990), 217; Kulish, P. et al., *Phys. Lett.*, **B234**(199), 291, 401; Song, X. C., *J. Phys.*, **A23**(1990), L821; Guo, H. Y. et al., *J. Phys.*, **A23**(1990), 4185; Hou, B. Y. et al., *Commun. Tehor. Phys.*, **13**(1990), 181.

[9] Sun, C. P., Lu, J. F. & Ge, M. L., *J. Phys.*, **A23**(1990), 1199; Sun, C. P. & Ge, M. L., *J. Math. Phys.*, **32**(1991), 595.

[10] Ma Z. Q., *J. Math. Phys.*, **33**(1990), in press; Hou, B. Y. et al., *Comm. Theor. Phys.*, **13**(1990), 71; Liao, L. & Song, X. C., *Comm. Theor. Phys.*, **13**(1990), 209.

[11] Ueno, K. et al., *Lett. Math· Phys.*, **18**(1989), 215.

[12] De Concini & Kac, V. G., preprint, 1990.

[13] Jimbo, M. et al., *RIMS*, **703**(1990).

[14] Parshell, B. & Wang, J. P., *AMS Meoirs*, 1990, 439.