High-Order Adiabatic Approximation for Non-Hermitian Quantum System and Complexification of Berry's Phase

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Abstract

In this paper the evolution of a quantum system driven by a non-Hermitian Hamiltonian depending on slowly-changing parameters is studied by building an universal high-order adiabatic approximation (HOAA) method with Berry's phase, which is valid for either the Hermitian or the non-Hermitian cases. This method can be regarded as a non-trivial generalization of the HOAA method for closed quantum system presented by this author before. In a general situation, the probabilities of adiabatic decay and non-adiabatic transitions are explicitly obtained for the evolution of the non-Hermitian quantum system. It is also shown that the non-Hermitian analog of the Berry's phase factor for the non-Hermitian case just enjoys the holonomy structure of the dual linear bundle over the parameter manifold. The non-Hermitian evolution of the generalized force harmonic oscillator is discussed as an illustrative example.

1. Introduction

Since Berry's phase factor (BPF) was discovered in evolution of a quantum system with adiabatically-changing parameters [1], a few methods studying non-adiabatic evolution of a quantum system driven by a Hermitian Hamiltonian depending on slowly (but not adiabatically)-changing parameters have been presented in connection with BPF's [2–5]. In these methods, the high-order adiabatic approximation (HOAA) method was proposed by this author for the first time [4] and has been used and developed for many cases [6–12]. However, none of these studies has been concerned with a kind of important quantum system, the quantum open system that possesses a non-Hermitian (nH) Hamiltonian. This paper will be devoted to the generalization of the HOAA method for such a nH quantum system.

In fact, though the Hamiltonian for a closed quantum system, which is usually considered as a basic object in quantum theory, must be a Hermitian operator, many theories, such as the Fock–Krylov theorem [13], show the probability to apply the nH Hamiltonian for those quantum phenomena with dissipation, decay and relaxation [14, 15]. Recently, many practical problems including the multi-photon ionization, the supermode free-electron laser and the transverse mode propagation in an optical resonator [16–19] have been concerned with the use of nH Schrödinger equation and nH Hamiltonian correspondingly. Since the occurrence of a BPF or its analog may be established when something in the considered system is varied, it is natural to generalize the concept of BPF for nH quantum system. More recently, some authors made this generalization and applied it to concrete physical problems [20–21], but their studies were only focused on the adiabatic case that the parameters change so slowly that transitions between any two (quasi-) energy levels do not happen. In this paper we especially emphasized the non-adiabatic evolution of the nH quantum system and the geometry of the nH analog of BPF.

The paper is arranged as follows. In Section 2, using a similarity transformation of the nH Hamiltonian, we build an universal formalism of the HOAA method for the nH quantum system. It is also available to the Hermitian quantum system. In Section 3, we explicitly analyse the conditions under which the lowest order approximation, namely the adiabatic approximation, can work well. We also compare our results with that obtained with the bi-orthonormal state method [20–21] in the adiabatic case. In Section 4, we apply the general result to a simple toy model – the nH forced oscillator to show the usefulness of our analysis. In Section 5, we show that the nH analog of BPF appearing in an adiabatic evolution of nH quantum system are non-unitary holonomy group element for the dual bundles over the parameter manifold. In the further studies, we will provide more applications of this generalized HOAA method to some physical problems.

2. Generalized HOAA method for nH quantum system

Let us begin by setting some notations. The Hamiltonian of the quantum open system we consider as follows is a nH operator

$$\mathscr{H} = \mathscr{H}(t) = \mathscr{H}[\mathbf{R}] = \mathscr{H}[\mathbf{R}(t)]$$

that depends on a set of slowly-changing parameters

$$\mathbf{R} = \mathbf{R}(t) : (R_1(t), R_2(t), \dots, R_L(t)).$$

We now assume that $\mathscr{H}(t)$ is diagonalizable at each instant t, i.e., there exists a similarity transformation

$$U(t) = U[\mathbf{R}] = U[\mathbf{R}(t)]$$

such that

$$U(t)^{-1}\mathscr{H}(t)U(t) = \begin{bmatrix} \varepsilon_1(t) & 0 & \cdots & 0 \\ 0 & \varepsilon_2(t) & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \varepsilon_N(t) \end{bmatrix} = \mathscr{H}_d(t) \tag{2.1}$$

where the "quasi-energy levels"

$$\varepsilon_k(t) = \varepsilon_k[\mathbf{R}], \quad k = 1, 2, \dots, N$$

may be complex and $U(t)$ is not unitary correspondingly.

Let

$$|\Psi(t)\rangle = U(t)|\Phi(t)\rangle$$

be a solution of the nH Schrödinger equation

$$i\hbar \frac{\partial}{\partial t}|\Psi(t)\rangle = \mathcal{H}(t)|\Psi(t)\rangle. \tag{2.2}$$

The equivalent wavefunction $|\Phi(t)\rangle$ must satisfy an equivalent Schrödinger equation (ESE)

$$i\hbar \frac{\partial}{\partial t}|\Phi(t)\rangle = \mathcal{H}_e(t)|\Phi(t)\rangle, \tag{2.3}$$

with the equivalent Hamiltonian

$$\mathcal{H}_e(t) = \mathcal{H}_d(t) - i\hbar U(t)^{-1} \frac{\partial U(t)}{\partial t}. \tag{2.4}$$

Separate $\mathcal{H}_e(t)$ into two parts, the diagonal part

$$\mathcal{H}_0(t) = \mathcal{H}_d(t) + \text{diagonal part of} \left[-i\hbar U(t)^{-1} \frac{\partial}{\partial t} U(t) \right]$$

and the off-diagonal part

$$\mathcal{V}(t) = \text{off diagonal part of} \left[-i\hbar U(t)^{-1} \frac{\partial}{\partial t} U(t) \right].$$

Then,

$$\mathcal{H}_e(t) = \mathcal{H}_0(t) + \mathcal{V}(t).$$

Later on we will show that the diagonal part $\mathcal{H}_0(t)$ governs an adiabatic evolution of the nH quantum system while the off-diagonal part $\mathcal{V}(t)$ governs the non-adiabatic transitions among the quasi-energy levels. In fact, since $\mathcal{V}(t)$ completely vanishes when $\mathcal{H}$ is independent of time t and $\mathcal{H} = \mathcal{H}(t) = \mathcal{H}[\mathbf{R}]$ is a smooth function of $\mathbf{R}(t)$, we can regard $\mathcal{V}(t)$ as a perturbation when $\mathcal{H}$ or $\mathbf{R}$ depends weakly on time, i.e., $\mathbf{R}(t)$ changes with t slowly enough.

Based on the above decomposition of $\mathcal{H}_e(t)$, we can use the standard time-dependent perturbation theory to solve the ESE (2.3), obtaining an approximate series solution

$$\begin{aligned}|\Phi(t)\rangle = &|\Phi^0(t)\rangle + |\Phi^1(t)\rangle \\ &+ |\Phi^2(t)\rangle + \cdots + |\Phi^l(t)\rangle + \cdots. \end{aligned} \tag{2.5}$$

Here, the lth order solution $|\Phi^l(t)\rangle$ is determined by

$$i\hbar \frac{\partial}{\partial t}|\Phi^0(t)\rangle = \mathcal{H}^0(t)|\Phi^0(t)\rangle;$$

$$i\hbar \frac{\partial}{\partial t}|\Phi^l(t)\rangle = H_0(t)|\Phi^l(t)\rangle + \mathcal{V}(t)|\Phi^{l-1}(t)\rangle.$$

$$l = 1, 2, \ldots. \tag{2.6}$$

Because the above equations of the lth order solution $|\Phi^l(t)\rangle$ only concerns $(l-1)$th order solutions $|\Phi^{l-1}(t)\rangle$, we can get approximate solutions of each order starting from the zeroth order:

$$|\Phi^0(t)\rangle = \exp\left[\frac{1}{i\hbar}\int_0^t \mathcal{H}_0(t')\, dt'\right]|\Phi(0)\rangle$$

$$|\Phi(0)\rangle = U(0)^{-1}|\Psi(0)\rangle. \tag{2.7}$$

In order to write these solutions in an explicit form, we use the eigenstates

$$|1\rangle = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}, |2\rangle = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}, \ldots, |N-1\rangle = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \end{bmatrix}, |N\rangle = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

of $\mathcal{H}_0(t)$ with the corresponding eigenvalues

$$E_n(t) = \varepsilon_n(t) - \hbar\dot{\gamma}_n(t),$$

where the additional term to the quasi-energy

$$\gamma_n(t) = i \int_0^t \langle n| U(t')^{-1} \frac{\partial U(t')}{\partial t'} |n\rangle \, dt' \tag{2.8}$$

will be proved to be the nH analog of Berry's phase. Then, each order solution

$$|\Phi^l(t)\rangle = \sum_{n=1}^N C_n^l(t) \exp(i\gamma_n(t)) \exp\left[\frac{1}{i\hbar}\int_0^t \varepsilon_n(t')\, dt'\right]|n\rangle, \tag{2.9}$$

follows from eq. (2.6) immediately. Here, the coefficients $C_n^l(t)$ satisfy

$$C_n^0(t) = \langle n|\Phi(0)\rangle = \langle n| U(0)^{-1}|\Psi(0)\rangle,$$

$$\begin{aligned}C_n^l(t) = \sum_{m=1}^N \frac{1}{i\hbar}\int_0^t &\langle n|\mathcal{V}(t')|m\rangle \\ &\times \exp\left[i\int_0^{t'} \omega_{n,m}(s)\, ds\right] C_m^{l-1}(t')\, dt', \end{aligned} \tag{2.10}$$

where

$$\omega_{n,m} = \frac{\varepsilon_n(t) - \varepsilon_m(t) + (\dot{\gamma}_m(t) - \dot{\gamma}_n(t))\hbar}{\hbar}.$$

Notice that the difficulty that the eigenstates of $\mathcal{H}$ are not orthogonal to each other due to non-Hermiticity of $\mathcal{H}$ has been avoided in the above discussion by a trick building the ESE (2.3) to find the perturbation $\mathcal{V}(t)$. This is the key to our studies in this paper.

3. Adiabatic approximation and comparison with the biorthonormal state method

In this section our first focus is on the adiabatic conditions that the zeroth order (adiabatic) approximation can well approach the true evolution of the nH quantum system. Consider the integral

$$\begin{aligned}\mathcal{I} = &\int_0^t \langle n|\mathcal{V}(t')|m\rangle \exp\left[i\int_0^{t'}\omega_{n,m}(s)\, ds\right] dt', \\ = &\int_0^t \langle n|\mathcal{V}(t')|m\rangle \exp\left\{-\text{Im}\left[\int_0^{t'}\omega_{n,m}(s)\, ds\right]\right\} \\ &\times \exp\left\{i\,\text{Re}\left[\int_0^{t'}\omega_{n,m}(s)\, ds\right]\right\} dt'\end{aligned}$$

appearing in the first order approximation

$$\begin{aligned}C_n^l(t) = \sum_{m=1}^N \frac{1}{i\hbar} &\langle n|\Phi(0)\rangle \\ &\times \int_0^{t'}\langle n|\mathcal{V}(t')|m\rangle \exp\left[i\int_0^{t'}\omega_{n,m}(s)\, ds\right] dt'.\end{aligned}$$

Here, we have separated the damping factor

$$\exp\left\{-\mathrm{Im}\left[\int_0^t \omega_{n,m}(s)\,\mathrm{d}s\right]\right\}$$

and the oscillating factor

$$\exp\left\{\mathrm{i}\,\mathrm{Re}\left[\int_0^t \omega_{n,m}(s)\,\mathrm{d}s\right]\right\}.$$

If the latter oscillates so fast that the conditions

$$\left|\frac{\langle n|\mathscr{V}(t)|m\rangle \exp\left\{-\mathrm{Im}\left[\int_0^t \omega_{n,m}(s)\,\mathrm{d}s\right]\right\}}{\mathrm{Re}\,[\omega_{n,m}(t)]\hbar}\right| \ll 1 \qquad (3.1)$$

hold for $m \neq n$, the integral $\mathscr{I}$ tends to zero. This statement can be proved by integrating $\mathscr{I}$ by part. Then, we get so-called adiabatic conditions (3.1). When they hold for a nH quantum system, we can ignore the higher-approximation solutions $|\Phi_l(t)\rangle$, $(l = 1, 2, \ldots)$.

To compare our analysis with the biorthonormal state method in Refs [20, 21] in the adiabatic case, we define

$$|\phi_n(t)\rangle = |\phi_n[\mathbf{R}]\rangle = U(t)|n\rangle,$$

$$|\chi_n(t)\rangle = |\chi_n[\mathbf{R}]\rangle = [U(t)^{-1}]^\dagger|n\rangle. \qquad (3.2)$$

Obviously, they are the eigenstates of $\mathscr{H}(t)$ and $\mathscr{H}(t)^\dagger$ respectively with the eigenvalues $\varepsilon_n(t)$ and $\varepsilon_n^*(t)$. Using the completeness relation and the orthonormal relations of states $|n\rangle$ $(n = 1, 2, \ldots, N)$:

$$\sum_{n=1}^N |n\rangle\langle n| = I \text{ (unit operator)}, \quad \langle n|m\rangle = \delta_{m,n},$$

we immediately get the generalized completeness relations

$$\sum_{n=1}^N |\chi_n(t)\rangle\langle\phi_n(t)| = \sum_{n=1}^N |\phi_n(t)\rangle\langle\chi_n(t)| = I \qquad (3.3)$$

and the biorthonormal relations

$$\langle\phi_m(t)\chi_n(t)\rangle = \langle\chi_m(t)|\phi_n(t)\rangle = \delta_{m,n}. \qquad (3.4)$$

In terms of the biorthonormal basis $\{|\phi_n(t)\rangle, |\chi_n(t)\rangle\}$, we reexpress the high-order approximations of the nH quantum evolution as

$$|\Psi(t)\rangle = \sum_{l=0}^\infty |\Psi^l(t)\rangle = \sum_{l=0}^\infty U(t)|\Phi^l(t)\rangle$$

$$= \sum_{l=0}^\infty \sum_{n=1}^N C_n^l(t)\exp\left[\mathrm{i}\gamma_n(t)\right]$$

$$\times \exp\left[\frac{1}{\mathrm{i}\hbar}\int_0^{t'}\varepsilon_n(t')\,\mathrm{d}t'\right]|\phi_n(t)\rangle, \qquad (3.5)$$

with the coefficient equations:

$$C_n^0(t) = \langle\chi_n(0)|\Psi(0)\rangle,$$

$$C_n^l(t) = -\sum_{m=1}^N \int_0^t \langle\chi_n(t')|\dot\phi_m(t')\rangle$$

$$\times \exp\left[\mathrm{i}\int_0^{t'}\omega_{n,m}(s)\,\mathrm{d}s\right]C_m^{l-1}(t')\,\mathrm{d}t', \qquad (3.5')$$

for $l = 1, 2, 3, \ldots$. Here, the nH analog of Berry's phase is rewritten as

$$\gamma_n(t) = \mathrm{i}\int_0^t\left\langle\chi_n(t')\left|\frac{\mathrm{d}}{\mathrm{d}t'}\phi_n(t')\right.\right\rangle\,\mathrm{d}t'. \qquad (3.6)$$

Correspondingly, the adiabatic conditions (3.1) are rewritten as

$$\left|\frac{\left\langle\chi_n(t)\left|\frac{\mathrm{d}}{\mathrm{d}t}\phi_n(t)\right.\right\rangle \exp\left\{-\mathrm{Im}\left[\int_0^t \omega_{m,w}(s)\,\mathrm{d}s\right]\right\}}{\mathrm{Re}\,[\omega_{n,m}(t)]}\right| \ll 1. \qquad (3.7)$$

Now, let us consider the adiabatic evolution of an instantaneous eigenstate of $\mathscr{H}(t)$ under adiabatic conditions. If the system is initially in the state

$$|\Phi(0)\rangle = U(t = 0)|n\rangle = |\phi_n(0)\rangle,$$

it will evolve into

$$|\Phi(t)\rangle = \exp\left[\mathrm{i}\gamma_n(t)\right]\exp\left[\frac{1}{\mathrm{i}\hbar}\int_0^t\varepsilon_n(t')\,\mathrm{d}t'\right]|\phi_n(t)\rangle$$

$$\equiv \exp\left[\mathrm{i}\Omega_n(t)\right]|\phi_n(t)\rangle \qquad (3.8)$$

where

$$\Omega_n(t) = \gamma_n(t) - \frac{1}{\hbar}\int_0^t\varepsilon_n(t')\,\mathrm{d}t'.$$

The above eq. (3.8) manifests that the adiabatically-evolving state is always an eigenstate of the instantaneous Hamiltonian $\mathscr{H}(t)$ if the initial state is such state at $t = 0$. This is the quantum adiabatic theorem for nH quantum systems, which is quite similar to that for the Hermitian case. It shows the quantum number labelling the quasi-energy level to be an adiabatic invariant. Since $\gamma_n(t)$ and $\varepsilon_n(t)$ are usually not real due to the non-Hermiticity of $\mathscr{H}(t)$, the damping factor $\exp\left[-\mathrm{Im}\,\Omega_n(t)\right]$ causes the adiabatic decay of the state. Especially, when $\varepsilon_n(t)$ is real, the decay only results from the nH-analog of BPF and is a purely geometrical effect. In the next section an example will be used to illustrate this situation.

When the adiabatic conditions (3.7) are broken, there may appear transitions from an instantaneous eigenstate $|\phi_n(0)\rangle$ to others $|\phi_n(t)\rangle$ for $m \neq n$. From eqs (3.5) or (2.5, 2.9, 2.10), we obtain the transition probabilities

$$P(n \to m) = \frac{1}{\hbar^2}\left|\int_0^t\langle m|\mathscr{V}(t')|n\rangle\exp\left[\mathrm{i}\int_0^{t'}\omega_{n,m}(s)\,\mathrm{d}s\right]\,\mathrm{d}t'\right|^2$$

$$\times \exp\left\{-2\,\mathrm{Im}\,[\Omega(t)]\right\} \qquad (3.9)$$

or

$$P(n \to m) = \left|\int_0^t\left\langle\chi_m(t')\left|\frac{\mathrm{d}}{\mathrm{d}t'}\phi_n(t')\right.\right\rangle\right.$$

$$\times \exp\left[\mathrm{i}\int_0^{t'}\omega_{m,n}(s)\,\mathrm{d}s\right]\,\mathrm{d}t'\Big|^2$$

$$\times \exp\left\{-2\,\mathrm{Im}\,[\Omega(t)]\right\}. \qquad (3.9')$$

Therefore, we reach the statement mentioned in section 2 that the diagonal and off-diagonal parts respectively describe the adiabatic evolution and the non-adiabatic transitions for the quantum system. Notice that Berry also discussed the geometrical phase effect in non-adiabatic transition in connection with the non-Hermiticity due to the complexification of time [22].

4. Example: generalized force harmonic oscillator

Now, let us apply the above general analysis to a simple model – the generalized forced harmonic oscillator. The

time-dependent Hamiltonian for this model is

$$\mathscr{H}(t) = \mathscr{H}[\alpha, \beta] = \hbar\omega(a^\dagger a + \beta a^\dagger + \alpha a) \qquad (4.1)$$

where $\alpha = \alpha(t)$ and $\beta = \beta(t)$ are slowly-changing complex parameters, the constant ω is real, a and $a^\dagger$ are respectively the creation and annihilation operators for a boson state, which satisfy

$$[a, a^\dagger] = 1.$$

For the usual forced harmonic oscillator (FHO), there is a constraint $\alpha = \beta^*$ and the corresponding Hamiltonian is Hermitian. However, here we make a generalization to remove this constraint so that the Hamiltonian (4.10) is non-Hermitian.

Using the translated boson operators

$$A(\alpha)^\dagger = a^\dagger + \alpha, \quad A(\beta) = a + \beta \qquad (4.2)$$

obeying the same boson commutation relation as the above

$$[A(\beta), A(\alpha)^\dagger] = 1,$$

we rewrite down the Hamiltonian (4.1) as

$$\mathscr{H}(t) = \mathscr{H}[\alpha, \beta] = \hbar\omega[A(\alpha)^\dagger A(\beta) - \alpha\beta].$$

Based on the above expression for $\mathscr{H}(t)$, we immediately obtain the instantaneous eigenstates of $\mathscr{H}(t) = \mathscr{H}[\alpha, \beta]$

$$|\phi_n(t)\rangle = |\phi_n[\alpha, \beta]\rangle = |n[\alpha, \beta]\rangle = \frac{1}{\sqrt{n!}} [A(\alpha)^\dagger]^n |0(\beta)\rangle \quad (4.3)$$

with the eigenvalues

$$\varepsilon_n(t) = \varepsilon_n[\alpha, \beta] = (n - \alpha\beta)\hbar\omega.$$

Here, the translated vacuum state $|0(\beta)\rangle$ obeys

$$A(\beta)|0(\beta)\rangle = 0, \quad \text{or} \quad a|0(\beta)\rangle = -\beta|0(\beta)\rangle,$$

that is to say, $|0(\beta)\rangle$ is a coherent state

$$|0(\beta)\rangle = C_{\alpha, \beta} e^{-\beta a^\dagger}|0\rangle, \qquad (4.4)$$

where $|0\rangle$ is the usual vacuum state satisfying $a|0\rangle = 0$ and $C_{\alpha, \beta}$ is to be determined by the biorthonormal conditions. According to the above analysis, we can finally write the explicit form of $|\phi_n(t)\rangle$

$$|\phi_n(t)\rangle = C_{\alpha, \beta} \sum_{k=0}^{n} \frac{n!\alpha^{n-k}}{(n-k)!\sqrt{k!}} e^{-\beta a^\dagger}|k\rangle, \qquad (4.3')$$

where

$$|k> = \frac{1}{\sqrt{k!}} [a^\dagger]^k |0\rangle$$

is the k boson state.

Noticing

$$\mathscr{H}[\alpha, \beta]^\dagger = \mathscr{H}[\beta^*, \alpha^*],$$

we easily obtain the dual vectors $|\chi_n(t)\rangle$ to $|\phi_n(t)\rangle$:

$$|\chi_n(t)\rangle = |\chi_n[\alpha, \beta]\rangle = |n[\beta^*, \alpha^*]\rangle$$

as the eigenvectors of $\mathscr{H}[\alpha, \beta]^\dagger$ with the corresponding eigenvalues $\varepsilon_n[\beta^*, \alpha^*]$. Then, the biorthonormal conditions

$$\langle \chi_n(t) | \phi_m(t)\rangle = \delta_{m, n}$$

define the biorthonormalization coefficient

$$C_{\alpha, \beta} = \exp(-\tfrac{1}{2}\alpha\beta).$$

Taking into account the translation transformation

$$e^{\xi a} a^\dagger e^{-\xi a} = a^\dagger + \xi,$$

$$e^{\xi a^\dagger} a e^{-\xi a^\dagger} = a - \xi, \qquad (4.5)$$

$$\xi \in \text{complex field } C$$

we get the $U(t)$-operator expression of the biorthonormal basis $\{|\phi_m(t)\rangle, |\chi_n(t)\rangle\}$

$$|\phi_n(t)\rangle = |\phi_n[\mathbf{R}]\rangle = U(t)|n\rangle,$$

$$|\chi_n(t)\rangle = |\chi_n[\mathbf{R}]\rangle = [U(t)^{-1}]^\dagger|n\rangle, \qquad (4.6)$$

where the $U(t)$-operator

$$U(t) = U[\alpha, \beta] = \exp(-\tfrac{1}{2}\alpha\beta)\exp(-\beta a^\dagger)\exp(-\alpha a) \qquad (4.7)$$

diagonalizes the Hamiltonian $\mathscr{H}(t)$, i.e.,

$$U(t)^{-1}\mathscr{H}(t)U(t) = \hbar\omega a^\dagger a.$$

Now, let us discuss the evolution governed by the nH Hamiltonian (4.1). To this end, we first calculate

$$\left\langle \chi_m(t) \left| \frac{d}{dt} \phi_n(t) \right.\right\rangle = \sqrt{n}\,\dot\alpha\delta_{m, n-1} - \sqrt{n+1}\,\beta\delta_{m, n+1}$$
$$+ \tfrac{1}{2}[\alpha\dot\beta - \beta\dot\alpha]\delta_{m, n}. \qquad (4.8)$$

Then, we obtain the nH analog of BPF

$$\exp[i\gamma_n(t)] \equiv \exp[i\gamma(t)]$$
$$= \exp\left\{\frac{i}{2}\int_0^t [\alpha\dot\beta - \beta\dot\alpha](t')\,dt'\right\}$$
$$= \exp\left\{\frac{i}{2}\int_{C; \mathbf{R}(0)}^{\mathbf{R}(t)} [\alpha\,d\beta - \beta\,d\alpha]\right\} \qquad (4.9)$$

where C is a curve $\{\mathbf{R}(t)\}$ in a four dimensional parameter manifold

$$\mathbf{M}: \{\mathbf{R} = (\alpha, \beta) = (\alpha_1, \alpha_2, \beta_1, \beta_2)|\alpha = \alpha_1 + i\alpha_2, \beta = \beta_1 + i\beta_2\}$$

where α_1, α_2, β_1 and β_2 are real. Because the factor $e^{i\gamma_n(t)}$ is independent of the quantum number n, the adiabatic evolution

$$|\Phi(t)\rangle = e^{i\gamma(t)} \sum_{n=1}^{N} \langle \chi_n(0)|\Psi(0)\rangle e^{-in\omega t}|\phi_n(t)\rangle$$

$$\equiv e^{-\Gamma(t)} e^{i\Theta(t)} \sum_{n=1}^{N} \langle \chi_n(0)|\Psi(0)\rangle e^{-in\omega t}|\phi_n(t)\rangle \qquad (4.10)$$

is accompanied with a global geometrical factor with the oscillating part $e^{i\Theta(t)}$:

$$\Theta(t) = \text{Re}[\gamma_n(t)] = \frac{i}{2}\int_0^t [\beta_1\dot\alpha_2 - \beta_2\dot\alpha_1 - \alpha_2\dot\beta_1 - \alpha_1\dot\beta_2](t')\,dt'$$

and the damping part $e^{-\Gamma(t)}$:

$$\Gamma(t) = \text{Im}[\gamma_n(t)] = \frac{i}{2}\int_0^t [\beta_2\dot\alpha_2 - \beta_1\dot\alpha_1 - \alpha_2\dot\beta_2 + \alpha_1\dot\beta_1](t')\,dt'.$$

Notice that, when $\alpha(t) = \beta(t)^*$, the BPF of a coherent state for FHO in Ref. [23] is given once again as a special case of eq. (4.9).

For the case with large quantum number n or rapidly-changing parameters $\alpha(t)$ and $\beta(t)$, the adiabatic conditions

$$\frac{\sqrt{n}\,|\dot\alpha(t)|}{\omega} e^{-\Gamma(t)} \ll 1; \quad \frac{\sqrt{n+1}\,|\dot\beta(t)|}{\omega} e^{-\Gamma(t)} \ll 1 \qquad (4.11)$$

do not hold and we need to consider the non-adiabatic effects caused by the first order approximation at least. If the system is initially in a state $|\phi_k(0)\rangle$, the initial conditions

$$C_n^0(0) = \delta_{n,k}, \quad C_n^l(0) = 0, \quad \text{for} \quad l \geqslant 1$$

leads to the first order approximate solution

$$
\begin{aligned}
|\Psi^1(t)\rangle = &\sqrt{k}\, \mathrm{e}^{-\Gamma(t)}\, \mathrm{e}^{\mathrm{i}[\Theta(t)-(k-1)\omega t]} \\
&\times \int_0^t \dot{\alpha}(t')\, \mathrm{e}^{-\mathrm{i}\omega t'}\, \mathrm{d}t'\, |\phi_{k-1}(t)\rangle \\
&+ \sqrt{k+1}\, \mathrm{e}^{-\Gamma(t)}\, \mathrm{e}^{\mathrm{i}[\Theta(t)-(k+1)\omega t]} \\
&\times \int_0^t \dot{\beta}(t')\, \mathrm{e}^{\mathrm{i}\omega t'}\, \mathrm{d}t'\, |\phi_{k+1}(t)\rangle.
\end{aligned}
\tag{4.12}
$$

The probabilities of the transition from $|\phi_n(0)\rangle$ to $|\phi_n(t)\rangle$ ($n = k-1, k+1$) are

$$
\begin{aligned}
P(k \to n) = &k\, \mathrm{e}^{-2\Gamma(t)} \left| \int_0^t \dot{\alpha}(t')\, \mathrm{e}^{-\mathrm{i}\omega t'}\, \mathrm{d}t \right|^2 \delta_{n,k-1} \\
&+ (k+1)\, \mathrm{e}^{-2\Gamma(t)} \left| \int_0^t \dot{\beta}(t')\, \mathrm{e}^{\mathrm{i}\omega t'}\, \mathrm{d}t \right|^2 \delta_{n,k+1}.
\end{aligned}
\tag{4.13}
$$

Obviously, the selection rule for such a transition is

$$\Delta n = +1 \quad \text{or} \quad -1, \tag{4.14}$$

in the first order approximation.

5. The nH-analog of BPF and holonomy in dual line fibre bundles

For Hermitian quantum systems, Simon recognized that the BPF is precisely the holonomy in a Hermitian line bundle defined by the adiabatic evolution. The adiabatic evolution can be interpreted as a parallel translation in such a bundle [24]. Now, a question naturally arises for the non-Hermitian case: what is a geometrical interpretation of the nH analog of BPF? The answer will be given in this section.

Let **M** be the parameter manifold formed by the parameters $\mathbf{R} = (R_1, R_2, \ldots, R_L)$. A line bundle defined by the nH Hamiltonian $\mathcal{H}[\mathbf{R}]$ is

$$\mathbf{F}_n = \{(\mathbf{R}, |\sigma_n[\mathbf{R}]\rangle)| \mathcal{H}[\mathbf{R}]|\sigma_n[\mathbf{R}]\rangle = \varepsilon_n[\mathbf{R}]|\sigma_n[\mathbf{R}]\rangle, \mathbf{R} \in \mathbf{M}\}.$$

Its fibre space is a one-dimensional linear space

$$V_n: \{|\sigma_n[\mathbf{R}]\rangle) = \mathrm{e}^{\mathrm{i}\theta[\mathbf{R}]}|\phi_n[\mathbf{R}]\rangle|$$

where $\theta[\mathbf{R}]$ is a real function depending on $\mathbf{R}$}.

Over the same base mainfold **M**, the dual bundle $\mathbf{F}_n^*$ is defined by

$$\mathbf{F}_n^* = \{(\mathbf{R}, |\sigma_n^*[\mathbf{R}]\rangle)| \mathcal{H}[\mathbf{R}]^\dagger|\sigma_n^*[\mathbf{R}]\rangle$$
$$= \varepsilon_n^*[\mathbf{R}]|\sigma_n^*[\mathbf{R}]\rangle, \mathbf{R} \in \mathbf{M}\},$$

where the fibre space

$$V_n^*: \{|\sigma_n^*[\mathbf{R}]\rangle = \mathrm{e}^{\mathrm{i}\theta'[\mathbf{R}]}|\chi_n[\mathbf{R}]\rangle|,$$

where $\theta'[\mathbf{R}]$ is a real function depending on $\mathbf{R}$},

is the dual space of V_n and $|\chi_n[\mathbf{R}]\rangle$ is the dual element to $|\phi_n[\mathbf{R}]\rangle$, i.e.,

$$\langle\chi_n[\mathbf{R}]|\phi_n[\mathbf{R}]\rangle = \delta_{m,n}.$$

Since the quantum number n, labelling an eigenstate $|\phi_n[\mathbf{R}]\rangle$ of the instantaneous Hamiltonian $\mathcal{H}[\mathbf{R}]$, is an adiabatic invariant, we can assume that

$$|\sigma_n(t)\rangle = |\sigma_n[\mathbf{R}]\rangle = C_n[\mathbf{R}]|\phi_n[\mathbf{R}]\rangle$$

is an evolution state in adiabatic case. Now, let us show that the holonomy group elements of $\mathbf{F}_n(\mathbf{F}_n^*)$ is the nH analog of BPF for the adiabatic evolution while $\{|\sigma_n(t)\rangle | t \in [0, T]\}$ ($\{|\sigma_n^*(t)\rangle | t \in [0, T]\}$), as a curve in $\mathbf{F}_n$ ($\mathbf{F}_n^*$), is a horizontal lift of the curve $C: \{\mathbf{R}(t)| t \in [0, T]\}$ on the base manifold **M**. To this end we consider a decomposition of a tangent vector

$$\frac{\mathrm{d}}{\mathrm{d}t}|\sigma_n(t)\rangle = \frac{\mathrm{d}_v}{\mathrm{d}t}|\sigma_n(t)\rangle + \frac{\mathrm{d}_h}{\mathrm{d}t}|\sigma_n(t)\rangle \tag{5.1}$$

where the vertical part along the fiber is

$$
\begin{aligned}
\frac{\mathrm{d}_v}{\mathrm{d}t}|\sigma_n(t)\rangle &= \langle\chi_n[\mathbf{R}]|\frac{\mathrm{d}}{\mathrm{d}t}|\sigma_n[\mathbf{R}]\rangle|\phi_n[\mathbf{R}]\rangle \\
&= \left(\frac{\mathrm{d}}{\mathrm{d}t}C_n[\mathbf{R}] + \langle\chi_n[\mathbf{R}]|\frac{\mathrm{d}}{\mathrm{d}t}|\phi_n[\mathbf{R}]\rangle\right. \\
&\quad \left. \times C_n[\mathbf{R}]\right)|\phi_n[\mathbf{R}]\rangle
\end{aligned}
\tag{5.2}
$$

and the horizontal part orthogonal to the fibre is

$$\frac{\mathrm{d}_h}{\mathrm{d}t}|\sigma_n(t)\rangle = \sum_{m \neq n} \langle\chi_m[\mathbf{R}]|\frac{\mathrm{d}}{\mathrm{d}t}|\sigma_n[\mathbf{R}]\rangle|\phi_m[\mathbf{R}]\rangle. \tag{5.3}$$

A parallel-translation implies

$$\langle\chi_n[\mathbf{R}]|\frac{\mathrm{d}_v}{\mathrm{d}t}|\sigma_n(t)\rangle = 0, \tag{5.4}$$

which results in a one-form equation

$$\mathrm{d}C_n[\mathbf{R}] + \langle\chi_n[\mathbf{R}]|\mathrm{d}\phi_n[\mathbf{R}]\rangle C_n[\mathbf{R}] = 0. \tag{5.5}$$

Its solution

$$C_n(t) = C_n[\mathbf{R}(t)] = \mathrm{e}^{\mathrm{i}\gamma_n(t)}$$

just gives the nH analog of BPF again.

For a cyclic evolution that $\mathbf{R}(0) = \mathbf{R}(T)$ and C is a closed curve, the complex phase $\gamma_n(T)$ in the nH analog of BPF can be expressed as a closed path integration

$$\gamma_n(T) = \gamma_n[C] = \oint_C A_n[\mathbf{R}] \tag{5.6}$$

of a complex potential one-form

$$A_n[\mathbf{R}] = \mathrm{i}\langle\chi_n[\mathbf{R}]|\mathrm{d}\phi_n[\mathbf{R}]\rangle.$$

Similarly, on dual Bundle $\mathbf{F}_n^*$, the parallel-translation condition

$$\langle\phi_n[\mathbf{R}]|\frac{\mathrm{d}_v}{\mathrm{d}t}|\sigma_n^*(t)\rangle = 0$$

results in the dual nH analog of BPF with the geometric phase

$$\bar{\gamma}_n(t) = i \int_0^t \langle \phi_n[\mathbf{R}(t')] | \frac{d}{dt'} | \chi_n[\mathbf{R}(t')] \rangle \, dt'. \qquad (5.7)$$

Obviously, the nH analog $\exp[i\gamma_n(t)]$ and its dual $\exp[i\bar{\gamma}_n(t)]$ occurring in the adiabatic evolution are holonomy group elements on the line bundle $\mathbf{F}_n$ and its dual $\mathbf{F}_n^*$ respectively. Except for the effect of dynamical factors

$$f(t) = \exp\left\{\frac{1}{i\hbar} \int_0^t \varepsilon[\mathbf{R}(t')] \, dt'\right\},$$

$$\bar{f}(t) = \exp\left\{\frac{1}{i\hbar} \int_0^t \varepsilon[\mathbf{R}(t')]^* \, dt'\right\},$$

the adiabatic effect in quantum evolution are equivalent to the parallel-translation on the line bundles $\mathbf{F}_n$ and its dual $\mathbf{F}_n^*$. This is a circumstance similar to that for the Hermitian quantum system.

Acknowledgements

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