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Exactly-Solvable Dynamics of Neutron in a Helical Magnetic Field and Geometric Effects in the Limit of Strong Field

Xiao-Guang Wang and Chang-Pu Sun

Institute of Theoretical Physics, Northeast Normal University, Changchun 130024, Peoples' Republic of China

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Abstract

By invoking the use of a space–spin coupling transformation, the exact Schrödinger evolution wavefunction is fortunately obtained for a neutron moving in a helical magnetic field. This result just enjoys the effect of Berry's phase or its induced gauge field in Bitter–Dubbers experiment as its limiting case with strong field. It is also predicted from the exactly-dynamical analysis that a pulse of polarized neutrons with monotonous momentum will be split into two pulses with different momentum shifts accompanying the spin-down and spin-up states respectively.

1. Introduction

It is well known that the features of processing neutrons in an external magnetic field can serve for the verification of some theoretical predictions in quantum mechanics. Recently, the polarization of neutrons in a helical inhomogeneous magnetic field [1] and in a slowly-changing magnetic field [2] has been used experimentally to test Berry's remarkable discovery [3] in modern quantum theory. For the case of the helical magnetic field in Bitter–Dubbers (B–D) experiment [1], the geometrical phase effect can be understood in terms of Berry's phase in a moving frame of reference along with the neutron [1] or the Aharonov–Bohm (A–B) phase of induced gauge field [4] through the adiabatic approximation or Born–Oppenheimer (B–O) approximation respectively [5]. Notice that the above-mentioned approximations can work effectively only for an extremely cold neutron that moves slowly enough, so that the adiabatic condition is satisfied. It is natural to question whether the solution to the problem can be obtained even for the case without adiabatic approximation, that is to say, whether we can find an exact solution of the Schrödinger equation for the neutron in an helical magnetic field. Such a solution is much appreciated to test the validity of the B–O approximation quantitatively.

To solve this problem, we consider a more conventional problem in parallel with the above-mentioned one. For the case of a slowly-changing magnetic field, if the field changes harmonically, the Schrödinger equation can be solved exactly according to the Rabi rotation [6] and hereby the Berry's phase appears in the adiabatic limit of the exact solution with a very strong field [7, 8]. Notice that this case is time-dependent but not space-dependent. For the space-dependent case, the exact solution is also expected to be found for the precession of neutrons in a harmonically-inhomogeneous (helical) magnetic field. Fortunately a space–spin coupling transformation can be found to separate the transformed Hamiltonian into two commuting parts, each of them can be solved exactly. This is the key point to the present studies.

2. Exactly solution for spin-$\frac{1}{2}$

In the helical external magnetic field

$$\boldsymbol{B}(z) = B\,[\sin\theta\cos(kz), \sin\theta\sin(kz), \cos\theta], \quad k = \frac{2\pi}{L}$$

where L is the length of the domain of $\boldsymbol{B}(z)$, the Hamiltonian of a single neutron for the B–D experiment is

$$\hat{H} = \frac{\hat{p}_z^2}{2m} + \tfrac{1}{2}g\boldsymbol{B}(\hat{z})\cdot\boldsymbol{\sigma}, \tag{1}$$

where $\boldsymbol{\sigma}$ is the Pauli matrix, g the coupling constant and $\hat{p}$ the z component of momentum satisfying

$$[\hat{p}, \hat{z}] = i\hbar$$

To diagonalize $\hat{H}$, we introduce a unitary transformation $\hat{W}(\hat{z})$ with space–spin coupling

$$\hat{W}(\hat{z}) = \exp\left(-i\tfrac{1}{2}k\hat{z}\sigma_z\right) \tag{2}$$

in the solution $|\psi\rangle = \hat{W}(\hat{z})|\chi\rangle$ of the Schrödinger equation $\hat{H}|\psi\rangle = E|\psi\rangle$. It is not difficult to prove that $|\chi\rangle$ satisfies an effective Schrödinger eigenequation by an effective Hamiltonian

$$\hat{H}_e = \frac{\hat{p}_z^2}{2m} + \frac{\hbar^2k^2}{8m} + \hbar\omega\,[\cos\alpha\sigma_z + \sin\alpha\sigma_x] \tag{3}$$

where

$$\hbar\omega = \hbar\omega(\hat{p}_z) = \tfrac{1}{2}gB\left[1 - \frac{2\hat{p}_z\hbar k}{mgB}\cos\theta + \frac{\hat{p}_z^2\hbar^2k^2}{m^2g^2B^2}\right]^{1/2}, \tag{4}$$

$$ctg\alpha = ctg\alpha(\hat{p}_z) = ctg\theta - \frac{\hbar k\hat{p}_z}{mgB\sin\theta}. \tag{5}$$

Because the coordinate operator $\hat{z}$ does not appear in the above effective Hamiltonian $\hat{H}_e$, $\hat{H}_e$ is obviously diagnalized in the momentum representation. It immediately follows from this fact that the eigenstates of $\hat{H}_e$ are

$$|\hat{p}_z, \chi_1(\hat{p}_z)\rangle = |\hat{p}_z\rangle \otimes |\chi_1(\hat{p}_z)\rangle = |\hat{p}_z\rangle \otimes \begin{pmatrix}\cos(\alpha/2)\\ \sin(\alpha/2)\end{pmatrix}, \tag{6}$$

$$|\hat{p}_z, \chi_2(\hat{p}_z)\rangle = |\hat{p}_z\rangle \otimes |\chi_2(\hat{p}_z)\rangle = |\hat{p}_z\rangle \otimes \begin{pmatrix}\sin(\alpha)/2\\ -\cos(\alpha/2)\end{pmatrix} \tag{7}$$

which correspond to the eigenvalues

$$E_\pm = \frac{\hat{p}_z^2}{2m} + \frac{\hbar^2k^2}{8m} \pm \hbar\omega \equiv \varepsilon_0 \pm \hbar\omega, \tag{8}$$

respectively. They are two-fold degenerate for $\pm p$.

If a neutron with the space wavefunction

$$|\phi(0)\rangle = \int \phi(p)|p\rangle \, dp$$

is initially polarized in the spin-up state

$$|\uparrow\rangle = \begin{pmatrix}1\\0\end{pmatrix},$$

the initial condition for the total wavefunction is

$$|\psi(0)\rangle = \int dp\phi(p)|p\rangle \otimes |\uparrow\rangle. \tag{9}$$

For the effect Schrödinger equation

$$i\hbar \frac{\partial}{\partial t}|\chi(t)\rangle = H_e|\chi(t)\rangle$$

governed by H_3, this condition can be expressed as

$$|\chi(0)\rangle = W^+(z)|\psi(0)\rangle$$
$$= \int dp\phi(p)\left[\cos\frac{\alpha'}{2}|p', \chi_1'(p)\rangle + \sin\frac{\alpha'}{2}|p', \chi_2'(p)\rangle\right] \tag{10}$$

where

$$f'(p) = f\left(p + \frac{\hbar k}{2}\right), \quad f = p, \chi_1, \chi_2, \alpha \text{ etc.}$$

From this initial condition, the evolving wavefunction is written immediately

$$|\psi(t)\rangle = \int dp \exp(-i\varepsilon_0' t/\hbar)\, \phi(p)\left[\left(\cos^2\frac{\alpha'}{2}\exp(-i\omega' t)\right.\right.$$
$$\left. + \sin^2\frac{\alpha'}{2}\exp(i\omega' t)\right)|p\rangle \otimes |\uparrow\rangle$$
$$\left. + i\sin\alpha' \sin(\omega' t)|p + \hbar k\rangle \otimes |\downarrow\rangle\right]. \tag{11}$$

Notice that the object of attention is the momentum shift accompanying the spin states. This fact means that the pulse of the polarized-up neutrons will become two pulses, one of which has spin-down and a momentum shift by an amount $\hbar k$ and another one spin-up with unchanged momentum.

3. Meaning of solution and adiabatic limit

Now, let us first consider the meaning of solution (9). Using eq. (9), we compute the polarization of neutron at time t

$$P_z = \mathrm{Tr}\,(\sigma_z|\psi(t)\rangle\langle\psi(t)|)$$

through the helical magnetic field. Explicitly, we have

$$P_z = \int dp|\phi(p)|^2(1 - 2\sin^2\alpha' \sin^2(\omega' t)\, dp. \tag{12}$$

This is the central result of this paper, which can be compared with experiment and still holds even for the non-adiabatic case in an alternative experiment with fast neutrons. If the initial neutron is a plane wave, i.e.

$$\phi(p) = \delta(p - p_0)$$

then,

$$P_z = 1 - 2\sin^2\alpha_0' \sin^2\omega_0' t \tag{13}$$

where

$$\alpha_0' = \alpha\left(p_0 + \frac{\hbar k}{2}\right), \quad \omega_0' = \omega\left(p_0 + \frac{\hbar k}{2}\right).$$

In the adiabatic case that the Rabi frequency $\omega_0 = qB/2\hbar$ is much higher than the frequency $\omega_1 = pk/m = vk$ of varying of magnetic field $\boldsymbol{B}(z)$ "seen" by the moving neutrons, the terms of order $O^l(\omega_1/\omega_0)$, $(l \geqslant 1)$ in the amplitudes and the terms of order $O^k(\omega_1/\omega_0)$, $(k \geqslant 2)$ in the oscillating phase can be neglected. In this adiabatic case, we have

$$P_z(t) \simeq 1 - 2\sin^2\theta \int_{-\infty}^{\infty} \sin\left(\tfrac{1}{2}gBt - \frac{pk\cos\theta t}{2m}\right)|\phi(p)|^2\, dp. \tag{14}$$

For the case of plane wave with $\phi(p) = \delta(p - p_0)$ and $t = 2\pi m/kp_0$, eq. (11) becomes

$$P_z \simeq 1 - 2\sin^2\theta \cdot \sin(\tfrac{1}{2}gBt - \pi\cos\theta)|_{p=p_0}. \tag{15}$$

Here, the phase $-\pi\cos\theta$ in addition to the dynamical phase $\frac{1}{2}gBt$ is just the Berry's phase. It can also be regarded as the A–B phase $\oint A_\mu\, dx_\mu$ of the induced U(1)-gauge potential

$$A_z = \frac{\pi}{L}\cos\theta, \; A_x = 0, \; A_y = 0. \tag{16}$$

4. Generalization and discussion

In the last section, we first point out that the analysis of this paper can be directly generalized to the case of arbitrary spin $j \geqslant \frac{1}{2}$. In this case, the Hamiltonian is

$$\hat{H} = \frac{\hat{p}_z^2}{2m} + g\boldsymbol{B}(z)\cdot\boldsymbol{J} \tag{17}$$

where $\hat{\boldsymbol{J}} = (\hat{J}_1, \hat{J}_2, \hat{J}_3)$ is the angular momentum operator. The space–spin coupling transformation is

$$W(z) = \exp(-ikz\hat{J}_z/\hbar) \tag{18}$$

and the corresponding effective Hamiltonian is written as

$$H_e = \frac{\hat{p}_z^2}{2m} + \frac{k^2 J_z^2}{2m} + \hbar\Omega[\cos\beta J_z + \sin\beta J_x] \tag{19}$$

where

$$\hbar\Omega = \hbar\Omega(\hat{p}) = gB\left[1 - \frac{2p_z\hbar k}{mgB}\cos\theta + \frac{\hat{p}_z^2\hbar^2k^2}{m^2B^2g^2}\right]^{1/2},$$

$$ctg\beta = ctg\beta(\hat{p}) = ctg\theta - \frac{\hbar k\hat{p}_z}{mgB\sin\theta}.$$

Notice that the nonlinear term is proportional to J_z^2 appearing in eq. (20). We cannot analytically obtain the explicit solution for the Schrödinger equation governed by H_e eq. (20) for a given arbitrary spin j. This analytical solution are only written fo some small j. However, it is very convenient to use this effective Hamiltonian to obtain the adiabatic approximation solution. In the adiabatic limit, we can ignore the terms $(\hbar^2k^2/2m)/\hbar\Omega$ and obtain the approximate Hamiltonian

$$H_e = \frac{\hat{p}_z^2}{2m} + \hbar\Omega(\cos\beta J_z + \sin\beta J_x)$$
$$= \frac{\hat{p}_z^2}{2m} + \hbar\Omega \exp\left(\frac{-i\beta J_y}{\hbar}\right) J_z \exp\left(\frac{i\beta J_y}{\hbar}\right) \tag{20}$$

with

$$ctg\beta = ctg\theta, \quad \Omega \simeq gB\left(1 - \frac{\hat{p}_z \hbar k}{mgB}\cos\theta\right).$$

Roughly, the effective dynamical phase

$$\Omega t = MgBt - M\frac{\hat{p}_z \hbar kt}{m}\cos\theta$$

includes the geometrical phase $M\pi\cos\theta$ for the magnetic quantum number M.

Finally, we use the exact solution (11) and its corollary to show how the Doppler effect appear for the resonance transition. If the initial state of a neutron is spin-up with definite momentum $\boldsymbol{p}_0$, we can have the probability finding the neutron with spin-down:

$$P(\downarrow) = \sin^2 \alpha\left(p_0 + \frac{\hbar k}{2}\right)\sin^2\left[\omega\left(p_0 + \frac{\hbar k}{2}\right)t\right]. \tag{21}$$

From eq. (4), we can see that only when

$$\frac{\hbar k p_0}{m} + \frac{\hbar^2 k^2}{2m} = gB\cos\theta$$

a resonance transition with frequency gB will happen. The first term of the above expression is the Doppler shift and the second term is related to the recoil of the particle in the helical magnetic field.

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