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Exact dynamics of a quantum dissipative system in a constant external field

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We study the quantum dynamics of the simplest dissipative system, a particle moving in a constant external field and interacting with a bath of harmonic oscillators with Ohmic spectral density. Applying the main idea and methods developed in our recent work [L.H. Yu and C.P. Sun, Phys. Rev. A **49**, 592 (1994)] to this system, we obtain the simple and exact solutions for the coordinate operator of the system in the Heisenberg picture and the wave function of the composite system of the system and the bath in the Schrödinger picture. An effective Hamiltonian for the dissipative system is explicitly derived from these solutions. The meaning of the wave function described by this effective Hamiltonian is clarified by analyzing the effect of the Brownian motion. In particular, the general effective Hamiltonian for an arbitrary potential is directly derived with this method for the case when the Brownian motion can be ignored. Using this effective Hamiltonian, we show an interesting result that the dissipation suppresses the wave-packet spreading.

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I. INTRODUCTION

In this paper we apply the idea and methods developed in our recent work [1] on a quantum dissipative system to a dissipative system with a constant external field. In Ref. [1] we worked on the case of an harmonic oscillator moving in a bath with Caldeira-Leggett's spectral density [2]. It is shown in Ref. [1] that the wave function of the system plus the bath is precisely described by a direct product of two independent Hilbert spaces. One of them is described by an effective Hamiltonian, while the other represents the effect of the bath, i.e., the Brownian motion. Therefore this result clarifies the structure of the wave function of the system whose energy is dissipated by its interaction with the bath and reveals the relationship between the different approaches for quantum dissipative systems. No path-integral technology is needed in this treatment.

The study of dissipative quantum systems, especially for the damped harmonic quantum oscillator (DHQO), has a rather long history and has been paid much attention more recently due to the work by Caldeira and Leggett [2]. Now let us briefly describe two main approaches for quantum dissipative systems before our work in Ref. [1]. To reproduce and quantize the phenomenological dissipative equation

$$M\ddot{q}(t) = -\eta\dot{q}(t) - \frac{\partial V(q)}{\partial q} \tag{1.1}$$

for a one-dimensional dissipative system S with coordinate q, mass M, and potential $V(q)$, one approach is to put it into an environment, a bath B of N harmonic oscillators interacting with the system S through certain coupling [3–12]. The bath and the system constitute a closed composite system $C = S + B$ and the quantization of C is direct. By eliminating the bath variables from its corresponding Heisenberg equation, through proper approximations such as the Markovian approximation and the Wigner-Wisskopf approximation [9], the phenomenological dissipative equation Eq. (1.1) in the operator form is derived. This approach provides both the friction and the fluctuation force in Brownian motion. The path-integral technique [2,13] and the field-theory method [14] are sometimes used in this approach. Along this direction, the work of Caldeira and Leggett reveals a remarkable fact that the dissipation occurs exactly, instead of approximately, if the spectral density of the bath is Ohmic (to be described later). It encourages people to reconsider many problems of dissipation in an exact manner. Another approach, for the DHQO, is the use of an effective Hamiltonian [15–17]

$$H_E = H_E(t) = \frac{1}{2M}e^{-\eta t/M}p^2 + \frac{1}{2}M\omega^2 e^{\eta t/M}q^2, \tag{1.2}$$

which is now called the Caldirora-Kani (CK) Hamiltonian [13]. With the canonical commutation relation

$$[q,p] = i\hbar, \tag{1.3}$$

this Hamiltonian automatically yields the dissipation equation (1.1) through the Heisenberg equation. Notice that alternative forms of the effective Hamiltonian have been given by many authors and an elegant example can be found in Ref. [18]. Though this approach is very convenient to treat some dynamical problems of dissipation process for both classical and quantum cases, such as tunneling and the motion of a wave packet, it is only a pure phenomenological method and the Brownian motion cannot be analyzed by this approach. In particular, the meaning of the wave function described by the CK Hamiltonian is ambiguous.

This paper applies the method and the idea developed in Ref. [1] to a disspative system with a constant external field to show the direct connection between the above two approaches and to see how the interaction between the bath and the system leads to an explicit description for the dissipative system in terms of the effective Hamiltonian. All the discussions in this paper proceed with a simple model, a particle moving in one dimension with a constant potential field, but the main idea and methods can be generalized to other cases.

This work answers the following questions: (i) How is the dissipative system quantized with the effective Hamiltonian, based on the basic principles of quantum mechanics instead of only a phenomenological treatment? (ii) What is the meaning of the wave function described by the effective Hamiltonian? (iii) How is the propagator constructed for the dissipative system? (iv) What happens to the spreading of the wave packet in the presence of dissipation?

II. EXACT CLASSICAL LANGEVIN EQUATION

In this section we reformulate the first approach about the quantum dissipative system in the Ohmic case with a simple example. For simplicity, the dissipative system S is considered to be a charged particle with unit mass, unit negative charge, and coordinate q in a constant electric field E. It interacts with a bath B of N harmonic oscillators B_i with coordinates x_j, mass m_j, and frequency ω_j. Let p and p_j be the corresponding momenta to q and x_j, respectively. The Hamiltonian of the composite system of B and S is

$$\begin{aligned} H &= \frac{1}{2M}p^2 - Eq + \sum_{j=1}^{N}\left[\frac{p_j^2}{2m_j} + \frac{1}{2}m_j\omega_j^2(x_j - q)^2\right] \\ &= \frac{1}{2M}p^2 - Eq + H_B - \sum_{j=1}^{N} C_j x_j q + \Delta V, \end{aligned} \tag{2.1}$$

where

$$H_B = \sum_{j=1}^{N}\left[\frac{p_j^2}{2m_j} + \frac{1}{2}m_j\omega_j^2 x_j^2\right].$$

The problem of the renormalization mentioned in Ref. [2] is treated by introducing the frequency-dependent coupling constant $C_j = m_j\omega_j^2$ and the renormalization potential

$$\Delta V = \sum_{j=1}^{N}\frac{C_j^2}{2m_j\omega_j^2}q^2 = \sum_{j=1}^{N}\frac{1}{2}m_j\omega_j^2 q^2$$

in the Hamiltonian (2.1). The Hamiltonian leads to a system of classical equations of motion

$$M\ddot{q} = E - \sum_{j=1}^{N} C_j x_j - \sum_{j=1}^{N}\frac{C_j^2}{m_j\omega_j^2}q \tag{2.2a}$$

$$\ddot{x}_j = -\omega_j^2 x_j - \frac{C_j}{m_j}q. \tag{2.2b}$$

Notice that the canonical equation for the closed composite system of S and B defines the usual momentum-velocity relations

$$p(t) = M\dot{q}(t), \quad p_j(t) = m_j\dot{x}_j(t).$$

After applying Laplace transformation to Eq. (2.2), a direct substitution of Eq. (2.2b) into Eq. (2.2a) yields an exact equation of motion for the system S

$$M\ddot{q} = E + L(q) + G(t), \tag{2.3}$$

where

$$G(t) = \sum_{j=1}^{N} C_j\left[x_j(0)\cos\omega_j t + \frac{\dot{x}_j(0)}{\omega_j}\sin\omega_j t\right] \tag{2.4}$$

and the second term of the right-hand side of Eq.(2.3)

$$L(s) = \wp^{-1}\left[-s^2\sum_{j=1}^{N}\frac{C_j^2 s^2}{m_j\omega_j^2(s^2+\omega_j^2)}\bar{q}(s)\right] \tag{2.5}$$

is determined by the inverse of Laplace transformation $\wp$

$$\wp[q(t)] = \bar{q}(s) = \int_0^\infty q(t)e^{-st}dt.$$

Usually, for finite N or for a general spectral distribution $\rho(\omega_j)$ of an infinitely large number of oscillators in a bath, the dissipative term $-\eta\dot{q}$ with a positive number η does not appear exactly. However, according to Caldeira and Leggett, for a specific spectral distribution of the bath oscillators

$$\rho(\omega_j) = \frac{2\eta\omega_j^2 m_j}{\pi C_j^2} = \frac{2\eta}{\pi m_j\omega_j^2}, \tag{2.6}$$

which is called the Ohmic distribution for the bath, the sum over index j

$$\sum_{j=1}^{N}\frac{C_j^2 s^2}{m_j\omega_j^2(s^2+\omega_j^2)}$$

becomes an integral

$$\begin{aligned}\int_0^\infty \rho(\omega_j)\frac{C_j^2 s^2}{m_j\omega_j^2(s^2+\omega_j^2)}d\omega_j &= \frac{2\eta s^2}{\pi}\int_0^\infty\frac{d\omega_j}{(\omega_j^2+s^2)} \\ &= s\eta\end{aligned}$$

and thus results in

$$L(q) = -\eta\dot{q} - \eta q(0)\delta(t)$$

by an inverse Laplace transformation. Then, the dissipative equation, the classical Langevin equation,

$$M\ddot{q} = E - \eta\dot{q} + G(t) \tag{2.7}$$

is derived. Because the dissipative process is irreversible, we shall pay attention only to the process with $t > 0$ in the following. Hence the impact $\delta(t)$ in the dissipative equation does not play a role in the dynamical problem.

Notice that the Ohmic distribution (2.6) is only an alternative; however, an explicit and convenient reformulation of the Caldeira-Leggett constraint

$$J(\omega) = \eta\omega$$

on the spectral density

$$J(\omega) = \frac{\pi}{2}\sum_{j=1}^{N}\frac{C_j^2}{m_j\omega_j}\delta(\omega-\omega_j). \tag{2.8}$$

We remark that the fluctuating external force $G(t)$ acting on the system S is generated by the bath and depends on the initial states of the oscillators in the bath. For the classically statistical average $\langle\rangle_{\text{classical}}$, $G(t)$ obeys the dissipation-fluctuation relation at temperature T

$$\langle G(t)G(t')\rangle_{\text{classical}} = \frac{1}{2}\eta KT\delta(t-t') \tag{2.9}$$

and $\langle G(t)\rangle_{\text{classical}} = 0$. This is actually the classical Brownian force in the disspative process.

III. QUANTIZATION OF A DISSIPATIVE SYSTEM WITH EFFECTIVE HAMILTONIAN

In the following, we first detail the description of the plus system bath in Ref. [1] by an example. Then, we further develop our theory to derive a general effective Hamiltonian for arbitrary potential.

It is observed from Eq. (2.7) that, in the Ohmic case, the action of the bath B on the system S can be described as two parts, the dissipative term $-\eta\dot{q}$ depending only on the state of S, and the Brownian force $G(t)$, depending on the initial state of the bath. In order to study various dynamical problems in the dissipative process, such as the tunneling and wave-packet spreading, it is necessary to determine in what sense the dissipative system can be isolated from the environment as a "quasiclosed" system depending only on its own variables. Such a quasiclosed system can evolve independently and the effect of the bath is described only by the friction coefficient η. For the classical case, it is quite clear that at zero temperature the Brownian force can be neglected. According to the fluctuation-dissipation relation Eq. (2.9), the classically statistical fluctuation of the bath is proportional to ηT. So when $T = 0$, the system is isolated, with an effective dissipation equation

$$M\ddot{q} = E - \eta\dot{q}, \tag{3.1}$$

and the whole effect of bath on the system is characterized only through the friction constant η. However, for the quantum case, the situation is not so direct. Now, we consider the composite system $C = S + B$. Since C is closed, its quantization process is well known.

Let us start with the exact solution to Eqs. (2.2b) and (2.7),

$$q(t) = Q(t) + X(t), \quad Q(t) = a(t)\dot{q}(0) + q(0) + g(t) \tag{3.2}$$

$$X(t) = \sum_{k=1}^{N} X_k(t), \quad X_k(t) = \alpha_k(t)x_k(0) + \beta_k(t)\dot{x}_k(0),$$

where

$$a(t) = t_\eta = \frac{M(1-e^{-\eta t/M})}{\eta}, \quad g(t) = \frac{E}{\eta}(t - t_\eta)$$

$$\alpha_k(t) = \frac{C_j}{\eta/M^2+\omega_k^2}\left[\frac{\eta}{\omega_{k/M}}\sin\omega_k t - (\cos\omega_k t - 1) - \eta/Ma(t)\right],$$

$$\beta_k(t) = \frac{C_j}{(\eta/M^2+\omega_k^2)\omega_k}\left[\frac{\eta}{\omega_{k/M}}(\cos\omega_k t - 1) - \sin\omega_k t + \omega_k a(t)\right].$$

The commutation relations at $t = 0$ for the closed composite system of S and B

$$[q(0), M\dot{q}(0)] = i\hbar, \quad [x_j(0), \dot{x}_j(0)] = \frac{i\hbar}{m_j} \tag{3.3}$$

quantize $q(t)$ and $\dot{q}(t)$ at any instant so that

$$[q(t), p(t)] = i\hbar, \quad [x_j(t), p_j(t)] = i\hbar \tag{3.4}$$

for the ordinary momentum-velocity relations

$$p(t) = M\dot{q}(t), \quad p_j(t) = m_j\dot{x}_j(t).$$

Notice that the quantized coordinate $q(t)$ is separated into two commuting parts $Q(t)$ and $X(t)$ depending on the system and the bath, respectively.

Now, we consider the quantum statistical problem of dissipation and fluctuation for the above linear system. Denote the quantum statistical average $\langle\rangle = \langle\rangle_{\text{quantum}}$ over the bath defined by

$$\langle A\rangle = \frac{\mathrm{Tr}[Ae^{-\beta H_B}]}{\mathrm{Tr}[e^{-\beta H_B}]}, \quad \beta = \frac{1}{KT}$$

for an observable A, where H_B is the Hamiltonian for the bath. A direct calculation gives the quantum fluctuation-dissipation relation

$$\langle G(t)\rangle = 0,$$

$$D(t-t')_T = \frac{1}{2}\langle\{G(t), G(t')\}\rangle = \frac{\eta\hbar}{\pi}\int_0^\infty \omega_j \coth\left(\frac{\beta\hbar\omega_j}{2}\right)\cos\omega_j(t-t')d\omega_j. \tag{3.5}$$

At the high temperature limit or $\hbar \to 0$, i.e., $\beta \to 0$ or $T \to \infty$, the above results approach the classical dissipation-fluctuation relation (2.9). In the zero temperature limit, it becomes

$$D(t-t')_{T\to 0} = \frac{\eta\hbar}{\pi}\int_0^\infty \omega_j\cos\omega_j(t-t')d\omega_j = \begin{cases} \frac{\eta\hbar}{2\pi}\lim\limits_{\omega\to\infty}\omega^2 \to \infty & \text{if } t = t' \\ \frac{\eta\hbar}{\pi}\lim\limits_{\omega\to\infty}\left[\frac{\cos\omega(t-t')}{(t-t')^2} + \frac{\omega\sin\omega(t-t')}{(t-t')}\right] \to \infty & \text{if } t \neq t'. \end{cases} \tag{3.6}$$

This equation shows that the Brownian force cannot be neglected even at zero temperature for the quantum case. This leads to a divergence in the correlation function of the fluctuation force. In the practical problem, there may be a cutoff frequency of the bath with the upper bound of frequency ω_M and then the correlations are proportional to ω_M^2 and ω_M for the cases $t = t'$ and $t \neq t'$, respectively. This means that the correlation at $t = t'$ is stronger than when $t \neq t'$.

The effective coordinate $Q(t)$ approaches the physical coordinate $q(t)$ only when this quantum fluctuation is neglected in a certain sense. In Sec. IV we will clarify further the exact meaning of this argument in terms of the wave-function description.

When the quantum fluctuation is ignored, the evolution of the system can be described by the variable $Q(t)$, independent of the bath. Then, the system is isolated from the bath to become a "closed" system. Now, let us derive the effective Hamiltonian governing this system. Our derivation is valid for both the classical and the quantum case.

Notice that the explicit expression (3.2) for $Q(t)$ determines the commutator

$$[Q(t), \dot{Q}(t)] = (i\hbar/M)e^{-\eta t/M} \tag{3.7}$$

and then we can choose a momentum-velocity relation

$$P(t) = e^{\eta t/M} M\dot{Q}(t) \tag{3.8}$$

for the effective variable Q so that the basic commutators for the Hamiltonian dynamics are

$$[Q(t), P(t)] = i\hbar. \tag{3.9}$$

Notice that the definition of the canonical momentum based on Eq. (3.7) is not unique. Furthermore, different definitions, e.g., the one given by Ref. [18], may give the effective Hamiltonian different forms. For the $Q(t)$ given by Eq. (3.2), one can explicitly obtain the expressions for $\dot{Q}(t)$ and $\dot{P}(t)$ in terms of $Q(t)$ and $P(t)$

$$\dot{P}(t) = -Ee^{\eta t/M}, \quad \dot{Q}(t) = e^{-\eta t/M}P(t)/M. \tag{3.10}$$

We are looking for an effective Hamiltonian for which the associated Heisenberg equations

$$\dot{Q}(t) = \frac{1}{i\hbar}[Q(t), H_e(t)],$$

$$\dot{P}(t) = \frac{1}{i\hbar}[P(t), H_e(t)]$$

lead to Eq. (3.10). Thus we obtain a simple system of partial differential equations about the effective Hamiltonian $H_e(t)$:

$$\frac{\partial H_e(t)}{\partial Q} = -Ee^{\eta t/M}, \quad \frac{\partial H_e(t)}{\partial P} = e^{-\eta t/M}P(t)/M. \tag{3.11}$$

The solution to Eq. (3.11) determines the effective Hamiltonian $H_e(t)$

$$H_e = H_e(t) = \frac{1}{2M}e^{-\eta t/M}P^2 - EQe^{\eta t/M} + F(t), \tag{3.12}$$

up to an arbitrary function $F(t)$ of t independent of P and Q. It can be regarded as a generalization of the CK Hamiltonian [14–16] of damped harmonic oscillator.

Motivated by the above discussion, for an arbitrary potential $V(Q)$, we prove that the general effective Hamiltonian for the dissipation problem can be obtained (in one among the different forms, corresponding to different definitions of momentum) as

$$H_E = H_E(t) = \frac{1}{2M}e^{-\eta t/M}P^2 + e^{\eta t/M}V(Q), \tag{3.13}$$

when the Brownian motion is ignored. Indeed, the Heisenberg equation $H_E = H_E(t)$, or the disspation equation

$$M\ddot{Q}(t) = -\eta\dot{Q}(t) - \frac{\partial V(Q)}{\partial Q}, \tag{3.14}$$

for arbitrary potential, is derived in the same way as we derived Eq. (2.7), with the same Brownian term $G(t)$ neglected.

Next, the generalized CK Hamiltonian (3.13) can be explicitly derived from the above equation (3.14). It follows from Eq. (3.14) that the commutator $[Q(t), \dot{Q}(t)]$ at time t satisfies the equation

$$\frac{d}{dt}[Q(t), \dot{Q}(t)] = -(\eta/M)[Q(t), \dot{Q}(t)] \tag{3.15}$$

for any potential $V(Q)$. For an arbitrary potential $V(q)$, it still leads to the commutator Eq. (3.7), which is the same as the one for the case of the harmonic oscillator and for the case of a constant external field:

$$[Q(t), \dot{Q}(t)] = i\hbar e^{-\eta t/M}/M. \tag{3.7$'$}$$

It also suggests the same canonical momentum-velocity relation Eq. (3.8) for the arbitrary potential

$$P(t) = e^{\eta t/M}M\dot{Q}(t), \tag{3.8$'$}$$

as a special solution of Eq. (3.7$'$) and

$$[Q(t), P(t)] = i\hbar. \tag{3.9$'$}$$

Using Eq. (3.8$'$) and the dissipative equation Eq. (3.14), again we have

$$\dot{P}(t) = -e^{\eta t/M}\frac{\partial V(Q)}{\partial Q}. \tag{3.16}$$

Equations (3.16) and (3.8$'$) and the Heisenberg equation give the equations for the unknown Hamiltonian H_E,

$$\begin{aligned} [Q(t), H_E(t)] &= i\hbar e^{-\eta t/M}P(t)/M, \\ [P(t), H_E(t)] &= -i\hbar e^{\eta t/M}\frac{\partial V(Q)}{\partial Q}. \end{aligned} \tag{3.17}$$

Obviously, the generalized CK Hamiltonian (3.13) is a solution of the above equations. Therefore, we have derived the generalized CK Hamiltonian for the arbitrary potential $V(q)$.

We like to point out here that to derive the general CK Hamiltonian we do not need to know the explicit analytical form of $Q(t)$ for general $V(q)$. This is because the basic commutator (3.9$'$) can be obtained directly from the equation of motion Eq. (3.14). Notice that the derivation of the generalized CK Hamiltonian (3.13) and its special case (3.12) in this section is also valid for the classical case without Brownian motion, as long as the commutators is replaced by the Poisson brackets.

We like to point out that a recent paper by Schuch [18] also discussed the theory of the effective Hamiltonian for the quantization of dissipative systems and gave the commutation relation Eq. (3.7) phenomenologically so that the dissipative equation (1.1) can be derived in the Lagrange-Hamiltonian formalism. Actually, Dekker [17] also discussed similar results. However, our recent studies reveal the connection between the two major categories of methods describing the dynamics of dissipative quantum systems: the system-plus-environment approach, on the one hand, and the approach using the time-dependent Hamiltonian such as the CK Hamiltonian, on the other hand.

IV. MEANING OF THE WAVE FUNCTION FOR THE EFFECTIVE HAMILTONIAN

In the preceding section we derived the effective Hamiltonian in terms of the Heisenberg equations. It serves as a starting point to study some dynamical problems of quantum dissipation in a certain limited sense. To understand the problem completely, one must investigate the physical meaning of the (effective) wave function described by the effective Hamiltonian (3.12) and clarify in what sense this effective Hamiltonian can be used correctly. Thus we turn to the Schrödinger picture and further detail the main ideas and methods developed in Ref. [1].

Let us first show that the wave function of the composite system $C = S + B$ can be reduced to a direct product of wave functions in two independent Hilbert spaces. For this purpose, it is observed that the coordinate operator

$$q(t) = Q(t) + X(t)$$

of S is a direct sum of two commuting parts $Q(t)$ and $X(t)$. So the eigenfunction of $q(t)$ with the eigenvalue q is expressed as a direct product

$$|q,t\rangle = |Q,t\rangle \otimes |\xi_1,t\rangle \otimes |\xi_2,t\rangle \otimes \cdots \otimes |\xi_N,t\rangle \tag{4.1}$$

of the eigenstates $|Q,t\rangle$ for $Q(t)$ and $|\xi_j,t\rangle$ for $X_j(t)$ with the eigenvalues Q and ξ_j, respectively. Notice that these eigenvalues satisfy

$$q = Q + \sum_{j=0}^{N} \xi_j. \tag{4.2}$$

Due to Eq. (4.2), for a given q, there exist many diferent sets of $\{\xi_1, \xi_2, ..., \xi_N\}$ corresponding to the same q, that is to say, these eigenstates $|q,t\rangle$ are degenerate. So, the new notation $|q,\{\xi_j\},t\rangle$ with an additional index $\{\xi_j\}$, is used instead of $|q,t\rangle$ to distinguish among the different degenerate eigenstates with the same eigenvalues q. Correspondingly, the following notation for arbitrary complex number ξ_j:

$$|Q,\{\xi_j\}\rangle = |Q\rangle \otimes |\xi_1\rangle \otimes |\xi_2\rangle \otimes \cdots \otimes |\xi_N\rangle \tag{4.3}$$

is used to represent the degenerate eigenstate of $q(0)$ with eigenvalue Q, which is, in Hilbert space,

$$V = V_S \otimes V_B = V_S \otimes \prod_{j=1}^{N} \otimes V_j$$

of the composite system C. Here V_S and $V_B = \prod_{j=1}^{N} \otimes V_j$ are the Hilbert spaces of S and B, respectively, V_j is the Hilbert space for the jth oscillator in the bath B, $|Q\rangle$ is the eigenstate with the eigenvalue Q, and $|\xi_j\rangle \in V_j$ are the eigenstates of $x_j(0)$.

Let the composite system be initially in a product state

$$|\psi(0)\rangle = |\phi\rangle \otimes |W\rangle = |\phi\rangle \otimes \prod_{j=1}^{N} \otimes |W_j\rangle, \tag{4.4}$$

at $t = 0$, where $|\phi\rangle$, $|W\rangle$, and $|W_i\rangle$ are in V_S, V_B, and V_j, respectively. The central problem is now whether, similar to Eq. (4.4), where the wave function is a direct product of two wave functions in two independent Hilbert spaces, the product form is still persevered during the Schrödinger evolution. A positive answer would imply that the system can be isolated from the bath and described by an effective Hamiltonian. To study this problem, it is necessary to calculate the evolution operator $U(t)$ or its matrix elements. Because the coordinate operator $q(t)$ is an unitary transformation of $q(t)$:

$$q(t) = U(t)^\dagger q(0) U(t),$$

the eigenstate

$$|q,\{\xi_j\},t\rangle = U(t)^\dagger |q,\{\xi_j\}\rangle \tag{4.5}$$

of $q(t)$ with eigenvalue q can be constructed in terms of the evolution operator from the eigenstate $|Q = q, \{\xi_j\}\rangle$ of $q(0)$ with the same eigenvalue q. Then, the coordinate component of the evolution state

$$|\psi(t)\rangle = U(t)|\psi(0)\rangle$$

can be calculated as

$$\begin{aligned}\Psi(Q,\{\xi_j\},t) &= \langle Q,\{\xi_j\}|\psi(t)\rangle \\ &= \langle Q,\{\xi_j\}|U(t)|\psi(0)\rangle \\ &= [\langle\psi(0)|U(t)^\dagger|Q,\{\xi_j\}\rangle]^* \\ &= [\langle\psi(0)|Q,\{\xi_j\},t\rangle]^*,\end{aligned} \tag{4.6}$$

using the eigenstate $|Q,\{\xi_j\},t\rangle$ of $q(t)$. In fact, this eigenstate can be directly solved in the coordinate representation with

$$q(0) = Q, \dot{q}(0) = -(i\hbar/M)\frac{\partial}{\partial Q},$$

$$x_j(0) = \xi_j, \dot{x_j}(0) = -(i\hbar/m_j)\frac{\partial}{\partial \xi_j}.$$

Obviously, the eigenstates $|Q,t\rangle$ of $Q(t)$ and $|\xi_j,t\rangle$ of $\xi_j(t)$ with eigenvalues q and $\xi_j(t)$, respectively,

$$\begin{aligned}\phi_Q(Q',t) &= \langle Q'|Q,t\rangle \\ &= e^{\frac{iM}{\hbar\alpha(t)}\{-\frac{1}{2}Q'^2+[Q-g(t)]Q'+\lambda(Q)\}}, \\ u_{\xi_j}(\xi_j',t) &= \langle\xi_j'|\xi_j,t\rangle \\ &= e^{\frac{im_j}{\hbar\alpha_j(t)}[-\frac{1}{2}\beta_j(t)\xi_j'^2+\xi_j\xi_j'+\mu(\xi_j)]},\end{aligned} \tag{4.7}$$

are explicitly obtained from the differential equations

$$\{[-i\hbar a(t)/M]\frac{\partial}{\partial Q'} + Q' + g(t)\}\phi_Q(Q',t) = Q\phi_Q(Q',t), \tag{4.8}$$

$$\{[-i\hbar\alpha_j(t)/m_j]\frac{\partial}{\partial \xi'_j} + \beta_j(t)\xi'_j\}u_{\xi_j}(\xi'_j,t) = \xi_j u_{\xi_j}(\xi'_j,t).$$

$\lambda(Q)$ and $\mu(\xi_j)$ are functions independent of Q' and ξ'_j, respectively.

Now, we can write

$$\begin{aligned}\langle Q',\{\xi'_j\}|q,\{\xi_j\},t\rangle &= \langle Q'|Q = q - \sum_{j=1}^{N}\xi_j,t\rangle \prod_{j=1}^{N}\langle\xi'_j|\xi_j,t\rangle \\ &= \phi_{q-\sum\xi_j}(Q',t)\prod_{j=1}^{N}u_{\xi_j}(x'_j,t),\end{aligned} \tag{4.9}$$

which results in

$$\begin{aligned}\Psi(q,\{\xi'_j\}) &= [\langle\psi(0)|q,\{\xi'_j\},t\rangle]^* \\ &= W\left(q-\sum\xi_j,t\right)^*\prod_{j=1}^{N}W_j(\xi_j,t)^*,\end{aligned} \tag{4.10}$$

where

$$W\left(q-\sum\xi_j\right) = \int dQ'\langle\phi(0)|Q'\rangle\phi_{q-\sum\xi_j}(Q',t),$$

$$W_j(\xi_j,t) = \int d\xi'_j\langle w_j|\xi'_j\rangle u_{\xi_j}(\xi'_j,t).$$

Notice that the variables ξ_j are related to the bath, but they are not the coordinates x_j of the bath. Therefore, when applying the above discussion to a practical problem, one should distinguish between ξ_j and x_j.

From Eq. (4.10), it is observed that the system is in a highly excited state such that the Brownian motion contribution $\sum\xi_j$ is negligible in comparison with q, the first factor is approximately independent of ξ_j. In this sense, the whole wave function for the composite system $S+B$ is factorized into two parts in V_S and V_B, respectively. The first part represents the wave function of the dissipative system, evolving according to the effective Hamiltonian (3.12). Because of the Brownian motion, the physical variable $q = Q + \sum\xi_j$ fluctuates about Q with a mean value of $(\sum\xi_j)^2$

$$\left\langle\left(\sum\xi_j(t)\right)^2\right\rangle = \sum_{j=0}^{N}\frac{\hbar}{2m_j\omega_j}[|\alpha_j(t)|^2 + \omega_j^2|\beta_j(t)|^2] \times\coth\frac{\hbar\omega_j}{2KT} \tag{4.11}$$

at temperature T. It is zero at $t=0$. At the low temperature limit, it approaches its final value at $t=\infty$

$$\begin{aligned}\left\langle\left(\sum\xi_j(t=\infty)\right)^2\right\rangle &= \lim_{\omega_0\to 0}\left\{\frac{\hbar}{2M\omega}\left(\frac{\pi}{2} + \arctan\left[\frac{\omega_0}{\eta\omega}\right]\right)\right\} \\ &\propto \frac{\hbar\pi}{2M\eta},\end{aligned}$$

$$w^2 = w_0^2 - \eta_4^2 \tag{4.12}$$

within a time of the order of $1/\eta$. Obviously, in practical problems, whether or not the effective Hamiltonian can work well mainly depends on whether or not the values $\langle[\sum\xi_j(t)]^2\rangle$ can be neglected.

To study further the dynamics of the dissipative system, we need to calculate its propagator. There are usually two ways to do that in principle: (i) by solving the effective Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}|\Psi(t)\rangle = H_E(t)|\Psi(t)\rangle$$

directly for the effective Hamiltonian H_E obtained and (ii) by using the reduced density matrix in terms of the path integral for the composite system $C = S+B$. Now, we shall deal with this problem a third way, a shortcut, in which we need not know what the effective Hamiltonian H_E is. Because we have known the time evolution of the observable $Q(t)$, we can derive the propagator from $Q(t)$ directly. It not only avoids the complexity of the calculations in the first two ways, but also gives us insight for the understanding of quantum dissipation.

According to the definition of the propagator, we have

$$\begin{aligned}G(q_2,\{\xi_{j,2}\}t_2,q_1,\{\xi_{j,1}\},t_1) &= \langle q_2,\{\xi_{j,2}\},t_2|q_1,\{\xi_{j,1}\},t_1\rangle \\ &= \langle Q_2,t_2|Q_1,t_1\rangle\prod_{j=1}^{N}\langle\xi_{j,2},t_2|\xi_{j,1},t_1,\rangle \\ &= G(Q_2,t_2,Q_1,t_1)\prod_{j=1}^{N}G_j(\xi_{j,2},t_2|\xi_{j,1}t_1).\end{aligned} \tag{4.13}$$

Because of the linearity of the Heisenberg equation in the variables $Q(t)$,$\dot Q(t)$ and $\xi_j(t)$,$\dot\xi_j(t)$, the operators $O(t_2)$ $(O = Q,\dot Q,\xi_j,\dot\xi_j)$ at time t_2 must be a linear combination of the operators $O(t_1)$ at t_1. For example,

$$\begin{aligned}Q(t_1) &= Q(t_2) + a(t_1,t_2)P(t_2) + b(t_1,t_2), \\ Q(t_2) &= Q(t_1) + a(t_2,t_1)P(t_1) + b(t_2,t_1),\end{aligned} \tag{4.14}$$

where

$$b(t_2,t_1) = g(t_2) - g(t_1) - Ea(t_1)e^{-\eta t_1/M}a(t_2,t_1),$$

$$a(t_2,t_1) = -a(t_1,t_2) = a(t_2) - a(t_1).$$

Then, the definitions of eigenstates of $Q_i(t)$

$$Q(t_i)|Q_i,t_i\rangle = Q_i|Q_i,t_i\rangle,\ i=1,2$$

and its Q_j representation $(j\neq i)$ leads to the partial differential equations for the propagator

$$\begin{aligned}[Q_2 + [a(t_1,t_2)i\hbar/M]\frac{\partial}{\partial Q_2} + b(t_1,t_2)]G(Q_2,t_2;Q_1,t_1) &= Q_1 G(Q_2,t_2;Q_1,t_1), \\ \left[Q_1 - [a(t_1,t_2)i\hbar/M]\frac{\partial}{\partial Q_1} + b(t_2,t_1)\right]G(Q_2,t_2;Q_1,t_1)^* &= Q_1 G(Q_2,t_2;Q_1,t_1)^*.\end{aligned} \tag{4.15}$$

Solving the above equations, we obtain the propagator

$$G(Q_2,t_2;Q_1,t_1) = \frac{1}{\sqrt{\frac{2iM\pi}{\eta}|e^{-\eta t_1/M} - e^{-\eta t_2/M}|}} \times \exp\left\{ \frac{Mi}{a(t_1,t_2)\hbar}\left[\frac{1}{2}Q_1^2 + \frac{1}{2}Q_2^2 - Q_1Q_2 + b(t_2,t_1)Q_1 + b(t_1,t_2)Q_2 + \theta(t)\right]\right\}, \qquad (4.16)$$

where $\theta(t)$ is an arbitrary function of time independent of Q_i $(i=1,2)$. Similarly, we can also calculate the factors $G_j(\xi_{j,2},t_2;\xi_{j,1}t_1)$, but here we do not need to write them down explicitly. For a highly excited q system where the Brownian motion is ignored, the first factor $G(Q_2,t_2;Q_1,t_1)$ can be regarded as the effective propagator for the dissipative system. Taking into account the fact that the propagator is the matrix elements of the evolution operator $U_Q(t)$, i.e.,

$$\begin{aligned}\langle Q_2|U(t)|Q_1\rangle &= \langle Q_1|U(t)^\dagger|Q_2\rangle^* \\ &= \langle Q_2,t|Q_1\rangle = G(Q_2,t;Q_1,0),\end{aligned}$$

the effective Hamiltonian (3.11) can also be derived again from

$$H_E = i\hbar\frac{\partial U_Q(t)}{\partial t}U_Q(t)^{-1}. \qquad (4.17)$$

V. SPREADING OF THE WAVE PACKET SUPPRESSED BY DISSIPATION

In this section the effective Hamiltonian (3.12) for the dissipative system is applied to study a quite simple dynamical problem: the motion of the wave packet for a "free" (E=0) particle of mass M in one dimension. An interesting result is that the dissipation suppresses the spreading of the wave packet if the breadth of the initial wave packet is so wide that the effect of the Brownian motion can be ignored. Usually, without dissipation, a Gaussian wave packet spreads into the full space infinitely and the localization of the wave packet is lost during the evolution of the system. Its breadth increases to infinity while its height decreases from its initial value to zero. Notice that the height and breadth of a wave packet are correlated through its normalization. However, for the present case with dissipation, there appears to be a significant difference about the wave-packet spreading. It will be shown that the final breadth and height have finite values as $t \to \infty$. In the following, we denote the operator Q by x, which approximates the physical coordinate q when Brownian motion is ignored.

Using the effective Hamiltonian in the absence of the external field, the Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}\Psi(x,t) = -\frac{\hbar^2}{2M}e^{-\eta t/M}\frac{\partial^2}{\partial x_2}\Psi(x,t) \qquad (5.1)$$

gives the the evolution of wave function

$$\Psi(x,t) = \sum_k \langle k|\Psi(x,0)\rangle \exp\left[ikx - \frac{iE_k t_\eta}{\hbar}\right], \qquad (5.2)$$

where

$$E_k = \frac{\hbar^2k^2}{2M}$$

is the energy of the momentum eigenstate $|k\rangle$

$$\langle k|x\rangle = \frac{1}{\sqrt{2\pi}}e^{ikx}$$

and with the definition

$$t_\eta = a(t) = \frac{M(1-e^{-\eta t/M})}{\eta}.$$

The only difference between Eq. (5.2) and a solution of free propagation without dissipation is that t is replaced by t_η. We have $t_\eta \to t$ when $\eta/M \to 0$. So one can regard t_η as a deformation of time t due to disspation. Notice that t_η approaches a limit $\frac{M}{\eta}$ as $t \to \infty$; this clearly shows the physical features of wave-packet spreading in the presence of dissipation.

If we assume that the initial state is a Gaussian wave packet

$$\Psi(x,0) = \langle x|\Psi(0)\rangle = \frac{1}{[2\pi d^2]^{\frac{1}{4}}}e^{ik_0x - \frac{x^2}{4d}}, \qquad (5.3)$$

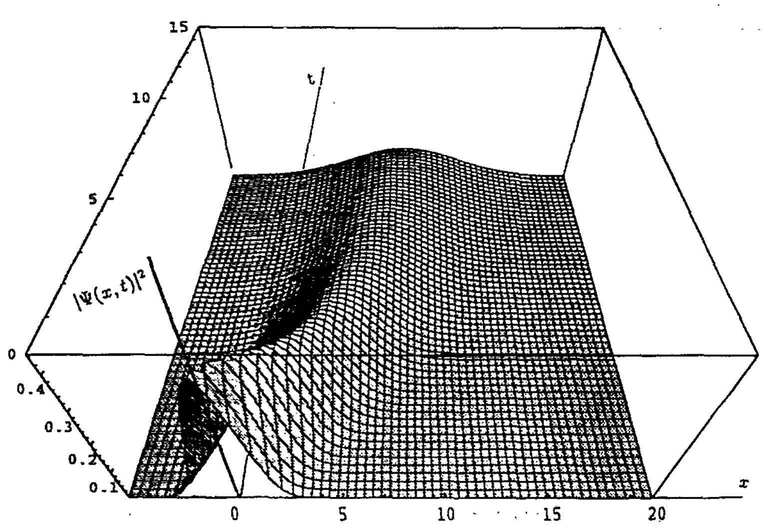

FIG. 1. Motion of a wave packet in the presence of dissipation: $F(x,t_\eta) = |\Psi(x,t)|^2$. The dissipation not only restricts the motion of the center of the Gaussian wave packet like a classical particle, but also suppress its spreading so that it has a limited Gaussian wave packet with finite breadth and height as $t \to \infty$.

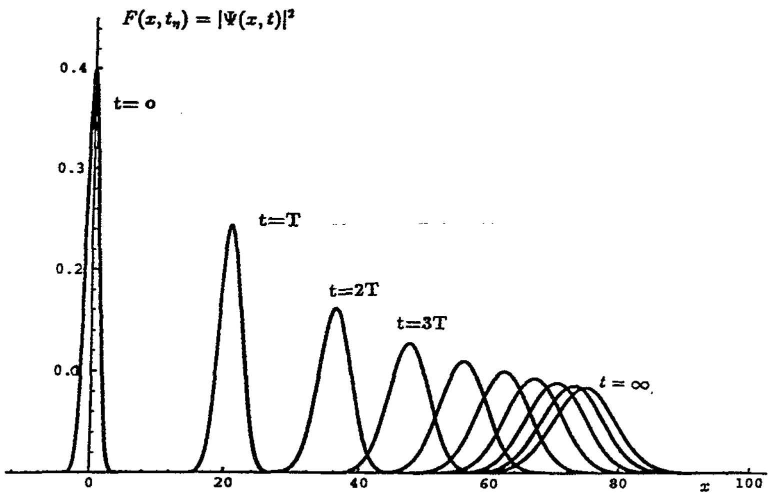

FIG. 2. Projection of Fig. 1 on the $|\Psi|^2$-x plane, which actually represents the function $F(x,t_\eta)=|\Psi(x,t)|^2$ at $t=0,\ T,\ 2T,\ 3T,\ldots,NT$. When $N\to\infty$, it becomes the Gaussian wave packet $|\Psi(x,t=\infty)|^2=F(x,1/\eta)$.

then we have

$$\langle\Psi(0)|x|\Psi(0\rangle=0,\quad \langle\Psi(0)|P|\Psi(0\rangle=\hbar k_0,$$

$$\langle\Delta x\rangle=\sqrt{\langle\Psi(0)|x^2|\Psi(0)\rangle-\langle\Psi(0)|x|\Psi(0)\rangle^2}=d. \tag{5.4}$$

Equation (5.4) shows that the wave packet is centered at x=0 with an average momentum $\hbar k_0$ and a breadth d. According to the wave equation (5.1), the Fourier transformation

$$\Psi(x,0)=\frac{1}{\sqrt{2\pi}}\int_{\infty}^{\infty}\psi(k)e^{ikx}dk,$$
$$\psi(k)=\frac{2d^2}{\pi}e^{-d^2(k-k_0)^2}$$

for the initial wave packet determines the wave function at t

$$\Psi(x,t)=\frac{\exp\left[ik_0x-\frac{iE_kt_\eta}{\hbar}\right]}{(2\pi)^{1/4}\sqrt{d+it_\eta\hbar/(2Md)}}\times\exp\left[-\frac{1}{4}(x-k_0t_\eta\hbar/M)^2\times\frac{1-it_\eta\hbar/2Md^2}{d^2+(t_\eta\hbar/2Md)^2}\right]. \tag{5.5}$$

To understand the physical meaning represented by the wave function (5.5), we write down the corresponding position probability

$$|\Psi(x,t)|^2=\frac{1}{\sqrt{2\pi[d^2+(t_\eta\hbar)^2/(2Md)^2]}}\times\exp\left[-\frac{(x-k_0t_\eta\hbar/M)^2}{2[d^2+(t_\eta\hbar)^2/(2Md)^2]}\right]. \tag{5.6}$$

This shows that the wave packet initially has a velocity

$$v_0=\langle P/M\rangle=\hbar k_0/M$$

and is centered at the position x=0. As $t\to\infty$, this Gaussian wave packet stops with its center at a limit position

$$x_{\text{limit}}=\frac{\hbar k_0}{\eta}.$$

In this process, the velocity of the center

$$v(t)=\frac{d}{dt}\langle\Psi(x,t)|x|\Psi(x,t)\rangle=\frac{\hbar k_0}{M}e^{-\eta t/M} \tag{5.7}$$

decreases from $\frac{\hbar k_0}{M}$ to zero. It is evident that the motion of the center of the wave packet is the same as that of a dissipative classical particle. However, a pure quantum picture is manifested by the change of its breadth

$$B(t)=\sqrt{d^2+(\hbar t_\eta/2Md)^2} \tag{5.8}$$

from d at t=0 to a limit value

$$B_{\text{limit}}=\sqrt{d^2+(\hbar/2\eta d)^2} \tag{5.9}$$

as $t\to\infty$. These results determine the final shape of the the spreading wave packet. These physical features are illustrated by Figs. 1 and 2.

Finally, we point out that the suppression of the wave packet spreading by dissipation possibly provides a mechanism to localize a quantum particle. It might be of interest to note that the finite value of the width of the damped particle wave packet for $t\to\infty$ leads to exactly the same final value for the uncertainty product of the damped free particle, found by Schuch using a nonlinear Schrödinger equation [19].

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