Hierarchy recurrences in local relaxation

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Inside a closed many-body system undergoing the unitary evolution, a small partition of the whole system exhibits a local relaxation. If the total degrees of freedom of the whole system is a large but finite number, such a local relaxation would come across a recurrence after a certain time, namely, the dynamics of the local system suddenly appears random after a well-ordered oscillatory decay process. It is found in this paper, among a collection of N two-level systems (TLSs), the local relaxation of one TLS inside has a hierarchy structure hiding in the randomness after such a recurrence: similar recurrences appear in a periodical way, and the later recurrence brings in stronger randomness than the previous one. Both analytical and numerical results that we obtained well explains such hierarchy recurrences: the population of the local TLS (as an open system) diffuses out and regathers back periodically due to the finite-size effect of the bath [the remaining $(N-1)$ TLSs]. We also find that the total correlation entropy, which sums up the entropy of all the N TLSs, approximately exhibit a monotonic increase; in contrast, the entropy of each single TLS increases and decreases from time to time, and the entropy of the whole N-body system keeps constant during the unitary evolution.

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I. INTRODUCTION

When an open system is in contact with an infinitely large bath, the open system would approach a certain steady state after a long time relaxation. However, such an irreversible behavior cannot be seen in the dynamics of one- or few-body systems. Thus the macroscopic irreversibility seems contradicted with the microscopic reversibility [1–5].

One useful way to look at this problem is to study the system relaxation in contact with a finite bath, namely, the bath contains a finite number of degrees of freedom (DoF), and then consider its transition to the thermodynamics limit [6–9]. The open system and the bath as a whole isolated system always follows the unitary evolution and keeps a constant entropy as the initial state, while the open system itself seems to relax towards a certain steady state, thus such relaxation behavior of the open system itself is called the *local relaxation* [6–8].

Due to the finite-size effect of the bath, the local relaxation of the open system would come across a recurrence after a certain time: at first the system dynamics shows a well-ordered oscillatory decay behavior, but then suddenly appears "random"[1] [6–8,10,11]. With the increase of the DoF number in the bath, such a recurrence time appears much later, thus it does not show up in practice. And it is worth noticing that such relaxation behavior does not require any average from disorder or time.

In this paper we find that, for a simple many-body system without disorder, in the region after such a recurrence, indeed there exists a hierarchy structure hiding in the randomness: similar recurrences appear in a periodical way, and the later recurrence brings in stronger randomness than the previous one, therefore we call them *hierarchy recurrences*.

Here we study the dynamics of a chain of N two-level systems (TLSs). One of the TLSs is treated as the open system, and all the other $(N-1)$ TLSs make up a finite bath. We obtain a Bessel function expansion for the system dynamics, which well explains the appearance of such hierarchy recurrences. Furthermore, we also find the physical reason for the appearance of such hierarchy recurrences: when time increases, the population of the open system diffuses out and propagates in the finite bath (the periodic TLS chain); once the population regathers back to the open system, the interference between the backward and outward propagations gives rise to such a recurrence, and this process happens again and again, leading to the hierarchy recurrences. Therefore, this property could also exist in some other many-body systems with certain proper propagation interferences.

We also study the dynamics of the *total correlation entropy* of the N-body system, which sums up the entropy of all the NTLSs [12–15]. It turns out the total correlation entropy approximately exhibits a monotonic increasing behavior, and the increasing curve becomes more and more "smooth" with the increase of the bath size. Thus the total correlation entropy

[1] Rigorously speaking, "quasirandomness" would be a more precise description. The "randomness" here means the dynamics appears "random" compared with the well-ordered decay behavior at the beginning, and our discussion below indeed gives a deterministic description. Here we still adopt the description random as in previous literatures.

exhibits a quite similar behavior as the irreversible entropy increase in the standard thermodynamics [16–19]. In contrast, the whole N-body system always keeps a constant due to the unitary evolution, and the entropy of each single TLS increases and decreases from time to time.

The paper is arranged as follows. In Sec. II we study the local relaxation dynamics of a chain of N TLSs. In Sec. III we show the hierarchy recurrences in the relaxation process and discuss its origin. In Sec. IV we show the dynamics of the total correlation entropy in this system, and then discuss its connection with the standard thermodynamics in Sec. V. The summary is drawn in Sec. VI.

II. LOCAL RELAXATION

We consider a chain of N TLSs. They have equal on-site energies ($\omega \geqslant 0$), and exchange energy with the nearest neighbors (interaction strengths $0 < g \ll \omega$):

$$\hat{\mathcal{H}} = \sum_{n=0}^{\mathrm{N}-1} \frac{1}{2}\omega\hat{\sigma}_n^z + g(\hat{\sigma}_n^+\hat{\sigma}_{n+1}^- + \hat{\sigma}_n^-\hat{\sigma}_{n+1}^+). \tag{1}$$

Here $\hat{\sigma}_n^+ := (\hat{\sigma}_n^-)^\dagger = |\mathsf{e}\rangle_n\langle\mathsf{g}|$, $\hat{\sigma}_n^z := |\mathsf{e}\rangle_n\langle\mathsf{e}| - |\mathsf{g}\rangle_n\langle\mathsf{g}|$, and $|\mathsf{e}\rangle_n$, $|\mathsf{g}\rangle_n$ are the excited and ground states of the nth TLS.

Here site 0 is regarded as an open "SYSTEM," while all the other (N − 1) TLSs make up a finite "BATH." Initially, the SYSTEM (site 0) starts from the excited state $\hat{\rho}_0(t=0) = |\mathsf{e}\rangle_0\langle\mathsf{e}|$ as its initial state, and all the TLSs in the BATH start from a thermal state $\hat{\rho}_n^{(T)} = \bar{n}_T|\mathsf{e}\rangle_n\langle\mathsf{e}| + (1-\bar{n}_T)|\mathsf{g}\rangle_n\langle\mathsf{g}| \sim \exp[-\frac{\omega}{2}\hat{\sigma}_n^z/T]$, with temperature T and $\bar{n}_T := 1/(1+e^{\omega/T}) \in [0, \frac{1}{2})$ as the initial population.

Now we study the dynamics of the open SYSTEM. The Hamiltonian (1) is a quantum *XX* model [20], and the dynamics of the whole chain is exactly solvable. Applying the Jordan-Wigner transform, the Hamiltonian (1) becomes a fermionic one,

$$\sigma_n^z = 2\hat{c}_n^\dagger\hat{c}_n - 1, \quad \hat{\sigma}_n^+ = \hat{c}_n^\dagger\prod_{i=0}^{n-1}(-\hat{\sigma}_i^z),$$

$$\hat{\mathcal{H}} = \sum_{n=0}^{\mathrm{N}-1}\omega\hat{c}_n^\dagger\hat{c}_n + g(\hat{c}_n^\dagger\hat{c}_{n+1} + \hat{c}_{n+1}^\dagger\hat{c}_n). \tag{2}$$

Under the periodic boundary condition, it can be further diagonalized by the Fourier transform $\hat{c}_n = \sum_{k=0}^{\mathrm{N}-1}\exp(i\frac{2\pi}{\mathrm{N}}nk)\hat{b}_k/\sqrt{\mathrm{N}}$, which reads $\hat{\mathcal{H}} = \sum\varepsilon_k\hat{b}_k^\dagger\hat{b}_k$, with the eigenmode energy $\varepsilon_k = \omega + 2g\cos\frac{2\pi k}{\mathrm{N}}$.

The N-body chain as a whole isolated system follows the unitary evolution. From the above transformations, the above initial condition gives $\langle\hat{c}_0^\dagger\hat{c}_0\rangle_{(t=0)} = 1$, and $\langle\hat{c}_m^\dagger\hat{c}_n\rangle_{(t=0)} = \bar{n}_T\,\delta_{mn}$ for the other m, n, and that gives the following dynamics:

$$\begin{aligned}\langle\hat{c}_m^\dagger\hat{c}_n\rangle_{(t)} &= \sum_{k,q=0}^{\mathrm{N}-1}\frac{1}{\mathrm{N}}e^{i\frac{2\pi}{\mathrm{N}}nq - i\frac{2\pi}{\mathrm{N}}mk}\langle\hat{b}_k^\dagger(0)e^{i\varepsilon_k t}\hat{b}_q(0)e^{-i\varepsilon_q t}\rangle\\ &= \sum_{kq,xy}\frac{\langle\hat{c}_x^\dagger\hat{c}_y\rangle_{(0)}}{\mathrm{N}^2}e^{i\frac{2\pi q}{\mathrm{N}}(n-y) - i\frac{2\pi k}{\mathrm{N}}(m-x) + i(\varepsilon_k-\varepsilon_q)t}\\ &:= (1-\bar{n}_T)\big[\Phi_m^{(\mathrm{N})}(2gt)\big]^*\,\Phi_n^{(\mathrm{N})}(2gt) + \bar{n}_T\,\delta_{mn},\end{aligned} \tag{3}$$

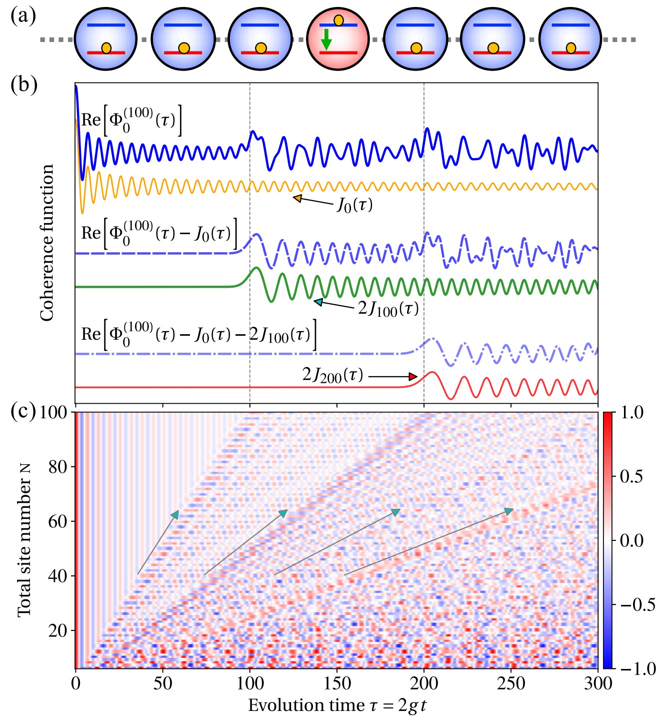

FIG. 1. (a) Demonstration for the NTLSs and their initial states. (b) The coherence function Re[$\Phi_0^{(\mathrm{N}=100)}(\tau)$], compared with the Bessel functions. (c) The scaling behavior of Re[$\Phi_0^{(\mathrm{N})}(\tau)$] with the site number N.

where we call $\Phi_n^{(\mathrm{N})}(2gt := \tau)$ as the coherence function, and[2]

$$\Phi_n^{(\mathrm{N})}(\tau) \;:=\; \frac{1}{\mathrm{N}}\sum_{k=0}^{\mathrm{N}-1}\exp\left[-i\tau\,\cos\frac{2\pi k}{\mathrm{N}} + i\frac{2\pi}{\mathrm{N}}kn\right] \tag{4a}$$

$$\xrightarrow{\mathrm{N}\to\infty}\int_0^{2\pi}\frac{dx}{2\pi}\,e^{-i\tau\cos x + inx} = (-i)^n J_n(\tau). \tag{4b}$$

In the thermodynamics limit $\mathrm{N}\to\infty$, $\Phi_n^{(\mathrm{N})}(\tau)$ becomes the Bessel function $J_n(\tau)$, which approaches zero when $\tau\to\infty$ [6–8,10,11].

It can be seen from Eqs. (2) and (3) that each site always keeps a diagonal density state $\hat{\rho}_n(t) = p_{n,\mathsf{e}}(t)|\mathsf{e}\rangle_n\langle\mathsf{e}| + p_{n,\mathsf{g}}(t)|\mathsf{g}\rangle_n\langle\mathsf{g}|$, where $p_{n,\mathsf{e}}(t) := \langle\hat{\sigma}_n^+\hat{\sigma}_n^-\rangle_{(t)} = \langle\hat{c}_n^\dagger\hat{c}_n\rangle_{(t)}$ is the excited population of site n. Therefore, if the BATH is infinitely large ($\mathrm{N}\to\infty$), all the NTLSs would reach and stay at the same thermal state $\hat{\rho}_n(t\to\infty)\sim\exp[-\frac{\omega}{2}\hat{\sigma}_n^z/T]$ after long time relaxation (here the limit $\mathrm{N}\to\infty$ is taken before $t\to\infty$).

III. SCALING BEHAVIOR OF RECURRENCES

If the BATH is a finite one composed of (N − 1) TLSs, due to the finite-size effect, the above coherence functions $\Phi_n^{(\mathrm{N})}(\tau)$ exhibit a *recurrence* behavior[3] [solid blue line in Fig. 1(b)]:

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[2]Utilizing $\exp[-i\tau\cos x] = \sum_{n=-\infty}^{\infty}(-i)^n J_n(\tau)e^{\pm inx}$.

[3]Precisely speaking, the recurrence behavior of the open system here is different from the Poincaré recurrence usually discussed in chaotic dynamics.

(1) Within the time $0 \leqslant t \lesssim t_{\text{rec}} := \text{N}/2g$, $\Phi_n^{(\text{N})}(\tau)$ exhibits a well-ordered oscillatory decay towards zero, and it fits the above Bessel function (4b) quite closely. (2) After $t \lesssim t_{\text{rec}}$, $\Phi_n^{(\text{N})}(\tau)$ shows a "sudden bump," and then starts to appear random, without showing any regular feature explicitly. Thus, such a sudden change around $t_{\text{rec}} \equiv \text{N}/2g$ was called a recurrence in previous studies [6–8,10,11].

Therefore, based on the local observation within a finite time smaller than t_{rec}, we may conclude the open SYSTEM itself is relaxing towards a certain steady state, but indeed the full N-body state always keeps a pure state during the unitary evolution. With the increase in the size of N, the recurrence time $t_{\text{rec}} \equiv \text{N}/2g$ becomes larger and larger, thus such a recurrence behavior does not show up in practice.

When looking at the scaling behavior for different sizes N, the "randomness" of $\Phi_0^{(\text{N})}(\tau)$ after the recurrence appears more explicit [see the bottom right patterns in Fig. 1(c)]. Moreover, besides the above recurrence appearing around $t \simeq t_{\text{rec}} \equiv \text{N}/2g$, it is worth noting that some well-organized recurrence patterns also appear in the region $t \lesssim t_{\text{rec}}$: similar recurrences also appear periodically around $t \simeq \text{q}\, t_{\text{rec}}$ for $\text{q} = 2, 3, 4, \ldots$ [see the arrows in Fig. 1(c)]. Moreover, each recurrence seems to bring in a stronger randomness to $\Phi_0^{(\text{N})}(\tau)$ than the previous one, which forms a hierarchy structure, thus we call them *hierarchy recurrences*.

We find that the appearance of such hierarchy recurrences can be explained by the following expansion of $\Phi_n^{(\text{N})}(\tau)$ [Eq. (4a)], that is footnote 2,

$$\begin{aligned}\Phi_n^{(\text{N})}(\tau) &= \frac{1}{\text{N}}\sum_{k=0}^{\text{N}-1}\left[\sum_{m=-\infty}^{\infty}(-i)^m J_m(\tau)e^{-im\frac{2\pi k}{\text{N}}}\right]e^{i\frac{2\pi}{\text{N}}kn}\\ &= \sum_{m=-\infty}^{\infty}(-i)^m J_m(\tau)\frac{1}{\text{N}}\sum_{k=0}^{\text{N}-1}e^{i\frac{2\pi k}{\text{N}}(n-m)}\\ &= \sum_{\text{q}=-\infty}^{\infty}(-i)^{n+\text{qN}}J_{n+\text{qN}}(\tau). \qquad (5)\end{aligned}$$

Here we used the relation $\sum_{k=0}^{\text{N}-1} e^{i\frac{2\pi k}{\text{N}}(n-m)} = \text{N}\delta_{n-m,\,\text{qN}}$, with q as an arbitrary integer.

For example, site 0 ($n = 0$) gives a simple Bessel function series [using $J_{-n}(\tau) = (-1)^n J_n(\tau)$][4]

$$\begin{aligned}\Phi_0^{(\text{N})}(\tau) &= J_0(\tau) + (-i)^{\text{N}}[1 + (-1)^{\text{N}}]J_{\text{N}}(\tau)\\ &\quad + (-i)^{2\text{N}}[1 + (-1)^{2\text{N}}]J_{2\text{N}}(\tau) + \cdots. \qquad (6)\end{aligned}$$

For a large N, the Bessel function $J_{\text{N}}(\tau) \simeq 0$ in the area $0 \leqslant \tau \lesssim \text{N}$, and starts to exhibit significant oscillations after $\tau \simeq \text{N}$ [see $J_{100}(\tau)$ in Fig. 1(b)]. Therefore, in the above expansion of $\Phi_0^{(\text{N})}(\tau)$, each term $J_{\text{qN}}(\tau)$ contributes a sudden bump around $\tau \simeq \text{qN}$, and this is just why the above recurrences appear around $t \simeq \text{q}\, t_{\text{rec}}$ ($\text{q} = 1, 2, 3, \ldots$).

Besides, in Fig. 2(b), the population dynamics of all the N TLSs is shown (the BATH temperature is set as $T \to 0^+$), i.e., $p_{n,\text{e}}(t) = \langle\hat{\sigma}_n^+\hat{\sigma}_n^-\rangle_{(t)} = |\Phi_n^{(\text{N})}(2gt \equiv \tau)|^2$, and a propagation

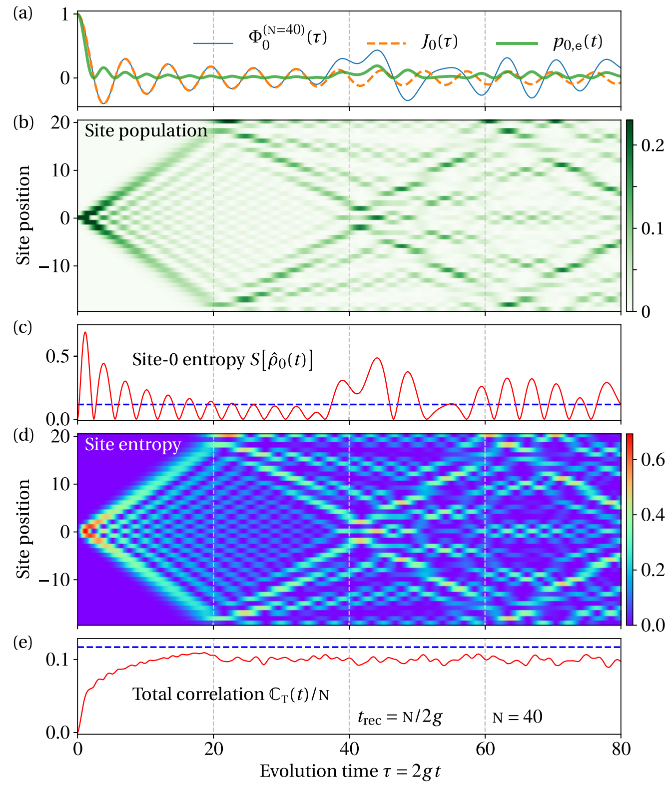

FIG. 2. (a) The coherence function $\Phi_0^{(\text{N})}(\tau)$ and site-0 population $p_{0,\text{e}}(t)$. (b) The population evolution of each TLS. (c) The entropy dynamics of site 0. (d) The entropy of each TLS. (e) The total correlation entropy $\mathbb{C}_{\text{T}}(t)/\text{N}$. The dashed blue lines in (c) and (e) are $\mathbb{C}_{\max}^{(\text{N})}/\text{N}$. Here the site number is $\text{N} = 40$. In all the above results, the initial BATH temperatures are set as $T \to 0^+$.

pattern is clearly seen. Initially, the population distribution of the N TLSs forms a "cusp" around site 0 [$p_{0,\text{e}}(t) = 1$, and $p_{n,\text{e}}(t) = 0$ for $n \neq 0$]. Within the time $t \lesssim t_{\text{rec}}$, the initial population cusp on site 0 propagates towards the two directions of the periodic chain, and the propagation "speed" is almost a constant [21,22]. This constant speed also can be seen from the leading terms of $p_{n,\text{e}}(t) = |\Phi_n^{(\text{N})}(2gt)|^2 \simeq |J_{|n|}(2gt)|^2 + \cdots$ [for $-\text{N}/2 < n < \text{N}/2$, see Eq. (5)]: the leading Bessel function indicates that the first sudden bump of site n appears around $t \simeq |n|/2g$, which linearly depends on the distance $|n|$ to site 0 [here site $(-n)$ and site $(\text{N} - n)$ are the same one due to the periodic boundary condition].

The two-side propagations would meet each other at the periodic boundaries at $n \sim \pm\text{N}/2$, and then regathers back to site 0 again. Notice that this is just the moment that $\Phi_0^{(\text{N})}(2gt)$ exhibits its first recurrence ($t \simeq t_{\text{rec}} \equiv \text{N}/2g$, see the dashed vertical lines in Fig. 2). The propagation regathered back would be superposed with the original one, which makes the system dynamics appear more random. Clearly, since such propagation and regathering happens again and again, the SYSTEM (site 0) experiences the above hierarchy recurrences periodically around $t \simeq \text{q}\, t_{\text{rec}}$.

Thus, it is also expectable that similar recurrence behaviors also exist in more many-body systems if there exists a certain superposition between a propagating wave and its reflections.

[4]When $\text{N} \to \infty$, the function series $\{\Phi_0^{(\text{N})}(\tau)\}$ converges pointwise to $J_0(\tau)$ but not uniformly.

Indeed similar propagation patterns as Fig. 2(b) also have been found in some other many-body systems, such as the Heisenberg chain [22] and Hubbard model [23,24], where it is called the quantum carpet [25,26].

Again we remark that all the above discussions are based on the exact unitary evolution, which is deterministic. Compared with the well-ordered decay behavior at the beginning, the dynamics in the recurrence region looks random.

IV. TOTAL CORRELATION ENTROPY

Now we consider the entropy dynamics in this system. The N-body chain as a whole isolated system follows the unitary evolution, thus its von Neumann entropy $S[\hat{\boldsymbol{\rho}}(t)] = -\mathrm{tr}[\hat{\boldsymbol{\rho}} \ln \hat{\boldsymbol{\rho}}]$ always keeps the same as the initial state. Especially, we focus on the situation that initially the BATH has a zero temperature $T \to 0^+$ and $\bar{n}_T = 0$, thus the entropy of the whole system always keeps at zero. Besides, the thermal entropy $đQ/T$ in the standard thermodynamics diverges when $T \to 0^+$ and thus cannot be used here [27].

Indeed, in practical observations, the full N-body state is usually not directly accessible for local measurements, and it is the few-body observables that can be directly measured [5,19,28]. Therefore, instead of the entropy of the whole system, here we consider the dynamics of the *total correlation entropy* of the N-body state $\hat{\boldsymbol{\rho}}(t)$, that is [12–15],

$$\mathbb{C}_{\mathrm{T}}[\hat{\boldsymbol{\rho}}(t)] := \sum_{n=0}^{\mathrm{N}-1} S[\hat{\rho}_n(t)] - S[\hat{\boldsymbol{\rho}}(t)], \tag{7}$$

where $\hat{\rho}_n$ are the reduced one-body states. $\mathbb{C}_{\mathrm{T}}[\hat{\boldsymbol{\rho}}]$ measures the total amount of all the correlations inside the N-body state $\hat{\boldsymbol{\rho}}$ [12,14]. For bipartite systems ($\mathrm{N} = 2$), it just returns the mutual information, which measures the bipartite correlation [17–19,29].

It turns out the entropy of each single TLS increases and decreases from time to time, and they also have the above recurrence behavior [Figs. 3(c) and 3(d)]. In contrast, their summation as the total correlation entropy $\mathbb{C}_{\mathrm{T}}(t)$ approximately exhibits a monotonic increasing behavior [except still carrying small fluctuations, see Fig. 2(e)]. Moreover, with the increase of the chain size N, the increasing curve of $\mathbb{C}_{\mathrm{T}}(t)$ appears more and more smooth [Figs. 3(a)–3(c)]. Clearly this is quite similar to the behavior of the irreversible entropy production during the relaxation process in the standard thermodynamics [30–33].

Now we consider the correlation maximum that $\mathbb{C}_{\mathrm{T}}(t)$ might achieve [34]. With the help of Lagrangian multipliers, under the constraints (1) $p_{n,\mathrm{g}} + p_{n,\mathrm{e}} = 1$ (probability normalization) and (2) $\sum_n p_{n,\mathrm{e}} = 1$ (excitation number conservation), the maximum of $\mathbb{C}_{\mathrm{T}}[\{p_{n,\mathrm{e(g)}}\}] = \sum_n -p_{n,\mathrm{e}} \ln p_{n,\mathrm{e}} - p_{n,\mathrm{g}} \ln p_{n,\mathrm{g}}$ is obtained as

$$\mathbb{C}_{\max}^{(\mathrm{N})} = \mathrm{N} \ln \frac{\mathrm{N}}{\mathrm{N}-1} + \ln(\mathrm{N}-1). \tag{8}$$

The maximum is achieved when all the NTLSs have the same populations $\tilde{p}_{n,\mathrm{e}} = 1/\mathrm{N}$.

Under the scaled time $2gt/\mathrm{N}$, the correlation evolutions $\mathbb{C}_{\mathrm{T}}(t)/\mathrm{N}$ for different sizes Nappear quite similar to each other [Figs. 3(a)–3(c)]. They all approach their upper bound $\mathbb{C}_{\max}^{(\mathrm{N})}/\mathrm{N}$ closely, and come across a sudden bump around half of the recurrence time $t \simeq t_{\mathrm{rec}}/2$ (indeed this is just the moment the two-side propagations meet each other at $n \sim \pm\mathrm{N}/2$).

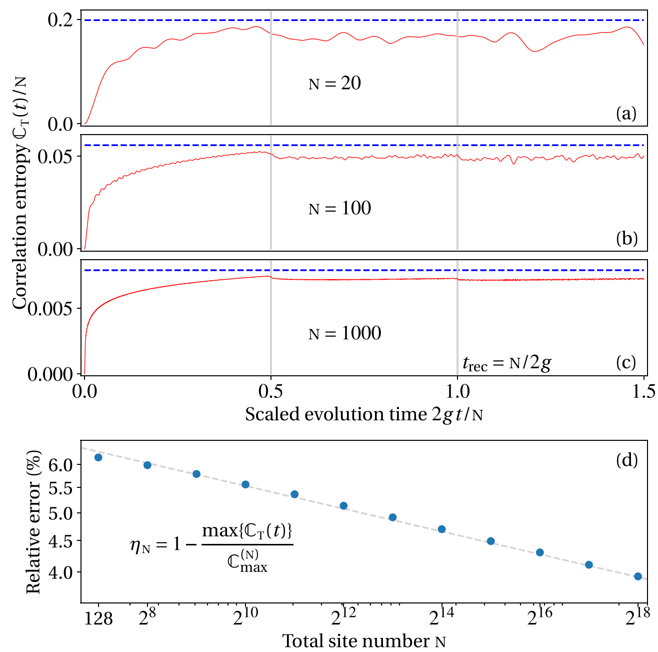

FIG. 3. (a)–(c) Evolution of the total correlation entropy $\mathbb{C}_{\mathrm{T}}(t)/\mathrm{N}$ with the scaled time $2gt/\mathrm{N}$ for $\mathrm{N} = 20, 100, 1000$. The blue dashed lines are the correlation maximum $\mathbb{C}_{\max}^{(\mathrm{N})}/\mathrm{N}$. (d) The relative error η_{N} between $\max\{\mathbb{C}_{\mathrm{T}}(t)\}$ (around $t \simeq t_{\mathrm{rec}}/2$) and the maximum $\mathbb{C}_{\max}^{(\mathrm{N})}$ decreases with the site number N. In all the above results, the initial BATH temperatures are set as $T \to 0^+$.

We denote $\eta_{\mathrm{N}} := 1 - \max\{\mathbb{C}_{\mathrm{T}}(t)\}/\mathbb{C}_{\max}^{(\mathrm{N})}$ as the relative error between $\max\{\mathbb{C}_{\mathrm{T}}(t)\}$ and the correlation maximum $\mathbb{C}_{\max}^{(\mathrm{N})}$. With the increase in the size of N, the error η_{N} decays slowly towards zero [approximately $\eta_{\mathrm{N}} \propto \mathrm{N}^{-\alpha}$ with $\alpha \simeq 0.062$, see Fig. 3(d)]. In the thermodynamic limit $\mathrm{N} \to \infty$, we may expect $\mathbb{C}_{\mathrm{T}}(t)$ could reach the maximum $\mathbb{C}_{\max}^{(\mathrm{N})}$.

In this sense, the above correlation maximization effectively gives a *pseudoequilibrium state* $\tilde{\boldsymbol{\rho}}_{\mathrm{eq}} \equiv \bigotimes_n \tilde{\varrho}_n$, where $\tilde{\varrho}_n := \frac{1}{\mathrm{N}}|\mathrm{e}\rangle_n\langle\mathrm{e}| + (1 - \frac{1}{\mathrm{N}})|\mathrm{g}\rangle_n\langle\mathrm{g}|$, and the whole N-body state $\hat{\boldsymbol{\rho}}(t)$ "looks" like approaching this pseudoequilibrium state during the unitary evolution [6]. But we emphasize indeed $\hat{\boldsymbol{\rho}}(t)$ and $\hat{\rho}_n(t)$ never have any steady states when $t \to \infty$, and $\hat{\boldsymbol{\rho}}(t)$ always keeps a pure state.

The increasing rate of the above total correlation entropy (7) also can be rewritten in the form of relative entropy [30,35–37]

$$\partial_t \mathbb{C}_{\mathrm{T}}(t) = \partial_t D[\hat{\boldsymbol{\rho}}(t) \parallel \textstyle\bigotimes_n \hat{\rho}_n(t)], \tag{9}$$

where $D[\rho \| \varrho] = \mathrm{tr}[\rho(\ln \rho - \ln \varrho)]$ is the relative entropy. Approximately, the reference state $\bigotimes_n \hat{\rho}_n(t)$ here can be replaced by the pseudoequilibrium state $\tilde{\boldsymbol{\rho}}_{\mathrm{eq}} \equiv \bigotimes_n \tilde{\varrho}_n$. Again, we emphasize that the pseudoequilibrium state $\tilde{\boldsymbol{\rho}}_{\mathrm{eq}}$ here is determined by the above correlation maximization, and it is different from a canonical state like $\hat{\boldsymbol{\rho}}_{\mathrm{th}} \sim \exp[-\hat{\mathcal{H}}/T]$.

V. CONNECTION WITH THE STANDARD THERMODYNAMICS

Here we show that the increase of the above total correlation entropy $\mathbb{C}_{\text{T}}(t)$ is closely connected to the irreversible entropy production in the standard thermodynamics.

In open system problems, the bath is often modeled as a collection of many noninteracting degrees of freedoms $\hat{H}_{\text{B}} = \sum_k \hat{\text{H}}_k$, which initially starts from a thermal equilibrium state $\hat{\rho}_{\text{B}}(0) \propto \exp(-\hat{H}_{\text{B}}/T)$ [38,39]. In most practical cases, if the system-bath interaction is quite weak, the bath state gives approximately [17–19,40,41]

$$\sum_k S[\hat{\varrho}_k(t)] \simeq S[\hat{\rho}_{\text{B}}(t)] := S_{\text{B}}, \tag{10a}$$

$$\partial_t S_{\text{B}} = -\text{tr}[\dot{\hat{\rho}}_{\text{B}}(t) \ln \hat{\rho}_{\text{B}}(t)] \simeq -\text{tr}[\dot{\hat{\rho}}_{\text{B}}(t) \ln \hat{\rho}_{\text{B}}(0)], \tag{10b}$$

and the bath entropy becomes

$$\partial_t S_{\text{B}} \simeq -\text{tr}\big[\dot{\hat{\rho}}_{\text{B}}(t) \ln e^{-\hat{H}_{\text{B}}/T}\big] = \frac{1}{T}\partial_t \langle \hat{H}_{\text{B}} \rangle \simeq -\frac{\dot{Q}}{T}. \tag{11}$$

Therefore, the changing rate of the total correlation entropy becomes $\partial_t \mathbb{C}_{\text{T}} = \partial_t[S_{\text{S}} + S_{\text{B}}] \simeq \dot{S}_{\text{S}} - \dot{Q}/T$, and this is just equivalent to the entropy production rate $dS - đQ/T$ in the standard thermodynamics [17–19]. In this sense, the irreversible entropy production in the standard thermodynamics also can be understood as the increase of the correlation between the system and the bath.

Another analogy with the standard thermodynamics is the Boltzmann entropy increase in an isolated classical gas composed of Nparticles. Since the full ensemble state $\rho(\vec{P}, \vec{Q})$ of the N-body system follows the Liouville theorem, its Gibbs entropy $S_{\text{G}}[\rho(\vec{P}, \vec{Q})]$ never changes [16,19]; on the other hand, because of the particle collisions, the distribution $\varrho(\mathbf{p}_n, \mathbf{x}_n)$ of each single particle approaches the Maxwell distribution as the steady state, with its entropy increasing monotonically (Boltzmann H theorem) [42]. Therefore, the increase of the total correlation entropy $\mathbb{C}_{\text{T}} = \sum_n S_{\text{G}}[\varrho(\mathbf{p}_n, \mathbf{x}_n)] - S_{\text{G}}[\rho(\vec{P}, \vec{Q})]$, which measures all the particle-particle correlations in the N-body system, just gives the irreversible entropy increase in the standard macroscopic thermodynamics [16,19,43,44].

In principle, the thermal entropy $đQ/T$ in the standard thermodynamics applies only for equilibrium states with a well-defined temperature T. When $T \to 0^+$, $đQ/T$ diverges and does not apply well here. Moreover, notice that the above dynamics of $\Phi_n^{(\text{N})}(\tau)$ [Eq. (3)] remains unchanged even if the initial populations in the BATH are reversed (setting $\frac{1}{2} < \bar{n}_T \leqslant 1$), which indicates the BATH has a negative temperature [42,45]. If all the TLSs in the BATH start from the excited state, effectively the BATH has a temperature $T \to 0^-$. In these situations, the application of the thermal entropy $đQ/T$ is questionable, while the total correlation entropy $\mathbb{C}_{\text{T}}(t)$ still applies and could be a meaningful generalization for the standard thermodynamics.

VI. SUMMARY

In this paper we study the local relaxation process of an open system in contact with a finite bath. We find that, due to the finite-size effect of the bath, the local relaxation of the open system exhibits hierarchy recurrences periodically, which makes the system dynamics appear more and more random. Essentially that is because the energy diffuses out of the open system and regathers back from the finite bath again and again. During the unitary evolution, the open system and the bath as a whole isolated system keeps a constant entropy, and the entropy of each single TLS increases and decreases from time to time, while the total correlation entropy approximately exhibits a monotonic increasing behavior, which is similar to the irreversible entropy increase in the standard thermodynamics [16–19]. We emphasize throughout the above discussions that there is no average on time or any random configurations. The quantum XX model here could be realized in many physical systems, such as optical lattices [46], superconducting circuits [47,48], and ion trap arrays [49,50].

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