Decoherence and relevant universality in quantum algorithms via a dynamic theory for quantum measurement

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It is well known that environment may decohere a quantum bit (qubit) system immersed in it, making a quantum computation invalid. But the quantitative features of the decoherence seem to depend on both the constitution of the environment and the details of its coupling with the qubit system. In this paper, based on the dynamic approach for quantum measurement developed from the Hepp-Coleman model [K. Hepp, Helv. Phys. Acta **45**, 237 (1972)], we generally model the environment as a collection of a large number of subsystems and then consider to what extent and in which way the environment and its coupling with the qubit system may affect a quantum computation process. In the weak-coupling limit, we find that as far as decoherence time is concerned, there is no essential difference between an environment of two-level subsystems and an environment of harmonic oscillators. This implies that there exists some universality independent of specific constitutions of environments. However, it is also shown that this is not true at finite temperature or in the case of strong coupling. So only if the coupling is weak and the temperature low does there exist the possibility of developing a universal scheme of controlling a qubit system such that the decoherence is avoided. The possible effect of environment on the efficiency of a quantum algorithm is also explicitly illustrated through the example of Shor's prime factorization algorithm. [S1050-2947(98)06909-1]

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I. INTRODUCTION

Quantum computations (QC) can be understood as quantum-mechanical evolution processes of certain quantum systems [quantum bits (so-called qubits)] [1–9], in which nonclassical dynamic features, such as quantum coherence of states, play a dominant role. Indeed, it is purely quantum characters that make it possible for a theoretical quantum computer to solve certain difficult mathematical problems efficiently. In this respect, perhaps the most important example is Shor's prime factorization algorithm [6]. As quantum computation is a quantum process, preserving coherence, at least to some extent, throughout the whole process, is thus an essential requirement. This is because the decoherence resulting from coupling with environment may make a quantum algorithm invalid and may cause unwelcome exponential increase of errors in output results [10–12]. Actually, a decoherence process was even regarded as a mechanism for enforcing classical behaviors in the macroscopic realm [13]. For this reason we may well view a decohered quantum computer as a classical one. To overcome the difficulties caused by decoherence, some schemes have been proposed in the last several years [14–20]. Among them are the quantum error-correcting technique developed from the classical error-correction theory and the decoherence-avoiding schemes reduced from the quantum measurement theory [21–24]. For the decoherence-avoiding schemes to work, in-

tuitively, one should know the details of the constitution of the environment and its coupling with the qubit system. But the environment may be very complicated with many unknown elements. So it seems impossible to control decoherence in a qubit system [23,24].

As in various quantum irreversible processes, such as quantum dissipation [25–30] and wave function collapse [27,31,32], the environment in quantum computation [11,12] was often modeled as a bath of harmonic oscillators with a linear coupling to the qubit system and some procedures to control decoherence have been presented based on such a model. Then a natural question is whether or not it is reasonable to model the environment universally as a harmonic oscillator bath for practical quantum computations. This is a main issue handled in this paper. Caldeira and Leggett [25] and Leggett *et al.* [26] have shown that, dealing with quantum dissipation in tunneling process in the weakly coupling limit, one can generally treat the environment as a harmonic oscillator bath. In this paper, by modeling the environment as a bath of a large number of two-level subsystems, we will consider the validity of this argument for the decoherence problem in quantum computation. If it is valid, one can design a universal decoherence-avoiding scheme in the weakly coupling limit. Otherwise one will have to design different decoherence-avoiding schemes to cope with different circumstances because different models may have very different behaviors. Therefore we have to consider to what extent the constitution of environment and the details of its coupling with the qubit system affect a quantum computation process. It is shown that, in the weakly coupling limit, the decoherence time derived from the two-level subsystem

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model of environment coincides with that derived from the harmonic oscillator model. This implies the existence of a universality independent of the constitution of environments. Nevertheless, there are some differences among different models not to be overlooked in the case of finite temperature or in the case of strong interaction between the environment and the qubit system. So only when the coupling is weak and the temperature is low can we expect to find a common decoherence-avoiding scheme for various systems.

The starting point of this paper is the quantum dynamic approach (QDA) [33–42] to the wave function collapse (WFC, also called von Neumann's reduction [43]) problem in quantum measurement. It is developed from the original contribution made by Hepp and Coleman (HC) [33] with a clarified physical presentation by Bell [34]. This approach may be thought of as a dynamic realization of von Neumann's theory [43] about the quantum measuring apparatus, which was proposed in contrast to Bohr's theory. Bohr believed that the apparatus must be classical. On the contrary, according to the quantum dynamic approach both the measured system and the measuring apparatus obey the Schrödinger equation and the dynamic evolution governed by their interaction is supposed to result in WFC under certain conditions. For example, the collapse happens if the detector contains a great number of particles or if the detector is in a state with a very large quantum number. These two cases are usually referred to as the macroscopic limit and as the classical limit, respectively [38]. We recall that, in the traditional theory of quantum measurement [24], the WFC postulate is only an extra assumption added to the ordinary quantum mechanics. Under this postulate, once we measure an observable and obtain a definite value a_k the state of the system must collapse into the corresponding eigenstate $|k\rangle$ from a coherent superposition $|\phi\rangle = \Sigma_k c_n |k\rangle\langle k|$. In the terminology of a density matrix this process is described by a projection $\rho = |\phi\rangle\langle\phi| \to \hat{\rho} = \Sigma_k |c_k|^2 |k\rangle\langle k|$ from a pure state to a mixed state. This projection, which was treated in the HC model [33] as a dynamic evolution process governed by the Schrödinger equation, means the loss of quantum coherence.

There is a strong resemblance between this phenomenon and the quantum decoherence of a quantum computer resulting from the coupling with the surrounding environment. It is then recognized that there exists a substantially close relation between the problem of decoherence in quantum computation and the problem of WFC in quantum measurement. Thus, when the environment surrounding a quantum computer corresponds to the measurement instrument monitoring the system to be measured, we can apply the known results in quantum dynamic models [31] for quantum measurement to discuss such problems concerned with decoherence in quantum computation as the dynamic mechanism of decoherence, quantum error-avoiding techniques, and calculation of decoherence time. We can also reconsider the strategy of grouping quantum states of qubits to form decoherence free subsets [21–23] and analyze the decohering behaviors of states not belonging to decoherence free subsets.

The arrangement of this paper is as follows. To study the influence of environment on quantum computation from a quantum measurement approach, in Sec. II we describe a general model of decoherence in quantum computation without referring to the concrete construction of environment. In

Sec. III, in the view of effective interaction, we show how certain superposition states lose their coherence and we suggest a decoherence-avoiding scheme. In Sec. IV, in the framework of the two-level subsystems model of environment, we study the decoherence problem of a two-qubit system at zero temperature by explicitly calculating the decohering factor. In Sec. V, in the weakly coupling limit, by comparing results from different models we point out the existence of a universality among different environments in respect to the decoherence properties of a qubit system at zero temperature. In Sec. VI we take into account the size of a qubit system and the effect of finite temperature and reveal a further universality. In Sec. VII we make a short discussion as a brief summary of this paper. Finally, in the Appendix, by the example of Shor's prime factorization algorithm, we demonstrate another fatal influence that environment may impose on quantum computation—destroying the efficiency of a quantum algorithm.

II. DECOHERENCE IN QUANTUM COMPUTATION VIA DYNAMIC FACTORIZATION

In this section we consider the effect of environment on quantum computation in general from the Hepp-Coleman approach for quantum measurement theory [33]. We recall that a quantum computer has at least one register. Usually a register is an array of qubits, each of which has two states $|0\rangle$ and $|1\rangle$, so its states can be represented as $|n\rangle = |n_0 n_1 n_2, \ldots, n_{L-1}\rangle$ where the labels satisfy the unique binary representation $n = \Sigma_{i=0}^{L-1} n_i 2^i$ ($n_i = 0,1$). The process of a quantum computation is none other than a series of transformations among the states of the register. Generally, in the beginning of a computation one puts the register in a superposition state $|\phi(0)\rangle = \Sigma_n c_n |n\rangle$ and then lets it evolve according to the Schrödinger equation with a specific Hamiltonian. Without the influence from environment, this evolution could be described with a unitary operator $U(t)$. Thus in the end the machine would be in the pure state $|\phi(t)\rangle = \Sigma_{n,m} c_n U(t)_{m,n} |n\rangle$. Then in some way the results of computation could be drawn from this coherent superposition.

When we take the influence of environment into account, the coherence in $|\phi(t)\rangle$ may be demolished, possibly leading to the failure of computation. According to Zurek [13], the influence of environment on quantum computation can be analyzed in the view of state entanglement or state correlation. Let $|\Phi(0)\rangle = |\phi(0)\rangle \otimes |e\rangle = \Sigma_n c_n |n\rangle \otimes |e\rangle$ be an initial state of the total system consisting of a qubit system and the environment. Here, $|e\rangle = |e_1\rangle \otimes |e_2\rangle \otimes \cdots \otimes |e_N\rangle$ is the initial state of the environment without correlation with the state of the machine. Notice that we have assumed that the environment consists of N particles. The interaction between the qubit system and the environment drives the total system into an entanglement pure state

$$|\Phi(t)\rangle = \sum_n c_n(t) |n\rangle \otimes |e[n]\rangle, \qquad (2.1)$$

where $|e[n]\rangle = U_n(t)|e\rangle$ and $U_n(t)$ is the effective evolution operator describing the correlation between the environment and the state $|n\rangle$. To proceed along with the discussion we should consider the reduced density matrix

$$\rho(t) = \mathrm{Tr}_e[|\Phi(t)\rangle\langle\Phi(t)|]$$

$$= \sum_n |c_n(t)|^2 |n\rangle\langle n| + \sum_{n \neq m} c_n(t)c_m^*(t)|n\rangle\langle m|F(n,m;t)$$

$$+ \mathrm{H.c.} \tag{2.2}$$

Here we have traced over the environment variables and defined the decohering factor $F(n,m;t) = \langle e[n]|e[m]\rangle$, which is a transition matrix element $\langle e|U_n^\dagger(t)U_m(t)|e\rangle$ of the environment. The contribution of environment is completely determined by the decohering factor. If $F(n,m;t) = 1$, we have $\rho(t) = |\phi(t)\rangle\langle\phi(t)|$. This is the ideal case. On the contrary, if $F(n,m;t) = 0$, the environment causes a complete decoherence. In such a case, a quantum computation may become invalid (see the Appendix for more precise information). It should be emphasized here that not only the norms of $F(n,m;t)$, but also their phases affect the quantum computation governed by $\rho(t)$.

If we index the elements of the reduced density matrix $\rho(t)$ by m and n, then the equation $F(n,m,t) = 0$ means that its off-diagonal elements vanish. Such a case appeared in the dynamic theory of quantum measurement as a consequence of a certain factorization structure of the effective evolution operator [38–42,31,32]. In fact, if $U_n(t)$ can be factorized as $U_n(t) = \Pi_j^N U_n^j(t)$, where $U_a^j(t)$ only concerns the jth particle in the environment, the decohering factor can be expressed as an N-multiple product

$$F(n,m;t) = \prod_j^N \langle e_j|U_m^{j\dagger}(t)U_n^j(t)|e_j\rangle \equiv \prod_j^N F^j(n,m;t) \tag{2.3}$$

of the single decohering factors $F^j(n,m;t) = \langle e|U_m^{j\dagger}(t)U_n^j(t)|e\rangle$ with norms less than unity. In the macroscopic limit $N \to \infty$, it is possible that $F(n,m;t) \to 0$, for $a' \neq a$, namely, $\langle e[m]|e[n]\rangle = \delta_{m,n}$. This factor reflects almost all the dynamic features of the influence of environment on a quantum computation process. For example, when it can be written in the form $\exp(-t/t_d)$, the coherence will experience a characteristic decay in the time scale t_d. t_d, which characterizes the speed of decoherence, is called the decoherence time. Its value depends on the physical features of the quantum system and their interaction with the environment. So further discussion should concern the microscopic dynamics of interaction between a qubit system and the environment. In the following we will deal with it in the context of a generalized HC model.

We assume the environment is made up of N particles and has a free Hamiltonian in the general form $\hat{H}_E = \Sigma_j^N \hat{H}_j$. Here, the single-particle Hamiltonian $\hat{H}_k$ only depends on dynamical variables x_j (such as canonical coordinate, momentum and spin, etc.). Not knowing further details of $\hat{H}_j$, we will generally consider the problem of to what extent the constitution of environment and its coupling with the qubit system can affect a quantum computation. If the state $|n\rangle(n = 1,2,\ldots,L)$ of the quantum register corresponds to the energy level E_n $(n = 1,2,\ldots,M)$, the Hamiltonian can

be written as a sum $\hat{H}_s = \Sigma_{n=1}^L E_n|n\rangle\langle n|$ of projections onto the subspace spanned by $|n\rangle$. Physically, to satisfy the basic requirement that the states of the qubit system should not change too much when coupled to the environment, the interaction $\hat{H}_I$ in the present model should be chosen to be of a form with the character of quantum nondemolition (QND) [44]

$$\hat{H}_I = \sum_n \sum_j g_{n,j}(x_j)|n\rangle\langle n|. \tag{2.4}$$

It satisfies $[\hat{H}_I, \hat{H}_s] = 0$, and generally, $[\hat{H}_I, \hat{H}_E] \neq 0$. It should be emphasized that, in the dynamic theory of quantum measurement, it is required that the interaction has different strengths for different states $|n\rangle$, i.e., it is required that $g_{nj} \neq g_{mj}$ for $m \neq n$. This is because the so-called measurement is a scheme to read out the states of the system from the number counting of the detector, different numbers corresponding to different states of the system. However, this requirement of nondegeneracy is not necessary when we turn to consider quantum computation.

If the coupling of the system to the environment is degenerate, namely, $g_{n,j}(x_j) = g_{m,j}(x_j)$ for certain $n \neq m$, we can regroup the coefficients of the interaction in the following way: $g_{1,j} = \cdots = g_{d_1,j} \equiv \kappa_{1,j}, g_{d_1+1j} = \cdots = g_{d_1+d_2j} \equiv \kappa_{2,j}, \cdots, g_{d_1+\cdots+d_{q-1}+1j} = \cdots = g_{d_1+\cdots+d_qj} \equiv \kappa_{q,j}, \ldots$. Correspondingly, the Hilbert space $V:\{|n\rangle|n = 1,2,\ldots,L\}$ of the qubit system is decomposed into a direct sum $V_s = \Sigma_q \oplus V^q$ of the subspaces

$$\{|n=m\rangle \equiv |1,m\rangle|m = 1,\ldots,d_1\} \quad (V^1),$$

$$\{|n=m+d_1\rangle \equiv |2,m\rangle|m = 1,\ldots,d_2\} \quad (V^2),$$

$$\{|n=d_1+\cdots+d_{q-1}+m\rangle \equiv |q,m\rangle|m = 1,\ldots,d_q\} \quad (V^q).$$

This decomposition enjoys the property that the coupling has the same strength k_{qj} for the states belonging to the same subspace V^q and has the different strengths κ_{qj} and $\kappa_{q'j}$ for the states $|q,m\rangle$ and $|q',m'\rangle$ belonging to the different subspaces V^q and $V^{q'}$. Now the interaction Hamiltonian can be rewritten as

$$H_I = \sum_{q,m} \sum_j \kappa_{q,j}(x_j)|q,m\rangle\langle q,m|. \tag{2.5}$$

As will be shown in the next section, the above general Hamiltonian $H = \hat{H}_s + \hat{H}_E + \hat{H}_I$ can indeed result in a factorization of the decoherence factors defined for any two different states $|q,m\rangle$ and $|q',m'\rangle$ of the qubit system belonging to different subspaces V^q and $V^{q'}$. Without referring to any concrete modeling of environment and further details of the interaction, we are able to get some useful dynamic information about decoherence in quantum computation from this general model. The obtained conclusions, which are indepen-

dent of concrete models, should be helpful to the further consideration of the decoherence problem.

Before going ahead, we would like to point out that environment may also cause another unwelcome effect on the computation process, namely, dissipating the energy of a qubit system into the environment. Mathematically, it can be described by adding an additional term H_I' not commuting with $\hat{H}_s$ to the interaction. This dissipation effect due to imperfect isolation is characterized by the relaxation time scale $\tau_{\rm rel}$. It is relatively easy to make systems having a very large $\tau_{\rm rel}$ and thus allowing a reasonable number of operations to complete [10]. In contrast, the effect of decoherence is much more insidious because the coherence information leaks out into the environment in a time scale τ_d much shorter than $\tau_{\rm rel}$ as a quantum system evolves [10,13]. Thus the sensibility of quantum computation mainly depends on τ_d rather than $\tau_{\rm rel}$. For this reason, the present discussions in this paper only focus on the decoherence problem rather than the dissipation effect.

III. STATE REDUCTION IN TIME EVOLUTION

In this section we first show that the above general structure of space decomposition indeed dynamically leads to a scheme of grouping the states of the qubit system to avoid decoherence. Let $V_d = V_1 \otimes V_2 \otimes \cdots \otimes V_{N-1} \otimes V_N$ denote the direct product Hilbert space for the environment. $V_k(k = 1,2,\ldots,N)$ denotes the Hilbert space of the kth particle in the environment. We will prove that, if the QND interaction (2.4) or (2.5) is assumed, any coherent superposition $\Sigma_m C_m |q,m\rangle$ of states belonging to the same subspace V^q is decoherence free and a coherent superposition $\Sigma_q D_q |q,m_q\rangle$ of states belonging to different subspaces may experience WFC or decoherence.

Let us choose an initial state $|\sigma(0)\rangle = \Pi_j^N |\sigma_j(0)\rangle \in V_d$, of the environment, and an initial state $|f(0)\rangle = \Sigma_{m,q} C_m^q |q,m\rangle$, of the qubit system. Then the initial state of the total system $|\Phi(0)\rangle = |f(0)\rangle \otimes |\sigma(0)\rangle$ will evolve into an entangling state

$$|\Phi(t)\rangle = \sum_{q,m} \exp[-iE_{qm}t]C_m^q |q,m\rangle \otimes \prod_j U_j^q(t)|\sigma_j(0)\rangle,$$

(3.1)

where $E_{qm} = E_{d_1 + \cdots + d_{q-1} + m}, m = 1,2,\ldots,d_q$. We observe that the effective evolution operator $U_q(t)$ of environment is determined by the evolution operators $U_j^q(t) = \exp\{-i[\hat{H}_j + \kappa_{q,j}(x_j)]t\}$ for particles through the factorized form $U_q(t) = \Pi_j U_j^q(t)$. It follows from Eq. (3.1) that states belonging to the same subspace V^q entangle with the environment through the same factor $\Pi_j U_j^q(t)|\sigma_j(0)\rangle$. This fact is the essential reason why coherence of the superposition $\Sigma_m C_m |q,m\rangle$ of states belonging to the same subspace V^q can be preserved in the dynamic evolution despite the fact that there is interaction between the qubit system and the environment. The similar conclusions can be specially obtained with a concrete model, such as the harmonic oscillator model of environment [20–24].

We can go further to calculate exactly the following reduced density matrix of the qubit system at time t:

$$\rho(t) = \mathrm{Tr}_d[|\Phi(t)\rangle\langle\Phi(t)|]$$

$$= \sum_q \left(\sum_m |C_m^q|^2 |q,m\rangle\langle q,m| \right.$$

$$\left. + \sum_{m \neq ,m'} \exp[iE_{qm'}t - iE_{qm}t]C_m^q C_{m'}^{q*}|q,m\rangle\langle q,m'| \right)$$

$$+ \sum_{q \neq q'} \sum_{m,m'} \exp[iE_{q'm'}t - iE_{qm}t]C_m^q C_{m'}^{q'*}$$

$$\times |q,m\rangle\langle q',m'| \prod_{j=1}^N \langle\sigma_j(0)|U_j^{q'\dagger}(t)U_j^q(t)|\sigma_j(0)\rangle,$$

(3.2)

where Tr_d means taking partial trace over the variables of the environment. From this expression we see that each off-diagonal element of $\rho(t)$, labeled by q and q', is accompanied by a factorized decohering factor

$$F_{q,q'}(N,t) = \prod_{j=1}^N \langle\sigma_j(0)|U_j^{q'\dagger}(t)U_j^q(t)|\sigma_j(0)\rangle \equiv \prod_{j=1}^N F_{q,q'}^j(t)$$

(3.3)

in the form of a factorized function. This kind of factorization structure in the evolution of a wave function is crucial to the occurrence of decoherence or WFC [38,31,32].

Obviously, if the initial state $|f(0)\rangle$ belongs to a single subspace V^q, then the terms accompanied by $F_{q,q'}(N,t)$ do not appear. Thus the system will remain in the pure state $\exp[-iH_s t]|f(0)\rangle\langle f(0)|\exp[iH_s t]$ throughout the evolution process. This fact is significant for developing schemes of error free quantum computations. The expression (3.3) also implies the occurrence of decoherence when a superposition of states mixes the vectors belonging to different subspaces. Intuitively, as $F_{q,q'}(N,t)$ is a multiplication of N factors $F_{q,q'}^j(t)$ with norms not larger than unity, it may approach zero in the macroscopic limit with very large N. To deal with this problem precisely, we define a real number not less than zero, $\Delta_j^{q,q'}(t) = -\ln|F_{q,q'}^j(t)|$. Then the norm of the accompanying factor $F_{q,q'}(N,t)$ is expressed as

$$|F_{q,q'}(N,t)| = \exp\left(-\sum_{j=1}^N \Delta_j^{q,q'} \right).$$

(3.4)

Obviously, the series $\Sigma_{j=1}^\infty \Delta_j^{q,q'}(t) \geq 0$ since each term is not less than zero. There are two cases in which the accompanying factor $F_{q,q'}(N,t)$ approaches zero in the macroscopic limit with very large N. The first case is that the series $\Sigma_{j=1}^\infty \Delta_j^{q,q'}(t)$ diverges on $(0,\infty]$. The second case is that the series converges to a monotonic function of t which approaches positive infinity as $t \to \infty$. Therefore it is possible that

$$\rho(t) \rightarrow \sum_q \left(\sum_m |C_m^q|^2 |q,m\rangle\langle q,m| \right.$$
$$\left. + \sum_{m \neq .m'} \exp[iE_{qm'}t - iE_{qm}t] C_m^q C_{m'}^{q*} |q,m\rangle\langle q,m'| \right)$$
$$(3.5)$$

as $N \rightarrow \infty$. In the next section, some examples will be presented to illustrate the above mentioned circumstances explicitly.

In some cases the above classification of state vectors is a reflection of the structure of some irreducible representation of a certain group chain $G \supset K$ where G, K are chosen such that H_I is G invariant and H_s is at most K invariant. As an example let us consider the group chain $SO(3) \supset SO(2)$. It defines the standard angular basis $|J,M\rangle$ through the Casimir operators $\hat{J}^2$ and $\hat{J}_3$ of SO(3) and SO(2). In this case, we can take $\hat{H}_s = H(\hat{J}^2, \hat{J}_3)$ to be the Zeeman Hamiltonian in a central force field (not Coulomb field) if the interaction $H_I = H_I(\hat{J}^2)$. A special case of the above general discussion has already been given in Ref. [21] where a totally factorized interaction of the form $H_I = \hat{Q} \otimes \Sigma_{j=1}^N f_j(x_j)$ is used. Here $\hat{Q}$ is a system variable commuting with the free Hamiltonian $\hat{H}_s$ of the qubit system and x_j $(j=1,2,\ldots,N)$ are the variables of the environment with the free Hamiltonian $\hat{H}_D = \Sigma_k^N \hat{H}_k$. The Hilbert space V_s for the system is spanned by $|q,m\rangle$ $(m=1,2,\ldots,d_q$ for a given $q)$, and the common eigenstates of $\hat{Q}$ and $\hat{H}_s$ are labeled by q and m.

$$\hat{Q}|q,m\rangle = e_q|q,m\rangle, \hat{H}_s|q,m\rangle = E_{qm}|q,m\rangle. \quad (3.6)$$

Then we have the direct sum decomposition $V_s = \Sigma_q \oplus V^q$ with the eigenspaces $V^q = \text{Span}\{|q,m\rangle | m = 1,2,\ldots,d_q\}$. This special interaction can be extended to a most general form with several system variables $\hat{Q}_j(j = 1,2,\ldots,K)$ that annihilate certain subspaces of the qubit system simultaneously. In fact a generalization along this line leads to an elegant mathematical structure [22,23]. For mathematical details we refer the readers to Ref. [23] where it was systematically described in the framework of error-avoiding quantum coding.

IV. DYNAMIC DECOHERENCE IN AN ENVIRONMENT CONSISTING OF N TWO-LEVEL SUBSYSTEMS

To make a deeper elucidation of the above mentioned quantum dynamic mechanism of decoherence in quantum computation and the relevant trick of grouping the states of a qubit system, in this section we model the environment as consisting of N two-level subsystems. We recall that Caldeira and Leggett [25] have pointed out that any environment weakly coupling to a system may be approximated by a bath of oscillators. On the condition that ''each environmental degree of freedom is only weakly perturbed by its interaction with the system,'' they have also justified describing the influence of environment by a coupling term linear in the bath variables up to the first order perturbation. We observe that any linear coupling only involves the transitions between the lowest two levels (ground state and the first excitation state)

of each harmonic oscillator in the environment though it has many energy levels. Therefore in such a case we can also describe the environment as a combination of many two-level subsystems without losing generality. In fact, for quantum computation, Unruh [11] and Palma et al. [12] have considered the harmonic oscillator environment. Their model is equivalent to the one introduced to explain the WFC in quantum measurement by Sun et al. [31,32]. A similar model has also been touched by Leggett and co-workers [25,26] and Gardiner [27] in studying the tunneling effect in a quantum dissipative process. Here we choose equivalently the two-level subsystem model to manifest some characters independent of environment in the weakly coupling limit and to demonstrate explicitly the qualitative calculation of decoherence time through a sample example without quantum dissipation.

Let $|g_j\rangle$ and $|e_j\rangle$ be the ground and excited states of the jth subsystem. We define the quasispin operators

$$\sigma_1(j) = |e_j\rangle\langle g_j| + |g_j\rangle\langle e_j|,$$
$$\sigma_2(j) = -i[|e_j\rangle\langle g_j| - |g_j\rangle\langle e_j|],$$
$$\sigma_3(j) = |e_j\rangle\langle e_j| - |g_j\rangle\langle g_j|.$$

Then we introduce the Hamiltonian of the environment $H_e = \Sigma_{j=1}^N \hbar \omega_j \sigma_3(j)$ and the interaction coupling to a qubit system $H_I = f(S)\Sigma_{j=1}^N \hbar g_j \sigma_2(j)$ where $f(S)$ is a function of the variable S of the qubit system. Let us for the time being focus on the simplest case where the system consists of two qubits with the Hamiltonian

$$H_s = \hbar \eta_1 S_3(1) + \hbar \eta_2 S_3(2). \quad (4.1)$$

Here $S(1) = \sigma_s \otimes 1, S_s(2) = 1 \otimes \sigma_s$ $(s=1,2,3)$ denote spin operators acting on the first and the second qubits, respectively; and σ_s $(s=1,2,3)$ is the usual Pauli matrix. We consider the special interaction given by

$$f(S) = S_3(1) + S_3(2). \quad (4.2)$$

It means that in our model the interaction has the same strength for different states. This model is very simple, or even too simple in some sense. But we would like to point out that some decoherence-avoiding scheme such as the free Hamiltonian elimination model in Ref. [21] is substantially only a plain generalization of the present example to the multipair case if one takes into account the SU(2) rotation transformation.

Let $|1\rangle$ and $|0\rangle$ be the qubit states that satisfy $S_3|k\rangle = (-)^{k+1}|k\rangle$ $(k=1,0)$. Then the Hilbert space, spanned by

$$\{|1,1\rangle = |1\rangle\otimes|1\rangle, |1,0\rangle = |1\rangle\otimes|0\rangle,$$
$$|0,1\rangle = |0\rangle\otimes|1\rangle, |0,0\rangle = |0\rangle\otimes|0\rangle\}$$

contains a null subspace V^0 of H_I spanned by $|1,0\rangle$ and $|0,1\rangle$. Any superposition $|\phi(0)\rangle = A|1,0\rangle + B|0,1\rangle$ in this subspace will preserve its purity in the evolution process despite the fact that there is interaction between the system and the environment. Precisely, the pure state $|\phi(0)\rangle\langle\phi(0)|$ will evolve into the pure state $U_0(t)|\phi(0)\rangle\langle\phi(0)|U_0^\dagger(t)$ where $U_0(t) = \exp[-i\eta_1 t S_3(1) - i\eta_2 t S_3(2)]$ is the free evolu-

tion operator of the qubit system. Physically, this fact implies that no useful information leaks out of the system in the process and the coherence is preserved. This analysis can be easily generalized to the many-bit case where the free-qubit Hamiltonian takes the form $H_s = \sum_{k=0}^{L-1} \hbar \, \eta_k S_3(k)$ and its interaction with the environment is determined by

$$f(S) = \sum_{k=0}^{L-1} \lambda_k S_3(k), \qquad (4.3)$$

where L is the number of qubits used and

$$S_s(k) = \overbrace{1 \otimes \cdots \otimes 1}^{k-1\,times} \otimes \sigma_s \otimes 1 \otimes \cdots \otimes 1.$$

The different λ_k's indicate that each single qubit has a different coupling to the same environment. In the Hilbert space of this L-qubit system with the basis

$$|q\rangle = |q_0\rangle \otimes |q_1\rangle \otimes |q_2\rangle \otimes \cdots \otimes |q_{L-1}\rangle,$$

$$q_k = 0,1, \quad k = 0,1,2,\ldots,L$$

the coherence-preserving subspace V^ξ can be spanned by those basis vectors $|q\rangle$ satisfying

$$\sum_{k=0}^{L-1} \lambda_k (-)^{q_k+1} = \mathrm{const} \times \xi.$$

Let us return to the two-qubit example. If a superposition contains a vector outside the decoherence free subspace, decoherence will happen due to the entanglement of system states with environment states. For example, if the initial state $|\varphi(0)\rangle = C|0,0\rangle + D|1,1\rangle$ of the qubit system involves states not belonging to V^0 and the environment is initially in the vacuum state $|0\rangle_e = |g_1\rangle \otimes |g_2\rangle \otimes \cdots \otimes |g_N\rangle$, the corresponding pure state density matrix $|\varphi(0)\rangle \langle \varphi(0)| \otimes |0\rangle_e \otimes_e \langle 0|$ of the total system will experience a unitary evolution to reach a pure state density matrix $\rho_T(t)$. The reduced density matrix of the qubit system

$$\rho(t) = \mathrm{Tr}_e \rho_T(t) = |C|^2 |0,0\rangle\langle 0,0| + |D|^2 |1,1\rangle\langle 1,1|$$
$$+ \{CD^* \exp[2i(\eta_1 + \eta_2)t] F(N,t)|0,0\rangle\langle 1,1| + \mathrm{H.c.}\}$$
$$(4.4)$$

is no longer pure because the environment state becomes correlated with the qubit system state. Here the decohering factor

$$F(N,t) \equiv \prod_{j=1}^{N} F_j(t) = \prod_{j=1}^{N} \langle g_j | U_{j1}^\dagger(t) U_{j0}(t) | g_j \rangle \quad (4.5)$$

is determined by the effective evolution operators of the form

$$U_{j\alpha}(t) = \exp[-i\omega_j \sigma_3(j)t - i\xi_\alpha g_j \sigma_2(j)t],$$

$$\xi_1 = 2, \quad \xi_0 = -2, \quad (\alpha = 0,1) \quad (4.6)$$

corresponding to the qubit states $|0,0\rangle$ and $|1,1\rangle$, respectively. Using the formula $\exp[i\vec{\sigma}\cdot\vec{A}] = \cos A + i\vec{\sigma}\cdot\vec{n}_A \sin A$ for a given vector $\vec{A}$ of norm A along the direction $\vec{n}_A$, we get the explicit form of $U_{j\alpha}(t)$,

$$U_{j\alpha} = \cos(\Omega_{j\alpha}t) - i[\sigma_2(j)\sin\theta_{j\alpha} + \sigma_3(j)\cos\theta_{j\alpha}]\sin(\Omega_{j\alpha}t),$$
$$(4.7)$$

where $\tan\theta_{j\alpha} = \xi_\alpha g_j/\omega_j$, $\Omega_{j\alpha} = \sqrt{(g_j\xi_\alpha)^2 + \omega_j^2}$. Then, using Eq. (4.7), after some straightforward calculation we get the real factorized decohering factor

$$F(N,t) = \prod_{j=1}^{N} F(j,t) = \prod_{j=1}^{N} [1 - 2\sin^2\theta_j\sin^2\Omega_j t], \quad (4.8)$$

which is an N-multiple product of the factors $F(j,t)$ of norms less than 1. Here, we have used the new definitions $\tan\theta_j = 2g_j/\omega_j$, $\Omega_j = \sqrt{4g_j^2 + \omega_j^2}$ for the special labels $\xi_1 = 2, \xi_0 = -2$. Therefore the temporal behavior of decoherence is described by

$$|F(N,t)| = e^{-S(t)} \equiv \exp \sum_{j=1}^{N} \ln \left| 1 - 8\frac{g_j^2}{\Omega_j^2}\sin^2(\Omega_j t) \right|.$$
$$(4.9)$$

This is an exact result without any approximation. A special case is that the subsystems constituting the environment are identical and the environment has a constant discrete spectrum, i.e., $\omega_k = \mathrm{constant}\ \omega$, $g_k = \mathrm{constant}\ g$. In this case, the decohering factor $|F(N,t)|$ becomes an exponential function $\exp[-\beta N]$ of N with a positive coefficient $\beta = -\ln|1 - (8g^2/\sqrt{4g^2+\omega^2})\sin^2(\sqrt{4g^2+\omega^2}t)| \geqslant 0$. So as $N \to \infty$, the off-diagonal elements with the decohering factor $|F(N,t)|$ approach zero for all t except those satisfying $\sqrt{4g^2+\omega^2} = (2k\pi/t)$ $(k = 0,1,2,\ldots)$. For more information, one needs a detailed analysis on the behavior of the series $S(t) = -\sum_{j=1}^{N}\ln|1 - 8(g_j^2/\Omega_j^2)\sin^2(\Omega_j t)|$ for various spectrum distributions of environments. Of special interest is the case with a continuous spectrum. In such a case $S(t)$ can be reexpressed in terms of a spectrum distribution $\rho(\omega_k)$ as

$$S(t) = -\int_0^\infty \rho(\omega_k)\ln\left| 1 - 8\frac{g_j^2}{\Omega_j^2}\sin^2(\Omega_j t) \right| d\omega_k. \quad (4.10)$$

If the above integral diverges to positive infinity, or converges to a monotonously increasing function of t, e.g., $S(t) \to \gamma t$, the norm of the decoherence factor decays to zero at infinite N as $t \to \infty$.

Physically, infinite N means that the environment is a macroscopic object since it is made of infinite numbers of subsystems in that case. Therefore the occurrence of decoherence of the qubit system at infinite N manifests a transition of the qubit system from the quantum realm to the classical realm as the environment surrounding it becomes macroscopic.

V. UNIVERSALITY OF ENVIRONMENTS
IN THE WEAK-COUPLING LIMIT

In the preceding section the analysis of decoherence for quantum computation is made by modeling the environment as a bath of a large number of two-level subsystems. The results seem to be different from those obtained from the harmonic oscillator model of environment [13,31]. Actually this is a specious observation. Indeed, an environment surrounding a qubit system for quantum computation may be very complicated. Intuitively, the dynamic process of decoherence in quantum computation should depend on the details of interaction between the qubit system and the environment. Different environments should cause different decoherence processes with distinct characters for the same qubit system. So generally it seems impossible to control decoherence in a qubit system. Nevertheless, one may well expect that in some limit situations there exists a certain universality in the dynamics of interaction so that the physical parameters dominating a quantum computation process would not depend on the details of environment. For the tunneling problem in the quantum dissipation process, this kind of universality has been considered by Caldeira and Leggett [25,26] by modeling the environment as a bath of harmonic oscillators with a linear coupling to the system.

In this section, we illustrate that, in the weakly coupling limit, the decoherence time obtained from the two-level subsystem model of environment coincides with that from the harmonic oscillator model. In the case of weak coupling, we have $g_j \ll \omega_j$. Thus the norm (4.8) of the decohering factor in the two-level subsystem model of environment becomes

$$|F(N,t)| = e^{-S(t)} \equiv \exp\left(-\sum_{j=1}^{N} \frac{8g_j^2}{\omega_j^2} \sin^2(\omega_j t) \right). \quad (5.1)$$

In the case of continuous spectrum, the sum $S(t) = \sum_{j=1}^{N} (8g_j^2/\omega_j^2)\sin^2(\omega_j t)$ can be reexpressed in terms of a spectrum distribution $\rho(\omega_k)$ as

$$S(t) = \int_0^\infty \frac{8}{\omega_k^2} \rho(\omega_k) g_k^2 \sin^2 \omega_k d\omega_k. \quad (5.2)$$

From some concrete spectrum distributions, interesting circumstances may arise. For instance, when $\rho(\omega_k) = (1/\pi)\gamma/g_k^2$ the integral converges to a negative number proportional to time t, namely, $S(t) = \gamma t$ [31]. This shows that the norm of the decoherence factor is exponentially decaying and as $t \to \infty$, the off-diagonal elements of the density matrix vanish simultaneously. Another example of spectrum distribution is $\rho(\omega_k) = 2\eta \omega_k^2/\pi g_k^2$ of Ohmic type [25,26], which leads to a divergent integral $S(t) \to \infty$ for $t \neq 0$. Therefore, in the present example, we can choose the spectrum distributions of the two-level subsystems in the environment such that the series $S(t)$ diverges to infinity. Then the dynamical evolution of the whole system will result in a complete decoherence in the reduced density matrix. This observation is quite similar to that made in the context of the harmonic oscillator model of environment.

To compare these models of environment in the weakly coupling limit, we should first briefly summarize conclusions from the harmonic oscillator model of environment in terms

of the present notations [31]. Let $a_i^\dagger$ and a_i be the creation and annihilation operators for the ith harmonic oscillator in the environment. The Hamiltonian of the environment takes the form $H = \sum_{j=1}^{N} \hbar \omega_j a_j^\dagger a$ and its interaction with the qubit system can be modeled as a linear coupling:

$$H_I = f(s) \sum_{j=1}^{N} \hbar g_j (a_j^\dagger + a_j), \quad (5.3)$$

where $f(s)$ is a linear or nonlinear function of the qubit system variable s. Let the initial state of the qubit system $|\varphi(0)\rangle = C|\alpha\rangle + D|\beta\rangle$ be a coherent superposition of two eigenstates of s, $s|\alpha\rangle = \alpha|\alpha\rangle$, $s|\beta\rangle = \beta|\beta\rangle$ and let the environment be initially in the vacuum state $|0\rangle_e = |0_1\rangle \otimes |0_2\rangle \otimes \cdots \otimes |0_N\rangle$ where $|0_j\rangle$ is the ground state of the jth single harmonic oscillator. The corresponding decohering factor $F(N,t) = \Pi_{j=1}^{N} {}_h\langle 0|U_j^{\beta\dagger}(t)U_j^\alpha(t)|0\rangle_h \equiv \Pi_{j=1}^{N} F_j(t)$ can be obtained by solving the Schrödinger equations of $U_j^\gamma(t)$ ($\gamma = \alpha\beta$) governed by the Hamiltonian of a forced harmonic oscillator

$$H_{j\gamma} = \hbar \omega_j a_j^\dagger a_j + f(\gamma) g_j (a_j^\dagger + a_j). \quad (5.4)$$

In fact, by the so-called Wei-Norman algebraic expansion technique one has the following exact result [12,31,35]:

$$F(N,t) = \exp\left[-[f(\alpha) - f(\beta)]^2 \sum_{j=1}^{N} \frac{2g_j^2}{\omega_j^2} \sin^2\left(\frac{\omega_j t}{2} \right) \right]$$
$$\times \exp\left[-i[f(\alpha)^2 - f(\beta)^2] \sum_{j=1}^{N} \frac{g_j^2}{\omega_j} \left(t + \frac{\sin(\omega_j t)}{\omega_j} \right) \right].$$
$$(5.5)$$

The decoherence time is decided by the norm part of $F(N,t)$, which is the same as that in Eq. (5.1) from the two-level subsystem model of environment in the weakly coupling limit. This can easily be seen if only one replaces $\omega_j/2$ in the above equation with ω_j in the case that we choose the initial state with $|\alpha\rangle = |0,0\rangle$, $|\beta\rangle = |1,1\rangle$ and the coupling function (4.2). This simply implies that in this case the details of environment do not affect the speed at which a quantum system approaches the classical kingdom. Therefore we have shown that there exists a universality of decoherence independent of the constitution of environments, that is, in the weakly coupling limit, the decoherence time, derived from the model of two-level subsystems, indeed coincides with that derived from the model of harmonic oscillators.

Notice that, for the specific choice $|\varphi(0)\rangle = C|0,0\rangle + D|1,1\rangle$ of the initial state, the phases of both decohering factors (4.7) and (5.5) obtained from two different models are zero due to $f(\alpha)^2 - f(\beta)^2 = 0 = \xi_0^2 - \xi_1^2$. But this is only accidental. If we start from a general initial state, even in the weakly coupling limit, the differences in these two different models of environment are reflected in the phases of the decohering factors. For instance, we take $|\varphi(0)\rangle = C|0,1\rangle + D|1,1\rangle$ in the weakly coupling limit, we show that the phase $\sum_{j=1}^{N} (g_j^2/\omega_j^2)\sin(2\omega_j t)$ of decohering factors

$$F_L(N,t) \simeq \exp\left(-\sum_{j=1}^{N} \frac{g_j^2}{\omega_j^2} [2\sin^2(\omega_j t) + i\sin(2\omega_j t)] \right)$$
(5.6)

obtained from the two-level subsystem model is essentially different from that $\Sigma_{j=1}^{N} 4(g_j^2/\omega_j)[t + \sin(\omega_j t)/\omega_j]$ in Eq. (5.5) from the harmonic oscillator model. In the large time scale limit, the latter is proportional to time t and thus leads to a fast oscillation factor $\lim_{t\to\infty} \exp[-4i(g_j^2/\omega_j)t]$ with a quite uncertain phase, but the former, with the time-dependent term $\sin(2\omega_j t)$, leads to the slow oscillation factor $\exp[i(g_j^2/4\omega_j^2)\sin(2\omega_j t)]$ with a finite phase. Thus the present result seems to be more reasonable. Furthermore, it will be shown in the next section that when the temperature is not low, different phase effects happen for different models of environment, or for different environments in practice. These differences do affect the success probability of a quantum computation. So only in the case of weak coupling at low temperature can there exist a common scheme of controlling a qubit system in various environments such that decoherence is avoided.

VI. DECOHERENCE FOR THE L-QUBIT SYSTEM AND AT FINITE TEMPERATURE

In this section the influences of the size of quantum register and the temperature of environment on the process of decoherence in quantum computation are investigated in the framework of the two-level model. In comparison with the harmonic oscillator model, it will be shown that as far as the size effect is concerned, the difference between the two models is reflected in the phases of the decohering factors. Even in the weakly coupling limit, this difference still exists. As for the effect of finite temperature, it is observed that, though the decoherence time may be independent of the temperature of environment, the decohering factors have different phases at different temperatures. Physically, the phase of a decohering factor is closely associated with the probability with which the machine ends in a specific state. Consequently, it will affect the success probability of a quantum computation (see the Appendix).

Usually, decoherence time t_d depends on the physical features of the quantum system and their interaction with the environment. For a single qubit system some numerical estimates of t_d have been made by DiVincenzo [45] for several physical realizations. It ranges from 10^4 s (for nuclear spins) to 10^{-12} s (for the electron-hole excitation in bulk of a semiconductor). In practice, to carry out a quantum computation, one needs a large number of qubits. The dynamic analysis in the following shows that the speed of decoherence becomes larger as the number of qubits increases. Notice that the dependence of decoherence time on the size of quantum register has already been discussed in various references [4,5,10,12] in the framework of the harmonic oscillator model of environment. But it should be emphasized that here the same dependence of decoherence time on the size of quantum register will be derived in the context of a general model to expose a universality. An important by-product of our analysis is the derivation of the imaginary part of the decohering factor. We would also like to point out that the

importance of the phase of the decohering factor cannot be overemphasized as to a large extent it actually determines the probability of success of a quantum computation.

Let us consider the L-qubit system coupling to the environment mentioned in the preceding section. The interaction constants λ_k are chosen such that the eigenvalues $\xi(q) = \Sigma_{k=0}^{L-1} \lambda_k(-1)^{q_k+1}$ are not degenerate for $\{q_k=0,1\}$. Starting from an initial state $|\varphi(0)\rangle = C|q\rangle + D|q'\rangle$, where $|p\rangle = \Pi_{k=1}^{L-1} \otimes |p_k\rangle, p=q,q'$, the initial pure state density matrix of the total system will experience a unitary evolution to reach a pure state density matrix $\rho_L(t)$. Imitating the calculation process in Sec. IV, we can obtain the reduced density matrix $\rho(t) = \mathrm{Tr}_e \rho_L(t)$. Its off-diagonal elements are proportional to the decohering factor

$$F_L(N,t) = \prod_{j=1}^{N} F_L(j,t) \equiv \prod_{j=1}^{N} \langle g_j | U_{jq}^{\dagger}(L,t) U_{jq'}(L,t) | g_j \rangle,$$
(6.1)

where $U_{jq}(L,t) = \exp[-i\omega_j\sigma_3(j)t - i\xi(q)g_j\sigma_2(j)t]$ is the effective evolution operator acting on the subsystem in the environment. Using the notions $\tan\theta_j(q) = \xi(q)g_j/\omega_j$, $\Omega_j(q) = \sqrt{[g_j\xi(q)]^2 + \omega_j^2}$, and the matrix representation of $U_{jq}(L,t)$, after a straight calculation we get

$$F_L(j,t) = \sin\theta_j(q)\sin[\Omega_j(q)t]\sin\theta_j(q')\sin[\Omega_j(q')t]$$
$$+ \{\cos[\Omega_j(q)t] - i\cos\theta_j(q)\sin[\Omega_j(q)t]\}$$
$$\times \{\cos[\Omega_j(q')t] + i\cos\theta_j(q')\sin[\Omega_j(q')t]\}.$$
(6.2)

Trivially, $F_L(N,t)$ becomes unity when $q=q'$. However, when $q \neq q'$, in the weakly coupling limit $g_j \ll \omega_j$, we have $\sin\theta_j(q) \simeq \theta_j(q)$, $\cos\theta_j(q) \simeq 1 - \frac{1}{2}\theta_j^2(q)$, and $\Omega_j(q) \simeq \omega_j$. Thus

$$F_L(j,t) \simeq 1 - \frac{1}{2}\{\theta_j(q) - \theta_j(q')\}^2 \sin^2(\omega_j t)$$
$$+ \frac{i}{4}\{\theta_j^2(q) - \theta_j^2(q')\}\sin(2\omega_j t).$$

Considering the approximations $\theta_j(q) \simeq \sin\theta_j(q) \simeq \xi(q)g_j/\omega_j$, we obtain the decohering factors

$$F_L(j,t) \simeq 1 - \frac{g_j^2}{2\omega_j^2}\{\xi(q) - \xi_j(q')\}^2 \sin^2(\omega_j t)$$
$$+ \frac{ig_j^2}{4\omega_j^2}\{\xi_j^2(q) - \xi_j^2(q')\}\sin(2\omega_j t),$$
(6.3)

which just has the same real part as that obtained from the harmonic oscillator model of environment. Consequently, the temporal behavior of the decoherence is described by $F(N,t)$, and actually determined by

$$|F_L(N,t)| = \exp[-S_L(t)]$$

$$= \exp\left(-[\xi(q)-\xi(q')]^2\sum_{j=1}^N \frac{g_j^2}{2\omega_j^2}\sin^2(\omega_j t)\right).$$

$$(6.4)$$

Here $S_L(t) = [\xi(q)-\xi(q')]^2\sum_{j=1}^N(g_j^2/2\omega_j^2)\sin^2(\omega_j t)$ is a non-negative series. When the qubits are identical we have $\lambda_k = 1$. Then the fastest decoherence happens if we choose $|q\rangle = |q_0=1\rangle\otimes|q_1=1\rangle\otimes\cdots\otimes|q_{L-1}=1\rangle$ and $|q'\rangle = |q_0=0\rangle\otimes|q_1=0\rangle\otimes\cdots\otimes|q_{L-1}=0\rangle$. In this case $|F_L(N,t)| = \exp[-L^2 S(t)]$. Thus for the instance with $S(t) = \gamma t$, we have $|F(N,t)| = \exp[-L^2\gamma t]$ where $\tau_d = \gamma^{-1}$ is the decoherence time for a single qubit. This shows that the characteristic time of the fastest decoherence happening in the L-qubit system is L^2 times that of a single qubit. This conclusion was obtained by Palma *et al.* [12]. But it should be stressed that the imaginary part $\sum_{j=1}^N(g_j^2/4\omega_j^2)\{\xi_j^2(q)-\xi_j^2(q')\}\sin(2\omega_j t)$ appearing here is also essentially different from $[f(\alpha)^2 - f(\beta)^2]\sum_{j=1}^N(g_j^2/\omega_j)[t+\sin(\omega_j t)/\omega_j]$. Thus the present result seems to be more reasonable. To conclude, Eqs. (6.2) and (5.5) demonstrate that different models of environment might lead to quite different results when the system is far away from the weakly coupling limit.

All of the above discussions about decoherence in quantum computation only concern the situation of zero temperature. Let us now take the influence of finite temperature into account. Suppose the environment is initially prepared in an equilibrium state described by the canonical density matrix

$$\rho_b(0) = \frac{\exp(-\beta\hat{H}_e)}{\mathrm{Tr}_b\exp(-\beta\hat{H}_e)} = \prod_{j=1}^N \rho_{jb}(0)$$

$$= \prod_{j=1}^N \frac{e^{-\beta\omega_j\sigma_3(j)}}{2\cosh(\beta\omega_j)}, \quad \beta = \frac{1}{K_B T} \quad (6.5)$$

and the initial state of the qubit system is a pure state $\rho_s(0) = |\phi(0)\rangle\langle\phi(0)|$. Then the initial state of the total system is described by the product density matrix $\rho(0) = \rho_s(0)\otimes\rho_b(0)$.

For a special initial state of the form $|\phi(0)\rangle = A|0,0\rangle + B|1,1\rangle$, we obtain the decohering factor $F_2(N,t) = \prod_j^N[1-2\sin^2\theta_j\sin^2(\Omega_j t)]$, which is the same as that derived under the condition of zero temperature. This is only an accidental situation owing to the special choice of the initial state. For a general initial state, we will see, the decoherence process indeed shows a temperature dependence. In fact, for a general initial state $|\varphi(0)\rangle = C|q\rangle + D|q'\rangle$, we can calculate the factor $F_L(N,t) = \prod_{j=1}^N F_L(j,t)$ as follows:

$$F_L(j,t) \equiv \mathrm{Tr}_b[U_{jq'}(L,t)\rho_{jb}(0)U_{jq}^\dagger(L,t)]$$

$$= \sin\theta_j(q)\sin[\Omega_j(q)t]\sin\theta_j(q')\sin[\Omega_j(q')t]$$

$$+ \cos[\Omega_j(q)t]\cos[\Omega_j(q')t]$$

$$+ \cos\theta_j(q)\sin[\Omega_j(q)t]\cos\theta_j(q')\sin(\Omega_j(q')t)$$

$$- \frac{i}{2}\tanh(\beta\omega_j)\sin[2\Omega_j(q)t]$$

$$\times\{\cos\theta_j(q)-\cos\theta_j(q')\}. \quad (6.6)$$

Notice that the effect of finite temperature only appears in the imaginary part of the decohering factor. In general the speed of decoherence characterized by the norm $\prod_{j=1}^N|F_L(j,t)|$ of $F_L(N,t)$ depends on temperature and there is not a global time scale characterizing the coherence decaying.

In the weakly coupling limit, it is not difficult to see that

$$F_L(j,t) \simeq 1 - \tfrac{1}{2}\{\theta_j(q)-\theta_j(q')\}^2\sin^2[\omega_j t]$$

$$- (i/4)\tanh(\beta\omega_j)\sin[2\omega_j t][\theta_j^2(q)-\theta_j^2(q')]$$

or

$$F_L(j,t) \simeq 1 - \frac{g_j^2}{2\omega_j^2}\{\xi(q)-\xi_j(q')\}^2\sin^2(\omega_j t)2$$

$$+ \frac{ig_j^2}{4\omega_j^2}\{\xi_j^2(q)-\xi_j^2(q')\}\tanh(\beta\omega_j)\sin(2\omega_j t).$$

$$(6.7)$$

This result can be rewritten as the following exponential form:

$$F_L(j,t) = \exp\left(-\frac{g_j^2}{2\omega_j^2}\{\xi(q)-\xi_j(q')\}^2\sin^2(\omega_j t)\right)$$

$$\times\exp\left(\frac{ig_j^2}{4\omega_j^2}\{\xi_j^2(q)-\xi_j^2(q')\}\right.$$

$$\left.\times\tanh(\beta\omega_j)\sin(2\omega_j t)\right), \quad (6.8)$$

where we have used the approximation $|1+\varepsilon+i\epsilon| = |\exp[\varepsilon + i\epsilon]| = e^\varepsilon \simeq 1+\varepsilon$ for real infinitesimal ϵ and ε. In the case that the series $\sum_j(g_j^2/2\omega_j^2)\{\xi(q)-\xi_j(q')\}^2\sin^2(\omega_j t)]$ converges to a linear function γt of time t, the decoherence time is $\tau_d = 1/\gamma$ by definition. This reveals the novel fact that, in an environment weakly interacting with the qubit system, the decoherence time does not depend on temperature, as a result of the temperature-independent norm of $F_L(j,t)$. Thus in this case thermal fluctuation plays a role in quantum computation only through affecting the phases of the off-diagonal elements of the reduced density matrix.

In conclusion, in the weakly coupling and low temperature limit, there exists an interesting universality in respect to the behavior of decoherence of a qubit system. In other words, under certain conditions the decoherence time or the speed of decoherence is independent of the adopted models of environment and the temperature of environment.

VII. DISCUSSION

From the above discussions about decoherence in quantum computation, we have seen, at least in the weakly coupling limit, that for a quantum register with L qubits the relevant coherence may experience a decay characterized by the factor $\exp(-[L^2/t_d]t)$ where t_d is the typical decoherence time for a single bit. Thus, in general, if a quantum algorithm calls for K elementary computation steps and each step takes

time τ on the average, in order that the algorithm could be feasible we should have the condition

$$L^2 \tau K < t_d. \tag{7.1}$$

Generally speaking, this would impose a strong restriction on L and K. We need to develop proper quantum error-correction schemes to cope with this difficulty caused by decoherence, which is unavoidable in the quantum kingdom. Along this line there has been some progress. Nevertheless there is another severe problem which may endanger the assumed great utility of quantum computers. In the Appendix we have shown that environment may affect the efficiency of a quantum algorithm. Although our discussion is not sophisticated enough it indeed gives us frustrating information. This problem, deeply rooted in the quantum kingdom, seems to have been ignored. We think it is now time to face it seriously.

Even if one can effectively control the speed of decoherence owing to the existence of a universality, so that the above condition is satisfied, there is still another stubborn problem to handle. Though different environments may lead to the same decoherence time, the decoherence processes may be quite different, characterized by different phases in decohering factors. This will make it almost impossible to bring a quantum computation under control in different environments. To be more precise, let us take Shor's prime factorization algorithm as an example. In this case, as reformulated in the Appendix, the effect of decoherence is reflected in the probability

$$p'(c,k) = \frac{1}{q^2} \sum_{a,a'}^{q-1} \exp\left(\frac{2\pi i(a-a')c}{q}\right) F(a,a'), \tag{7.2}$$

with which the machine ends in a particular state $|c,x^k \mathrm{mod}(n)\rangle \equiv |c,k(x;n)\rangle$ (for the notations and their meanings see the Appendix). Here, the decohering factors $F(a,a') = F(a,a';t)$ are usually complex. It is easily seen that the probability closely depends on both their norms and their phases. Since this probability is at the core of Shor's prime factorization algorithm, the phases of decohering factors play a crucial role here. At present as far as we know, how to control the influences of different phases due to different environments is still an open problem in quantum computation.

APPENDIX: DECOHERENCE IN THE SHOR FACTORIZATION ALGORITHM

In this appendix we illustrate the possible influence of environment on the validity of a quantum algorithm through the example of Shor's prime factorization algorithm. In this case, we recall, the so-called quantum computer has two registers in use. According to Shor's method [6], to factorize a number n one should first of all choose a number x. Then the first step is to put the first register in the uniform superposi-

tion of q states $|a\rangle$ $(a=0,1,\ldots,q-1)$ and the second one in a single state $|0\rangle$. This leaves the machine in the state

$$|\phi(0)\rangle = \frac{1}{\sqrt{q}} \sum_{a=0}^{q-1} |a\rangle \otimes |0\rangle. \tag{A1}$$

Next, one computes $x^q \mathrm{mod}(n)$ in the second register, leaving the machine in the state

$$|\phi(t)\rangle = \frac{1}{\sqrt{q}} \sum_{a=0}^{q-1} |a\rangle \otimes |x^a \mathrm{mod}(n)\rangle \equiv \frac{1}{\sqrt{q}} \sum_{a=0}^{q-1} |a,x^a \mathrm{mod}(n)\rangle. \tag{A2}$$

Then one performs a Fourier transform A_q on the first register. This leaves the machine in the state

$$|\phi_F(t)\rangle = \frac{1}{q} \sum_{a=0}^{q-1} \sum_{c=0}^{q-1} \exp\left(\frac{2\pi iac}{q}\right) |c,x^a \mathrm{mod}(n)\rangle. \tag{A3}$$

Finally one observes the machine. One easily finds that the probability that the machine ends in a particular state $|c,x^k \mathrm{mod}(n)\rangle \equiv |c,k(x;n)\rangle$ is

$$p(c,k) = \frac{1}{q^2} \left| \sum_{a:x^a = x^k \mathrm{mod}(n)}^{q-1} \exp\left(\frac{2\pi iac}{q}\right) \right|^2. \tag{A4}$$

Shor shows that if c lies in a particular region one can determine a nontrivial factor of n from the value of c. Denote the one-try-success probability of this method by p_s. Then one has the following result:

$$p_s \geq r\phi(r)p(c,k) \geq 1/\ln n \geq 1/3 \ln n, \tag{A5}$$

where r is the least integer such that $x^r \equiv 1(\mathrm{mod}\ n)$ and ϕ is Euler's quotient function.

Now let us take the influence of environment into account to some extent. Assume that the environment consists of N particles. In this case we denote by $\phi'(0), \phi'_F(t), p'(c,k)$, and p'_s the correspondences of $\phi(0), \phi_F(t), p(c,k)$, and p_s, respectively. Then we have

$$|\phi'(0)\rangle = \frac{1}{\sqrt{q}} \sum_{a=0}^{q-1} |a\rangle \otimes |0\rangle \otimes |e\rangle, \tag{A6}$$

where $|e\rangle = |e_1\rangle \otimes |e_2\rangle \otimes \cdots \otimes |e_N\rangle$ is the initial state of the environment without correlation with the state of the machine. Here, $|e_k\rangle$ $(k=1,2,\ldots,N)$ denotes the initial state of an individual particle in the environment. Accordingly,

$$|\phi'(t)\rangle = \frac{1}{\sqrt{q}} \sum_{a=0}^{q-1} |a\rangle \otimes |x^a \mathrm{mod}(n)\rangle \otimes |e[a]\rangle$$

$$\equiv \frac{1}{\sqrt{q}} \sum_{a=0}^{q-1} |a,x^a \mathrm{mod}(n)\rangle \otimes |e[a]\rangle, \tag{A7}$$

where $|e[a]\rangle = U_a(t)|e\rangle$ and $U_a(t)$ is the effective evolution operator of the environment correlated with the state $|a\rangle$. For simplicity we do not consider the influence of environment in the process of the Fourier transform A_q. Thus we have

$$|\phi_F'(t)\rangle = \frac{1}{q}\sum_{a=0}^{q-1}\sum_{c=0}^{q-1}\exp\left(\frac{2\pi iac}{q}\right)|c,x^a\mathrm{mod}(n)\rangle\otimes|e[a]\rangle.$$
(A8)

As the only difference between the present model and the original one is the involvement of the environment variables $|e[a]\rangle$ in the entanglement, to proceed along with the discussion we should consider the reduced density matrix

$$\rho(t) = \mathrm{Tr}_e[|\phi_F(t)\rangle\langle\phi_F(t)|]$$
$$= \frac{1}{q^2}\sum_{a=0}^{q-1}\sum_{c=0}^{q-1}\sum_{a'=0}^{q-1}\sum_{c'=0}^{q-1}\exp\left(\frac{2\pi i(ac-a'c')}{q}\right)$$
$$\times\langle e[a']|e[a]\rangle|c,x^a\mathrm{mod}(n)\rangle\langle c',x^{a'}\mathrm{mod}(n)|.$$
(A9)

Here we have traced over the environment variables. We notice that the contribution of environment is given by the decohering factor

$$F(a,a')\equiv\langle e[a']|e[a]\rangle = \langle e|U_{a'}^{\dagger}(t)U_a(t)|e\rangle.$$
(A10)

Now it directly follows that

$$p'(c,k) = \mathrm{Tr}\{\rho(t)|c,k\rangle\langle c,k|\}$$
$$= \frac{1}{q^2}\sum_{a,a':x^a=x^k\mathrm{mod}(n)=x^{a'}}^{q-1}\exp\left(\frac{2\pi i(a-a')c}{q}\right)F(a,a').$$
(A11)

This general expression directly shows the effect of environment on Shor's algorithm. Both the norms and the phases are crucial to the success of Shor's algorithm. We have argued that the norms decide the decoherence time while the phases can affect the probability in the text.

We are now in a position to consider two extreme cases. For the first case, suppose that the qubit system is completely isolated. In this case we have $U_{a'}=U_a$ for $a'\neq a$, so $\langle e[a']|e[a]\rangle=1$. As a result we get

$$p'(c,k) = \frac{1}{q^2}\left|\sum_{a:x^a=x^k\mathrm{mod}(n)}^{q-1}\exp\left(\frac{2\pi iac}{q}\right)\right|^2 = p(c,k).$$
(A12)

For the second case, suppose that the environment causes a complete decoherence. If we indexed the elements of the reduced density matrix by a and a', then this means that its off-diagonal elements vanish completely. Such a case appeared in the dynamical theory as a consequence of a certain factorizable structure of the effective evolution operator. In fact, if U_a can be factorized as

$$U_a(t) = \prod_j^N U_a^j(t),$$
(A13)

where $U_a^j(t)$ only concerns the jth particle in the environment, the decohering factor can be expressed as N-multiple product

$$F(a,a') = \prod_j^N \langle e_j|U_{a'}^{j\dagger}(t)U_a^j(t)|e_j\rangle \equiv \prod_j^N |F^j(a,a')$$
(A14)

of the decohering factors $F^j(a,a')=\langle e|U_{a'}^{j\dagger}(t)U_a^j(t)|e\rangle$ with norms less than unity. In the macroscopic limit $N\to\infty$, it is possible that $F(a,a')\to 0$, for $a'\neq a$, namely, $\langle e[a']|e[a]\rangle = \delta_{aa'}$. Then we have

$$p'(c,k) = \frac{1}{q^2}[(q-1-k)/r] \leqslant \frac{1}{q^2}\frac{q}{r} = \frac{1}{qr}$$
(A15)

and

$$p'(s) = r\phi(r)p'(c,k) = \phi(r)/q \leqslant \phi(r)/n^2 \leqslant 1/n.$$
(A16)

Let us proceed to discuss the possible influence of environment on the efficiency of Shor's algorithm. Generally speaking, a deterministic algorithm is said to be efficient if the number of the computation steps taken to execute it increases no faster than a polynomial function of $\ln N$ where N is the input. For a randomized algorithm this definition should be modified to fit in the probability character. Suppose the one-try-success probability of a randomized algorithm A is s, then A is said to be efficient if $\forall\varepsilon>0,\exists\ p(x)$ such that $\forall N(1-s)^{p(\ln N)}<\varepsilon$, where $p(x)$ is a polynomial. Obviously, the polynomial $p(x)$ here should have real coefficients and satisfy $p(\ln N)>0$. All the polynomials appearing in the following are tacitly assumed to have this property. It is also clear that in quantum computations all algorithms should be randomized ones.

Let A be a quantum algorithm. Suppose for an input N the one-try-success probability of A is $f(N)$ where f is a real continuous function defined on the real line. Then we have the following lemma.

Lemma. If there exists a polynomial $p(x)$ such that

$$\lim_{N\to\infty}[1-f(N)]^{p(\ln N)}<1$$
(A17)

then A is efficient. Conversely, if for an arbitrary polynomial $p(x)$ we have

$$\lim_{N\to\infty}[1-f(N)]^{p(\ln N)}\geqslant 1$$
(A18)

A is not efficient.

Proof. Let $p(x)$ be a polynomial such that $\lim_{N\to\infty}[1-f(N)]^{p(\ln N)}<1$. Then $\forall\varepsilon>0,\exists\alpha(\varepsilon)$ such that $\{\lim_{N\to\infty}[1-f(N)]^{p(\ln N)}\}^{\alpha(\varepsilon)}<\varepsilon$. Namely, $\lim_{N\to\infty}[1-f(N)]^{p(\ln N)\alpha(\varepsilon)}<\varepsilon$. Defining

$$p'(x)\equiv\alpha(\varepsilon)p(x)$$

we come to the conclusion that there exists some N_0 such that $\forall N>N_0$, $[1-f(N)]^{p'(\ln N)}<\varepsilon$. It is now evident that one can choose a suitable polynomial $q(x)$ such that $\forall N$, $[1-f(N)]^{q(\ln N)}<\varepsilon$. This proves the first part of the lemma.

For the second part of the lemma, if the conclusion were not true, $\forall\varepsilon>0$, there would exist a polynomial $p_\varepsilon(x)$ such that $\forall N$, $[1-f(N)]^{p_\varepsilon(\ln N)}<\varepsilon$. Thus we would have

$\lim_{N\to\infty}[1-f(N)]^{p_\varepsilon(\ln N)}\leq\varepsilon$, leading to the contradiction $1\leq\varepsilon$. The lemma is consequently proved.

Before concluding this Appendix let us take A to be Shor's prime factorization algorithm and return to the above mentioned two extreme cases. For the first case, we have $f(N)>1/3\ \ln N$. As a result,

$$\lim_{N\to\infty}[1-f(N)]^{3\ \ln N}\leq\lim_{N\to\infty}[1-1/(3\ \ln N)]^{3\ \ln N}=1/e<1.$$

(A19)

So according to the lemma, A is efficient. For the second

case, we have $f(N)\leq1/N$. It is easy to prove $\lim_{N\to\infty}(1-1/N)^{\ln^m N}=1$ for all integers m so for all polynomials $p(x)$, $\lim_{N\to\infty}(1-1/N)^{p(\ln N)}=1$. Consequently, for all polynomials $p(x)$

$$\lim_{N\to\infty}[1-f(N)]^{p(\ln N)}\geq\lim_{N\to\infty}(1-1/N)^{p(\ln N)}=1.$$

(A20)

This means that the algorithm is no longer efficient in this case.

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