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## ARTICLES

# Quantum dynamical model for wave-function reduction in classical and macroscopic limits

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In this paper, a quantum dynamical model describing the quantum-measurement process is presented as an extensive generalization of the Coleman-Hepp model. In both the classical limit with very large quantum number and the macroscopic limit with very large particle number in the measuring instrument, this model generally realizes the wave-packet collapse in quantum measurement as a consequence of the Schrödinger time evolution in either the exactly solvable case or the non-exactly-solvable case. For the latter, the quasiadiabatic case is explicitly analyzed by making use of the high-order adiabatic-approximation method, which manifests the wave-packet collapse as well as in the exactly solvable case. By highlighting these analyses, it is finally found that an essence of the dynamical model of wave-packet collapse is the factorization of the Schrödinger evolution rather than the exact solvability. Therefore many dynamical models including the previous well-known ones, whether they are exactly solvable or not, can be shown to be only the concrete realizations of this factorizability.

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## I. INTRODUCTION

Though quantum mechanics has been experimentally proven as a quite successful theory, its interpretation is still an important problem that the physicist cannot avoid completely [1–4]. In order to interpret its mathematical formulism physically, one has to introduce the wave-packet-collapse (WPC) postulate as an extra assumption added to the closed system of rules in quantum mechanics. This postulate is also called von Neumann's projection rule or wave-function reduction process. Let us now describe it briefly. It is well known in quantum physics that, if a measured quantum system $S$ is in a state $|\phi\rangle$ that is a linear superposition of the eigenstates $|k\rangle$ of the operator $\hat{A}$ of an observable $A$ just before a measurement, i.e.,

$$|\phi\rangle=\sum_k c_k|k\rangle\ ,\quad \text{where the } c_k\text{'s are complex numbers,} \tag{1.1}$$

then a result of the measurement of $A$ is one $a_k$ of the eigenvalues of $\hat{A}$ corresponding to $|k\rangle$ with the probability $|c_k|^2$. The von Neumann's postulate tell us that, once a well-determined result $a_k$ about $A$ has been obtained, the state of $S$ is no longer $|\phi\rangle$ and it must collapse into $|k\rangle$ since the immediately successive measurement of $A$ after the first one should repeat the same result. Using the density matrix

$$\rho=|\phi\rangle\langle\phi|=\sum_{k,k'} c_k c_{k'}^*|k\rangle\langle k'| \tag{1.2}$$

for the state $|\phi\rangle$, the above WPC process can be expressed as a projecton or reduction

$$\rho\to\hat{\rho}=\sum_k |c_k|^2|k\rangle\langle k|\ . \tag{1.3}$$

However, to realize the WPC, the external classical measuring apparatus must be used to detect the result. Then, one thinks the WPC postulate is not quite satisfactory since quantum mechanics is expected to be a universal theory valid for the whole "universe" because the detector, as a part of the universe, behaves classically in the von Neumann's postulate. A reasonable description of the detector should be essentially quantum and exhibit the classical or macroscopic features in certain limits. If one deals with the detector as a subsystem of the closed system (the universe is the measured system $S$ plus the detector $D$), it is possible that the quantum dynamics of the universe can result in the WPC through the interactions between $S$ and $D$. Up to now, some exactly solvable models have been presented to analyze this problem [5–10]. Among them, the Coleman-Happ (CH) model is a very famous one and has been extensively studied in the last twenty years [5–9]. In order to describe the studies in this paper clearly, we need to see some details of this model.

In the original CH model, an ultrarelativistic particle is referred to the measured system $S$ while a one-dimensional array of scatterers with spin $\frac{1}{2}$ is referred to the detector $D$. The interaction between $S$ and $D$ is represented by a homogeneous coupling

$$H_I=\sum_{n=1}^{N} V(x-a_n)\sigma_1^{(n)}\ , \tag{1.4}$$

 

where $\sigma_1^{(n)}$ is the first component of the Pauli matrix; $a_n$ is the position of the scatterer assigned to the $n$th site in the array. The Hamiltonian for $D$ is

$$H_s = c\hat{P} , \tag{1.5}$$

where $c$, $\hat{P}$, and $x$ are the light speed, the momentum, and the coordinate operators, respectively, for $S$. This model is quite simple, but it can be exactly solved to produce a deep insight on the dynamical description of the quantum-measurement process. Starting with the initial state

$$|\psi(0)\rangle = \sum c_k |k\rangle \otimes |D\rangle , \tag{1.6}$$

where $|D\rangle$ is pure state of $D$ (it is usually taken to be the ground state), the evolution state $|\psi(t)\rangle$ for the universe equal to $S+D$ is defined by the exact solution to this model. Then, the reduced density matrix

$$\rho_s(t) = \mathrm{Tr}_D(|\psi(t)\rangle\langle\psi(t)|) \tag{1.7}$$

of the measured system is obtained by taking the trace of the density matrix for pure state

$$\rho(t) = |\psi(t)\rangle\langle\psi(t)| \tag{1.8}$$

of the universe to the variables of $D$. Obviously, $\rho_s(t)$ depends on the particle number $N$ of $D$. When $N\to\infty$, i.e., in the macroscopic limit, $\rho_s(t)\to\hat{\rho}$ after long enough time $t$ as in Eq. (1.2). Namely, the Schrödinger evolution of the universe, equal to $S+D$, leads to the WPC for the measured system. More recently, the original CH model was improved to describe the energy exchange between $S$ and $D$ by adding a free-energy Hamiltonian [9]

$$H_0 = \hbar\omega \sum_{n=1}^{N} \sigma_3^{(n)} , \tag{1.9}$$

and correspondingly improving the interaction slightly. Notice that the improved model remains exactly solvable.

However, because the spin quantum number is fixed to be $\frac{1}{2}$ in the original CH model or its improved versions, they cannot describe the *classical characters* of the measurement. Usually, the classical feature of a quantum object is determined by taking a certain value for some internal quantum numbers of the detector $D$ or $\hbar=0$. In the case of the angular momentum, this classical limit corresponds to infinite spin. In an informative paper [10], this problem was analyzed by using another exactly solvable dynamical model for quantum measurement. So it is expected that the WPC in the classical limit can be incorporated in an extensive generalization of the CH model. The first step of this paper is to establish such a generalized CH model manifesting the WPC as the dynamical process in the classical limit as well as in the macroscopic limit simultaneously. Then, we attempt to find the essence for this model substantially resulting in the realization of the WPC as a quantum dynamical process as well as for those well-established ones before. To this end, we will explicitly study the dynamics of this generalized CH model in both the exactly solvable case and the nonsolvable case. For the latter, we will apply the high-order adiabatic-approximation (HOAA) method [11–13] to the special case where the coupling parameter depends on the position of the measured ultrarelativistic particle quite slightly. Finally, we point out the possible essence in the dynamical realization of the WPC, which is largely independent of the concrete forms of model Hamiltonians.

## II. GENERALIZATION OF THE CH MODEL AND ITS EXACT EVOLUTION OPERATOR

Based on the original CH model, the present generalizations are to assign an arbitrary spin $j_n$ to each scatterer on the one-dimensional array as the detector $D$ and to take an inhomogeneous coupling of the scatterers to the ultrarelativistic particle as the measured system $S$. In this case the spin couplings have different directions on different sites of the array. Let

$$\mathbf{J}(n) = (\hat{J}_x(n), \hat{J}_y(n), \hat{J}_z(n))$$

be the angular momentum operator acting on the $n$th site and the angular momentum operators on different sites $n=1,2,\ldots,N$ commute with each other. Then, we write the interacting Hamiltonian for the present generalized model

$$H_I = \sum_{n=1}^{N} \mathbf{J}(n)\cdot\mathbf{B}(x-a_n) , \tag{2.1}$$

in terms of the three-vectors $\mathbf{B}(x-a_n)$ depending on the coordinate $x$ of $S$ and the fixed coordinates $a_n$ of the scatterers in the spin array. As the energy exchanging between $D$ and $S$ is studied in Ref. [9], we introduce a free Hamiltonian for the spin array $D$

$$H_D = \sum_{n}^{N} B_0(x-a_n)\hat{J}_z(n) , \tag{2.2}$$

to distinguish the states of the detector $D$ via energy levels. Then, we have a Hamiltonian

$$H = c\hat{P} + \sum_{n}^{N} \mathbf{J}(n)\cdot\mathbf{R}(x-a_n) , \tag{2.3}$$

for the universe equal to $S+D$, where

$$\mathbf{R}(x) = (B_1(x), B_2(x), B_3(x)+B_0(x)) .$$

In the above model, because of the introduction of the *arbitrary spin* $j_n$, which labels any $(2j_n+1)$-dimensional irreducible representation of rotation group SO(3), we are able to consider the behaviors of the quantum dynamics governed by this model Hamiltonian in the classical limit with infinite spin $j_n$. It will be proved that, as in the macroscopic limit with infinite $N$, the quantum dynamical evolution of the universe also leads to the WPC for the measured system in the classical limit. The reason that the limit with infinite $j_n$ is called classical is that the mean-square deviations of the components $\hat{J}_x(n)$ and $\hat{J}_y(n)$ possess the limit feature [17]

$$\frac{\Delta\hat{J}_x(n)}{j_n} = \frac{\Delta\hat{J}_y(n)}{j_n} = \frac{1}{\sqrt{2j_n}} \to 0 \quad \text{as } N\to 0 .$$

To solve the dynamical evolution of the universe,

$S+D$, exactly, the polar coordinate $(R,\theta,\phi)$ for the space $\{\mathbf{R}\}$ of the coupling parameter

$$\mathbf{R}=R(\sin\theta\cos\phi,\ \sin\theta\cos\phi,\ \cos\theta)$$

is introduced, where

$$R(x)=\{B_1^2(x)+B_2^2(x)+[B_0(x)+B_3(x)]^2\}^{1/2},$$
$$\tan[\theta(x)]=\frac{[B_1^2(x)+B_2^2(x)]^{1/2}}{B_3(x)+B_0(x)}, \tag{2.4}$$
$$\tan[\phi(x)]=\frac{B_2(x)}{B_1(x)}.$$

Notice that the functions $R$, $\theta$, and $\phi$ usually depend on $x$ through the coupling parameters $\mathbf{R}$. According to the quantum rotation theory, the interaction Hamiltonian $H_I$ can be rewritten as

$$H_I=S^\dagger(\theta,\phi)\sum_{n=1}^{N}R_n\hat{J}(n)S(\theta,\phi), \tag{2.5}$$

where

$$S(x)=S(\theta(x-a_n),\phi(x-a_n))$$
$$=\prod_{n=1}^{N}e^{-i\hat{J}_z(n)\phi(x-a_n)/\hbar}e^{-i\hat{J}_y(n)\theta(x-a_n)/\hbar} \tag{2.6}$$

is a global rotation of the spin array generated by the local rotations

$$S_n(x)=e^{-i\hat{J}_z(n)\phi(x-a_n)/\hbar}e^{-i\hat{J}_y(n)\theta(x-a_n)/\hbar} \tag{2.7}$$

for each site. Later on, we will show that it is just this factorization of the Hamiltonian that leads to the WPC in quantum measurement through the factorization of the evolution operator.

For the evolution operator $U(t)$ of the universe satisfying the Schrödinger equation with the Hamiltonian (2.3), we introduce the "interaction" picture by

$$U(t)=e^{-ict\hat{P}/\hbar}U_e(t), \tag{2.8}$$

where $e^{-ict\hat{P}}$ is the generator for the coherent state as Gaussian wave packet [17]. In this picture, the reduced evolution operator obeys a time-dependent Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}U_e(t)=H_e(t)U_e(t), \tag{2.9}$$

with the time-dependent Hamiltonian

$$H_e(t)=\sum_{k=1}^{N}h_{ek}(t)$$
$$=\sum_{k=1}^{N}\mathbf{J}(n)\cdot\mathbf{R}(x-a_n+ct). \tag{2.10}$$

Notice that the Schrödinger equation governed by the Hamiltonian $H$ is exactly solvable only for the harmonic case with

$$\theta(x)=\text{const}=\theta,\quad R=\text{const},$$
$$\phi(x)=\frac{\omega x}{c},\quad \omega=(\text{real const}). \tag{2.11}$$

To solve Eq. (2.9) in this exactly solvable case, we use the Rabi-Ramsy-Schwinger rotating coordinate technique. We carry out the transformation on $U_e(t)$,

$$U_e(t)=W(t)U_R(t)$$
$$=\prod_{n=1}^{N}e^{-i\hat{J}_z(n)(\omega/c)(x-a_n+ct)/\hbar}U_R(t). \tag{2.12}$$

Here, the rotated evolution operator $U_R(t)$ is governed by the rotated Hamiltonian

$$H_R=W(t)^{-1}H_e(t)W(t)-\omega\sum_{k=1}^{N}J_z(n)$$
$$=\sum_{n=1}^{N}R\left[\hat{J}_x(n)\sin\theta+\hat{J}_z(n)\left[\cos\theta-\frac{\omega}{R}\right]\right]. \tag{2.13}$$

Notice that this is a time-independent Hamiltonian.

In terms of

$$\Omega=R\left[1+\frac{\omega^2}{R^2}-2\cos(\theta)\frac{\omega}{R}\right]^{1/2},\quad \sin\alpha=\frac{R\sin\theta}{\Omega}, \tag{2.14}$$

we rewrite the above rotated Hamiltonian as

$$H_R=\sum_{n=1}^{N}\Omega[\hat{J}_x(n)\sin\alpha+\hat{J}_z(n)\cos\alpha]$$
$$=\Omega\sum_{n=1}^{N}e^{-i\hat{J}_y\alpha/\hbar}\hat{J}_z(n)e^{i\hat{J}_y\alpha/\hbar}. \tag{2.15}$$

From the above expression for $H_R$, the rotated evolution operator $U_R(t)$ follows immediately,

$$U_R(t)=e^{-iH_Rt/\hbar}=\prod_{n=1}^{N}e^{-i\hat{J}_y\alpha/\hbar}e^{-i\Omega\hat{J}_z(n)/\hbar}e^{i\hat{J}_y\alpha/\hbar}. \tag{2.16}$$

Therefore, the evolution operator for the universe

$$U(t)=e^{-ict\hat{p}/\hbar}\prod_{n=1}^{N}e^{-i\hat{J}_z(n)(\omega x/c)(x-a_n+ct)/\hbar}e^{-i\hat{J}_y\alpha/\hbar}$$
$$\times e^{-i\Omega\hat{J}_z(n)/\hbar}e^{i\hat{J}_y\alpha/\hbar}, \tag{2.17}$$

finally is obtained from the above Eqs. (2.8), (2.12), and (2.16).

Here, we should remark that the exact solvability of the above generalized CH model mainly depends on the harmonic form of the function $\mathbf{R}(x)$ of $x$. If it is not harmonic, the above method cannot work well and certain approximation methods should be used to deal with the evolution operators approximately in various cases. If the coupling function $\mathbf{B}(x)$ depends on $x$ quite slightly then the measured ultrarelativistic particle may move so slowly that the spin states of the scatterer in the detector can hardly be excited, the adiabatic (Born-Oppenheimer) approximation or its generalization can make sense for the problem. Thereby, the Berry's geometric phase [14,15] and the corresponding induced gauge field can be incorporated in this dynamical model of the WPC for the quantum measurement in the adiabatic case.

## III. DYNAMICAL REALIZATION OF WAVE-PACKET COLLAPSE: EXACTLY SOLVABLE CASE

To consider the dynamical realizability of the WPC in the above model for quantum measurement, we consider an ideal double-slit interference experiment. Let a coherent beam of the ultrarelativistic particles be split into two branches represented by the wave functions $|\psi_1\rangle$ and $|\psi_2\rangle$, respectively. In the same time, the detector is assigned to its ground state.

$$|0\rangle=|j_1,m_1=-j_1\rangle\otimes|j_2,m_2=-j_2\rangle\otimes\cdots\otimes|j_N,m_N=-j_N\rangle\ , \tag{3.1}$$

where $|j_k,m_k\rangle$ $(k=1,2,\ldots,N)$ are standard angular-momentum states. The choice of ground state is required by the stable measurement $D$. Like the authors in Refs. [5–9], we also suppose that only the second branch wave $|\psi_2\rangle$ interacts with $D$. Starting with the initial state

$$|\psi(0)\rangle=(C_1|\psi_1\rangle+C_2|\psi_2\rangle)\otimes|0\rangle\ , \tag{3.2}$$

where

$$|C_1|^2+|C_1|^2=1\ ,$$

the evolution operator (2.17) defines the evolution state at an instant $t$ in the interaction picture with the interaction $H_I+H_D$,

$$|\psi(t)\rangle=C_1|\psi_1\rangle\otimes|0\rangle+C_2|\psi_2\rangle\otimes U_e(t)|0\rangle\ . \tag{3.3}$$

Then, we get the corresponding density matrix

$$\begin{aligned}\rho(t)=|\psi(t)\rangle\langle\psi(t)|=|C_1|^2|\psi_1(t)\rangle\langle\psi_1(t)|\otimes|0\rangle\langle 0|+|C_2|^2|\psi_2(t)\rangle\langle\psi_2(t)|\otimes U_e(t)|0\rangle\langle 0|U_e^\dagger(t)\\ +C_1C_2^*|\psi_1(t)\rangle\langle\psi_2(t)|\otimes U_e(t)|0\rangle\langle 0|+C_2C_1^*|\psi_2(t)\rangle\langle\psi_1(t)|\otimes|0\rangle\langle 0|U_e^\dagger(t).\end{aligned} \tag{3.4}$$

In the problem of WPC, because we are only interested in the behaviors of the system $S$ and the effect of the detector $D$ on it, we only need the reduced density matrix for $S$,

$$\rho(t)_S=\mathrm{Tr}_D\rho(t)=|C_1|^2|\psi_1(t)\rangle\langle\psi_1(t)|+|C_2|^2|\psi_2(t)\rangle\langle\psi_2(t)|+(C_1C_2^*|\psi_1(t)\rangle\langle\psi_2(t)|+C_2C_1^*|\psi_2(t)\rangle\langle\psi_1(t)|)\langle 0|U_e(t)|0\rangle\ , \tag{3.5}$$

where $\mathrm{Tr}_D$ represents the trace to the variables of the detector.

Obviously, under certain conditions to be determined, if the vacuum-vacuum transition amplitude $\langle 0|U_e(t)|0\rangle$ vanishes for the detector $D$, the coherent terms in Eq. (14) vanish and thus the quantum dynamics automatically leads to the wave-function reduction

$$\rho(t)_S\to\hat{\rho}(t)=|C_1|^2|\psi_1(t)\rangle\langle\psi_1(t)|+|C_2|^2|\psi_2(t)\rangle\langle\psi_2(t)|\ . \tag{3.6}$$

Namely, the WPC occurs as a quantum dynamical process under these conditions.

Now, let us prove that these conditions are just the macroscopic limit and the classical limit, which, respectively, correspond to the cases with very large particle number $N$ and very large quantum number $j_n$. To this end, we evaluate the norm of vacuum-vacuum transition amplitude $\langle 0|U_e(t)|0\rangle$. Using the explicit expression of the $d$ function

$$d^j_{m,m'}(\alpha)=\langle j,m|e^{-i\hat{J}_y(n)/\hbar}|j,m'\rangle\ ,$$

we have

$$\begin{aligned}|\langle 0|U_e(t)|0\rangle|&=\left|\prod_{n=1}^{N}\sum_{m_n=-j_n}^{j_n}d^{j_n}_{-j_n,m_n}(\alpha)d^{j_n}_{m_n,-j_n}(-\alpha)e^{-im_n\Omega t}\right|\\ &=\left|\prod_{n=1}^{N}\sum_{m_n=j_n}^{j_n}\frac{(2j_n)!}{(j_n+m_n)!(j_n-m_n)!}\left[\cos^2\frac{\alpha}{2}\right]^{j_n-m_n}\left[\sin^2\frac{\alpha}{2}\right]^{j_n+m_n}e^{-im_n\Omega t}\right|\\ &=\prod_{n=1}^{N}\left|\cos^2\left[\frac{\alpha}{2}\right]e^{-i\Omega t}+\sin^2\left[\frac{\alpha}{2}\right]\right|^{2j_n}\ ,\end{aligned}$$

that is

$$|\langle 0|U_e(t)|0\rangle| = \left| \prod_{n=1}^{N} \left[1-\sin^2\left[\frac{\Omega t}{2}\right]\sin^2\alpha\right]\right|^{j_n} . \quad (3.7)$$

The above formula is a main result of this paper, which directly manifests the WPC in the classical and macroscopic limits. Let us now go into some details for this conclusion. Notice that in a nontrivial case $\Omega$, $\alpha \neq 0$ and so

$$\left|1-\sin^2\left[\frac{\Omega t}{2}\right]\sin^2\alpha\right|$$

is usually a positive number less than 1. Thus, in the classical limit with $j_n \to \infty$ mentioned before,

$$|\langle 0|U_e(t)|0\rangle| \to 0 \quad \text{as } j_n \to \infty .$$

This means $\langle 0|U_e(t)|0\rangle \to 0$, as $j_n \to \infty$, that is to say, the WPC occurs as a quantum dynamical process in the classical limit. This is just what we expected. Then, we reach a concise statement that *if the detector behaves classically, but need not behave macroscopically, the WPC can be dynamically realized in the measurement.* The classical detector was required as a purely classical object before, but here it is proved to be a classical limit of a quantum object and the quantum mechanics can work well on it for quantum measurement. We should also stress that the macroscopic limit with very large $N$ is not necessary for the WPC. So long as the detector is in the classical limit, the WPC still appears as a dynamical evolution even for small $N$.

Now, we turn to discuss the macroscopic limit behaviors of the problem in detail. In Eq. (3.3), let us define the positive number $\Delta_n(t)$ by

$$e^{-\Delta_n(t)} = \left[1-\sin^2(\alpha)\sin^2\left[\frac{\Omega t}{2}\right]\right]^{j_n} \leq 1 . \quad (3.8)$$

Then,

$$|\langle 0|U_e(t)|0\rangle| = \exp\left[-\sum_{n=1}^{N}\Delta_n(t)\right] . \quad (3.9)$$

Usually, $\Delta_n(t)$ is nonzero and positive and thus the series $\sum_{n=1}^{\infty}\Delta_n(t)$ diverges to infinity, that is to say, $\langle 0|U_e(t)|0\rangle$ as well as its norm approach zero as $N \to \infty$. This just shows that the WPC can be realized as a quantum dynamical process for the generalized CH model in the macroscopic limit.

## IV. ADIABATIC APPROXIMATION FOR NONSOLVABLE CASE

As in most of the previous studies about the dynamical realization of the WPC for quantum measurement, the above discussions in this paper only concern an extremely idealized case where the model is exactly solvable. So it seems that the exact solvability is necessary for this problem. However, it is not really true. We will observe that the WPC can also happen in the nonsolvable case of the above generalized CH model. In such a case, the parameter $\mathbf{R}(t)$ is not harmonic and so some approximation methods are needed to probe the evolution of the universe, $S+D$. As an example of the nonsolvable model, the adiabatic case where the parameter $R(x+ct-a_n)$ in $H_e(t)$ depends on time "slightly" will be used to illustrate the above-mentioned observation. Because the quasienergy state of $H_e(t)$ can hardly be excited by the variation of $H_e(t)$ as $t$ in this case, the so-called high-order adiabatic-approximation method in connection with Berry's geometric phase [14,15] can effectively be employed to this end. This method was recently developed by this author [11–13] and is now reformulated in the evolution operator form in the Appendix. This reformulation of the HOAA method is quite convenient for application in this paper.

Defining the functions

$$f_n(t) = f(x-a_n+ct)$$

for $f=R,\theta,\phi$, etc., we first factorize the effective evolution operator $U_e(t)$ into

$$U_e(t) = S(t)U'(t) = \prod_{n=1}^{N} e^{-i\hat{J}_z(n)\phi_n(t)/\hbar} e^{-i\hat{J}_y(n)\theta_n(t)/\hbar} U'(t) \quad (4.1)$$

according to the HOAA method. Then, in the equivalent Hamiltonian governing $U'(t)$

$$H'(t) = H_0(t) + V(t): ,$$

$$H_0 = \sum_n^N \left[R_n(t) - \cos[\theta_n(t)]\frac{\partial}{\partial t}\phi_n(t)\right] J_z(n) , \quad (4.2)$$

$$V(t) = \sum_n^N \left[-\frac{\partial\theta_n(t)}{\partial t}J_y(n) + \sin[\theta_n(t)]\frac{\partial\phi_n(t)}{\partial t}\hat{J}_x(n)\right] \quad (4.3)$$

can be regarded as a perturbation. The standard perturbation theory determines the first-approximate evolution operator

$$U_0'(t) = \prod_{n=1}^{N} e^{-i\int_0^t R_n(t')dt'/\hbar} e^{-i\int_0^t \cos[\theta_n(t')](\partial\phi_n(t')/\partial t')dt' \hat{J}_z(n)} = \prod_{n=1}^{N} e^{-i\int_0^t R_n(t')dt'} e^{i\gamma_n(t)J_z(n)} , \quad (4.4)$$

which describes the geometric feature of the evolution in terms of the Berry's phase

$$\gamma_n(t) = -\int_0^t \frac{\partial\phi_n(t')}{\partial t'}\cos[\theta_n(t')]dt' , \quad (4.5)$$

When the parameter $\mathbf{R}$ is subject to a cyclic evolution that $\mathbf{R}(0)=\mathbf{R}(T)$, the Berry's phase

$$\gamma_n(T) = \int_0^{2\pi}[1-\cos\theta_n]d\phi_n \quad (4.6)$$

is just a solid angle spanned by the closed curve traced by the parameter $\mathbf{R}$. To consider whether the WPC happens or not for the adiabatic evolution, we explicitly calculate

$$|\langle 0|U_e(t)|0\rangle| = \prod_{n=1}^{N} |\langle 0|e^{-i\hat{J}_y(n)\theta_n(t)/\hbar}|0\rangle|$$
$$= \prod_{n=1}^{N} |d^{j_n}_{-j_n,-j_n}(\theta_n(t))|^{2j_n}$$
$$= \prod_{n=1}^{N} |\cos[\theta_n(t)/2]|^{2j_n} . \qquad (4.7)$$

By the proof similar to that in the last section, we see that $|\langle 0|U_e(t)|0\rangle|\to 0$ as $N\to\infty$. Namely, even in a nonsolvable case, the generalized CH model still realizes the WPC quantum dynamically for the adiabatic evolution.

Furthermore, let us prove that it does so for the nonadiabatic evolution. In fact, if the parameter **R** does not change slowly enough for the adiabatic condition

$$\left|\hbar\frac{\partial\phi_n(t)}{\partial t}\Big/R_n(t)\right| , \left|\hbar\frac{\partial\theta_n(T)}{\partial t}\Big/R_n(t)\right| \ll 1 , \qquad (4.8)$$

we at least consider the second-order approximation

$$U_e'(t)=U_0'(t)[1+U_1'(t)]=U_0'(t)\prod_{n=1}^{N}[1+U_1'^n(t)]$$
$$=\prod_{n=1}^{N} e^{-i\int_0^t R_n(t')\hat{J}_z(n)dt'/\hbar}\, e^{-i\int_0^t \cos[\theta_n(t')](\partial\phi_n(t')/\partial t')dt'\hat{J}_z(n)}$$
$$\times\left\{1+\frac{1}{i\hbar}\int_0^t U_0'^{\dagger}(t)\left[-\frac{\partial}{\partial t'}\theta_n(t')\hat{J}_y(n)+\sin[\theta_n(t')]\frac{\partial}{\partial t'}\phi_n(t')\hat{J}_x(n)\right]U_0'(t)dt'\right\} . \qquad (4.9)$$

Because of the cutoff in the Dyson series for the approximate evolution operator, the unitarity of the evolution operator is broken and so its leaded evolution state is not normalized to unity. Thus, when we calculate the vacuum-vacuum transition amplitude $\langle 0|U_e'(t)|0\rangle$, we should first renormalize it. Let us by $\tilde{U}_e'(t)$ denote the renormalized evolution operator defined by

$$\tilde{U}_e'(t)|\phi\rangle=\frac{U_e'(t)|\phi\rangle}{\langle\phi|U_e'^{\dagger}(t)U_e'(t)|\phi\rangle} \qquad (4.10)$$

for any state vector $|\phi\rangle$. This renormalization results in a reasonable vacuum-vacuum transition amplitude satisfying

$$|\langle 0|\tilde{U}_e'(t)|0\rangle|=\left|\frac{\langle 0|U_e'(t)|\phi\rangle}{\langle 0|U_e'^{\dagger}(t)U_e'(t)|0\rangle}\right|=\left|\prod_n^N\frac{\langle j_n,-j_n|U_1'^n(t)|j_n,-j_n\rangle}{\langle j_n,-j_n|U_1'^{n\dagger}(t)U_1'^n(t)|j_n,-j_n\rangle}\right| . \qquad (4.11)$$

As the formula given by Eq. (4.11), the above equation also explicitly defines the dynamical realization of the WPC in the classical limit with $N\to\infty$. Here, we have taken it into account that

$$\left|\frac{\langle j_n,-j_n|U_1'^n(t)|j_n,-j_n\rangle}{\langle j_n,-j_n|U_1'^{n\dagger}(t)U_1'^n(t)|j_n,-j_n\rangle}\right|\le 1$$
$$\text{for } n=1,2,\ldots,N . \qquad (4.12)$$

Based on the above discussions on the first- and second-order approximations, we guess that the WPC can be realized in an arbitrary-order approximation. Trying to prove this guess, we find some essential properties related to the WPC closely in the next section.

## V. COMMENTS ON ESSENCE OF DYNAMICAL REALIZABILITY

Including the above discussion in this paper, the previous investigations on the dynamical realization of the WPC in terms of quantum dynamical models only proceeded with the concrete form of the model Hamiltonians, especially of the interactions between $S$ and $D$. It seems that the dynamical realizability of the WPC depends on the choice of concrete forms of interaction. However, motivated by the above discussions, we will show a more universal fact that it is the factorizability of the evolution, other than its exact solvability, that leads to the WPC in quantum measurement. Now, let us describe what is the factorization of the evolution. Let $x$ and $p$ be the coordinate and momentum operator of the measured system, respectively; $x_n$ ($n=1,2,\ldots,N$) be the variables for the measuring instrument. Usually, the evolution operator $U(t,p,x,x_i)$ for the universe, $S+D$, depends on $x$, $p$, and $x_n$ ($n=1,2,\ldots,N$). If this operator can be expressed as the following factorizable form

$$U(p,x,x_i)=U_s(p,x,t)\prod_{n=1}^{N}U^{[n]}(x,x_n,t) , \qquad (5.1)$$

then we say that the evolution characterized by $U(t,p,x,x_i)$ is factorizable. Here, $U_s(p,x,t)$ is the evolution operator of $D$ in absence of the interaction with the detector $D$, and the unitary operator $U^{[n]}(x,x_n,t)$ only depends on $x_n$ and $x$ for fixed $n$. In this case, the reduced density matrix of $S$ for the above-mentioned double-slit interference experiment in the interaction picture is obtained as

$$\rho(t)_S = \mathrm{Tr}_D \rho(t) = |C_1|^2 |\psi_1(t)\rangle\langle\psi_1(t)| + |C_2|^2 |\psi_2(t)\rangle\langle\psi_2(t)| + [C_1 C_2^* |\psi_1(t)\rangle\langle\psi_2(t)| + C_2 C_1^* |\psi_2(t)\rangle\langle\psi_1(t)|]\langle 0|U_e(t)|0\rangle \;, \quad (5.2)$$

where

$$|0\rangle = |0_1\rangle \otimes |0_2\rangle \otimes \cdots \otimes |0_N\rangle \;,$$

and $|0_k\rangle$ is the ground state of each single particle in $D$.

Because

$$|\langle 0_k|U^{[k]}(t)|0_k\rangle| = \left[1 - \sum_{n\,(\neq 0)} |\langle n|U^{[k]}|0_k\rangle|^2\right]^{1/2} \leq 1 \;, \quad (5.3)$$

for the positive function

$$\Delta_k(t) = -\ln(|\langle 0_k|U^{[k]}|0_k\rangle|) \;,$$

the series $\sum_{k=1}^{\infty}\Delta_k(t)$ diverges to infinity. That is to say, $\langle 0|U_e(t)|0\rangle$ as well as its norm

$$|\langle 0|U_e(t)|0\rangle| = \prod_{k=1}^{N} |\langle 0_k|U^{[k]}|0_k\rangle| = \exp\left[-\sum_{k=1}^{N}\Delta_k(t)\right] \;, \quad (5.4)$$

approach zero as $N \to \infty$. Then, the WPC appears in the macroscopic limit if we can incorporate a quantum number $J_n$ into $U^{[n]}(x, x_n, t)$ such that $\Delta_k(t) \to \infty$ as $J_n \to \infty$. When $J_n$ enjoys the classical limit at $J_n = \infty$, like the spins $j_n$ in this paper, the WPC also occurs in this limit as a quantum dynamical process. Therefore, we conclude that the essence of the dynamical realizability of WPC is the factorization of the evolution operator for the appreciated model of quantum measurement.

Naturally, the question that follows is what is the general form of the model Hamiltonian which can realize this factorizable evolution. The answer is that the Hamiltonian should be decomposable in a certain sense. The following Hamiltonian sufficiently enjoys the answer:

$$H = H_0 + H' = H_0 + H_I + H_D : \;, \qquad H_I = \sum_{k=1}^{N} V_k(x, x_k), \quad H_D = \sum_{k=1}^{N} h_k(x_k) \;. \quad (5.5)$$

Here, the measured system $S$ is still represented by an ultrarelativistic particle with the free Hamiltonian $H_0 = c\hat{P}$, but the detector $D$ is made of $N$ particles with the quite general single-particle Hamiltonian $h_k(x_k)$, $(k = 1, 2, \ldots, N)$, which is Hermitian. $S$ is assumed to be *independently* subjected to the interaction $V_k(x, x_k)$ of each particle $k$. Here, $x$ and $x_k$ are the coordinates of $S$ and the single particle $k$ in $D$, respectively, and the $k$th interaction potential $V_k(x, x_k)$ only depends on $x$ and $x_k$ and $h_k(x_k)$ on the single-particle variable $x_k$. To prove the above statement, we take the transformation (2.8). Then the reduced evolution operator $U_e(t)$ obeys an effective Schrödinger equation with the effective Hamiltonian

$$H_e(t) = \sum_{k=1}^{N} h_{ek}(t) = \sum_{k=1}^{N} [h_k(x_k) + V_k(x + ct, x_k)] \;, \quad (5.6)$$

depending on time. Since $H_e(t)$ is a direct sum of the time-dependent Hamiltonians $h_{ek}(t)$ $(k = 1, 2, \ldots, N)$ parametrized by $x$, the $x$-dependent evolution operator, as the formal solution to the effective Schrödinger equation

$$U_e(t) = \prod_{k=1}^{N} \otimes U^{[k]}(t) = U^{[1]}(t) \otimes U^{[2]}(t) \otimes \cdots \otimes U^{[N]}(t) \;, \quad (5.7)$$

is factorizable; that is to say, $U_e(t)$ is a direct product of the single-particle evolution operators

$$U^{[k]}(t) = \mathcal{T}\exp\left[1/i\hbar \int_0^t h_{ek}(t)dt\right] \;, \quad (5.8)$$

where $\mathcal{T}$ denotes the time-order operation. As proved in the following, it is just the above factorizable property of the reduced evolution operator that results in the quantum dynamical realization of the WPC. Notice that some results of this section were announced by this author more recently [16].

Before concluding this paper, we shall give some remarks on the results and method of this paper. We first point out that this paper emphasizes the unified description of the classical limit and the macroscopic limit for quantum measurement. Because the macroscopic phenomena in quantum mechanics cannot be identified with those classical ones completely (e.g., the magnetic-flux quantization is a macroscopic quantum phenomenon, but it is not definitely classical), it is quite necessary to distinguish between these two cases. We should also remark that, in practical problems, there must exist interactions among the particles constituting the detector $D$, but in the present discussions there are no interactions among the particles in the detector. We understand it as an ideal case. How to realize the quantum measurement for the WPC in the interaction case is an open question we must face. It is expected, at least for some special case, that the certain canonical (or unitary) transformation possibly enables these particles to become the quasifree ones. This is just similar to the system of harmonic oscillators with quadratic coupling. In this case, we can imagine that the detector is made of free quasiparticles that do not interact with each other.

## APPENDIX: REFORMULATION OF THE HIGH-ORDER ADIABATIC APPROXIMATION METHOD

In order to use it in this paper conveniently, we now reformulate the high-order adiabatic approximation method in Refs. [11–13] in a general form, which can work well on the evolution operator for both the Hermitian and non-Hermitian Schrödinger time evolutions.

Let the Hamiltonian $H_e(t)$ of the quantum system depend on time $t$ through a set of the slowly changing parameters $\mathbf{R}(t)=(R_1(t),R_2(t),\ldots,R_K(t))$. We also assume the quasi-energy-levels $E_k(t)$ $(k=1,2,\ldots,K)$ of the time-dependent Hamiltonian $H(t)=H(R(t))$ for a frozen time $t$ are not degenerate. We diagonalize $H(t)$ by a similarity transformation $S(t)=S(R(t))$ in the following way:

$$S(t)H_e(t)S(t)^{-1}=H_d(t)$$

$$=\begin{bmatrix} E_1(t) & 0 & \cdots & 0 \\ 0 & E_2(t) & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & E_K(t) \end{bmatrix} . \quad \text{(A1)}$$

The corresponding quasienergy state to $E_k(t)$ $(k=1,2,\ldots,k)$ is denoted by $|k(t)\rangle$.

If we determine a solution of the Schrödinger equation of the evolution operator $U_e(t)$ governed by $H_e(t)$ as the following form

$$U_e(t)=S(t)U'(t) , \quad \text{(A2)}$$

then $U'(t)$ obeys the Schrödinger-type equation

$$i\hbar\frac{\partial}{\partial t}U'(t)=H'(t)U'(t) , \quad \text{(A3)}$$

where the equivalent Hamiltonian

$$H'(t)=H_d(t)-i\hbar S(t)^{-1}\frac{\partial}{\partial t}S(t) \quad \text{(A4)}$$

can be decomposed into the diagonal part

$$H_0(t)=H_d(t)+\left[\text{diagonal part of }\left[-i\hbar S(t)^{-1}\frac{\partial}{\partial t}S(t)\right]\right] \quad \text{(A5)}$$

and the off-diagonal part

$$V(t)=\left[\text{off-diagonal part of }\left[-i\hbar S(t)^{-1}\frac{\partial}{\partial t}S(t)\right]\right] . \quad \text{(A6)}$$

Physically, since $V(t)$ completely vanishes when $H(t)$ is independent of time, we deduce that $V(t)$ is a perturbation in the case that $H(t)$ depends on time quite "slightly." Later on, we will give the analytic condition where $V(t)$ can be regarded as a perturbation. Then, the adiabatic Dyson series solution of $U'(t)$:

$$U'(t)=U'_0(t)\left[1+\sum_{k=1}^{\infty}U'_k(t)\right]: ,$$
$$U'_0(t)=e^{(-i/h)\int_0^t H_0(s)ds} , \quad \text{(A7)}$$
$$U'_k(t)=\left[\frac{-i}{\hbar}\right]^k\int_0^t\int_0^{s_1}\int_0^{s_2}\cdots\int^{s_{k-1}}\overline{V}(s_1)\overline{V}(s_2)\cdots\overline{V}(s_{k-1})\overline{V}(s_k)ds_1ds_2\cdots ds_{k-1}ds_k ,$$

immediately follows from the standard time-dependent perturbation theory. Here,

$$\overline{V}(t)=e^{(i/\hbar)\int_0^t H(s)ds}V(t)e^{(-i/\hbar)\int_0^t H(s)ds} . \quad \text{(A8)}$$

The first-order approximation solution $U'_0(t)$ can be decomposed into the dynamical factor

$$\begin{bmatrix} e^{-i\int_0^t dt'E_1(t')/\hbar} & 0 & \cdots & 0 \\ 0 & e^{-i\int_0^t dt'E_2(t')/\hbar} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & e^{-i\int_0^t dt'E_K(t')/\hbar} \end{bmatrix} \quad \text{(A9)}$$

and the geometric factor

$$\begin{bmatrix} e^{i\int_0^t dt'A_1(t')} & 0 & \cdots & 0 \\ 0 & e^{i\int_0^t dt'A_2(t')} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & e^{i\int_0^t dt'A_K(t')} \end{bmatrix} , \quad \text{(A10)}$$

where

$$A_n(t)=i\left\langle n\left|S(t)^{-1}\frac{\partial}{\partial t}S(t)\right|n\right\rangle \quad \text{(A11)}$$

and each diagonal element in the above matrix is just the Berry's phase factor, i.e.,

$$\gamma_n(t)=\int_0^t A_n(s)ds=i\left\langle n\left|\int_0^t S(s)^{-1}\frac{\partial}{\partial s}S(s)ds\right|n\right\rangle . \quad \text{(A12)}$$

In terms of the concept of differential manifold, this phase can be rewritten as a curve integral

$$\gamma_n(t)=\gamma_n(\mathbf{R}(t))=\int_{\mathbf{R}(t)}A_\mu[\mathbf{R}]dR^\mu \quad \text{(A13)}$$

of the potential one-form $A_\mu[\mathbf{R}]dR^\mu$:

$$A_\mu[\mathbf{R}]=i\left\langle n\left|S[\mathbf{R}]^{-1}\frac{\partial}{\partial R^\mu}S[\mathbf{R}]\right|n\right\rangle , \quad \text{(A14)}$$

on the parameter manifold

$$\mathbf{M}=\{\mathbf{R}=(R_1,R_2,\ldots,R_K)| ,$$
$$R_i\in(\text{real number field}),i=1,2,\ldots,K\} .$$

Here, $A_\mu[\mathbf{R}]$ is called induced gauge potential. In this sense, the Berry's phase factor $e^{i\gamma_n[\mathbf{R}(T)]}$ can be understood as an element of the holonomy group for a closed parameter curve $C$: $\{\mathbf{R}(t)|\mathbf{R}(0)=\mathbf{R}(T)\}$.

It is pointed out that the transformation $S[\mathbf{R}]$ diagonalizing $H_e(T)$ is not unique, i.e., $S'[\mathbf{R}]=S[\mathbf{R}]X[R]$ also diagonalizes $H_e(t)$ if the matrix $X[R]$ commutes with $H_e(t)$. This means that state vectors $|n[\overline{\mathbf{R}}]\rangle=(S[\mathbf{R}]X[R])^{-1}|n\rangle$ as well as $|n[\mathbf{R}]\rangle=(S[\mathbf{R}])^{-1}|n\rangle$ are the instantaneous eigenfunctions of $H_e[R(t)]$. The above indeterminacy of $S[\mathbf{R}]$ results in the gauge transformation for induced gauge potential

$$A_\mu[\mathbf{R}]\to A'_\mu[\mathbf{R}]=A_\mu[\mathbf{R}]+i\langle n|X[\mathbf{R}]^{-1}\frac{\partial}{\partial R^\mu}X[\mathbf{R}]|n\rangle. \tag{A15}$$

From the second-order approximation

$$U(t)'\simeq U'_0(1)[1+U(t)'_1] =e^{(-i/h)\int_0^t H_0(s)ds}\left[1-\frac{i}{\hbar}\int_0^t\overline{V}(s)ds\right], \tag{A16}$$

we observe that the adiabatic condition under which the adiabatic-approximation solution $U'_0(t)$ works well, is

$$\left|\frac{\hbar\langle n|V(t)|m\rangle}{E_n(t)-E_m(t)}\right|\ll 1 \quad \text{for } m\neq n . \tag{A17}$$


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[1] J. von Neumann, *Mathematische Grundlage de Quantummechanik* (Springer, Berlin, 1932).
[2] V. B. Braginsky and F. Y. Khalili *Quantum Measurement* (Cambridge University Press, Cambridge, England, 1992).
[3] R. Omnes, Rev. Mod. Phys. **64**, 339 (1992).
[4] M. Gell-Mann and J. B. Hartle, Phys. Rev. D **47**, 3345 (1993).
[5] K. Hepp, Helv. Phys. Acta **45**, 237 (1972).
[6] J. S. Bell, Helv. Phys. Acta **48**, 93 (1975).
[7] T. Kobayashi, Found. Phys. Lett. **5**, 265 (1992).
[8] M. Namik and S. Pascazio, Phys. Rev. A **44**, 39 (1991).
[9] H. Nakazato and S. Pascazio, Phys. Rev. Lett. **70**, 1 (1993).
[10] M. Cini, Nuovo Cimento B **73**, 27 (1983).
[11] C. P. Sun, J. Phys. A **21**, 1595 (1988).
[12] C. P. Sun, Phys. Rev. D **38**, 2908 (1988).
[13] C. P. Sun, Phys. Rev. D **41**, 1318 (1990).
[14] M. V. Berry, Proc. R. Soc. London Ser. A **392**, 45 (1984).
[15] A. Shapere and F. Wilczek, *Geometrical Phases in Physics* (World Scientific, Singapore, 1990).
[16] C. P. Sun, State University of New York at Stony Brook, Institute of Theoretical Physics Report No. ITP.SB-93-13 (1993) (unpublished).
[17] W. M. Zhang, D. H. Feng, and R. Gilmore, Rev. Mod. Phys. **62**, 867 (1990).