# Uncertainties of genuinely incompatible triple measurements based on statistical distance

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We investigate the measurement uncertainties of a triple of positive-operator-valued measures based on statistical distance and formulate state-independent tight uncertainty inequalities satisfied by the three measurements in terms of triplewise joint measurability. In particular, uncertainty inequalities for three unbiased qubit measurements are presented with analytical lower bounds which relates to the necessary and sufficient condition of the triplewise joint measurability of the given triple. We show that the measurement uncertainties for a triple measurement are essentially different from the ones obtained by pairwise measurement uncertainties by comparing the lower bounds of different measurement uncertainties.

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## I. INTRODUCTION

The uncertainty principle is arguably one of the most famous features of quantum mechanics [1], which limits the accuracy of measuring some properties of a quantum system. The well-known Heisenberg-Robertson uncertainty relation says that [2], for any observables $A$ and $B$, $\Delta A\Delta B \geqslant \frac{1}{2}|\langle[A,B]\rangle|$, where $\Delta\Omega = \sqrt{\langle\Omega^2\rangle - \langle\Omega\rangle^2}$ is the standard deviation for observable $\Omega$, $\langle\cdot\rangle$ denotes the expectation of an operator with respect to a given state $\rho$, and $[A,B] = AB - BA$. This state-dependent inequality implies the impossibility of simultaneously determining the definite values of noncommuting observables. Such uncertainty relations based on the product form or summation form of deviation have been generalized and studied [3–11]. The entropic uncertainty relations [11–18] and measurement-probability-based universal uncertainty relations [19–25], with or without quantum memory, have been extensively investigated. In addition, uncertainty relations based on measurement noise and disturbance have been also derived and experimentally verified [26–29].

Since the influence of the measurement on quantum systems is not always the reason for uncertainty [30], there are uncertainty relations whose uncertainties are described by the approximation error for probabilities of joint measurements [23,24,31–35]. In [23,24] the approximation error for probabilities is quantified by the sum of relative entropies, while in [31–35] the corresponding approximation error for probabilities is quantified by $L_1$ distances. In addition, in [23,24] multispin-1/2-component measurement uncertainty relations were studied. In [31–35] two measurement uncertainty relations were investigated. Since a triple measurement uncertainty relation deduced from a two-observable uncertainty relation [4] is usually not tight, triple measurement uncertainty relations are essentially different from the ones obtained by pairwise measurement uncertainties: There exist genuinely incompatible triple measurements such that they are pairwise jointly measurable, just like the case of genuine tripartite entanglement or genuine nonlocal correlations.

In this paper, based on statistical distance, we formulate state-independent tight uncertainty relations satisfied by three measurements in terms of their triplewise joint measurability. By approximating a given triple of unbiased qubit measurements to all possible triple measurements that are triplewise jointly measurable, we show that the approximation error is lower bounded by a quantity which relates to the necessary and sufficient condition of the triplewise joint measurability of the given triple. We also compare the different uncertainty relations which are obtained by approximation of triplewise jointly measurable measurements and pairwise jointly measurable measurements, respectively. Examples are given to illustrate the merit our uncertainty relation.

## II. TRIPLE MEASUREMENT UNCERTAINTY RELATION

Consider three positive-operator-valued measures $\{M^i\}_{i=1}^3$, given by the semipositive measurement operators $\{M_k^i \mid M_k^i \geqslant 0, \sum_k M_k^i = \mathbb{I}\}$, $i = 1, 2, 3$, where $\mathbb{I}$ stands for the identity operator. Let $\{N_k^i \mid N_k^i \geqslant 0, \sum_k N_k^i = \mathbb{I}\}$, $i = 1, 2, 3$, be another set of three positive-operator-valued measures which are triplewise jointly measurable. For an arbitrary given state $\rho$, the measurement probabilities with respect to $M_k^i$ ($N_k^i$) are given by $p_k^{M^i} = \mathrm{Tr}(\rho M_k^i)$ [$p_k^{N^i} = \mathrm{Tr}(\rho N_k^i)$]. The approximation error between measurements $M^i$ and $N^i$ is given by $d_\rho(M^i;N^i) := \sum_k |p_k^{M^i} - p_k^{N^i}|$. By maximizing $d_\rho$ over all $\rho$, we obtain a state-independent approximation error, which is the worst case for all states, between the triple measurements

 

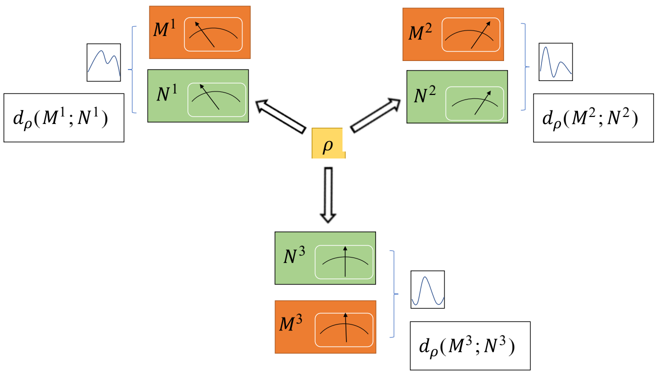


FIG. 1. Approximation of $\{M^i\}_{i=1}^3$ to triplewise jointly measurable measurements $\{N^i\}_{i=1}^3$.

$\{M^1, M^2, M^3\}$ and the triplewise jointly measurable measurements $\{N^1, N^2, N^3\}$, i.e.,

$$\Delta(M^1, M^2, M^3; N^1, N^2, N^3) := \max_\rho \sum_{i=1}^{3} d_\rho(M^i; N^i). \tag{1}$$

Let $\Delta_{lb}(M^1, M^2, M^3)$ denote the minimal value of $\Delta(M^1, M^2, M^3; N^1, N^2, N^3)$ over all possible triplewise jointly measurable triples $N^1$, $N^2$, and $N^3$. Then the quantity $\Delta_{lb}(M^1, M^2, M^3)$ quantifies the degree of incompatibility of the triple measurements $\{M^i\}_{i=1}^3$ (see Fig. 1). It is apparent that $\Delta_{lb}(M^1, M^2, M^3) = 0$ if and only if $M^1$, $M^2$, and $M^3$ are triplewise jointly measurable.

Consider now three unbiased qubit measurements $\{M^i\}_{i=1}^3$ described by positive-operator-valued measures

$$M^i_+ = \frac{I + \vec{m}_i \cdot \vec{\sigma}}{2}, \quad M^i_- = \frac{I - \vec{m}_i \cdot \vec{\sigma}}{2}, \quad i = 1, 2, 3,$$

where the three-dimensional vectors $\vec{m}_i$ satisfy $|\vec{m}_i| \leqslant 1$, $I$ is the $2 \times 2$ identity matrix, and $\vec{\sigma}$ is the vector with the Pauli matrix $\sigma_i$ as the $i$th entry. Let $\rho$ be a qubit state with Bloch vector representation $\rho = (I + \vec{r} \cdot \vec{\sigma})/2(|\vec{r}| \leqslant 1)$. Maximizing $\sum_{i=1}^3 d_\rho(M^i; N^i)$ over all $\rho$, we obtain

$$\Delta(M^1, M^2, M^3; N^1, N^2, N^3) = 2 \max_{\vec{r}} \sum_{i=1}^{3} |\vec{r} \cdot (\vec{m}_i - \vec{n}_i)|. \tag{2}$$

For simplicity, in the following we define $\vec{m}_{123} = \vec{m}_1 + \vec{m}_2 + \vec{m}_3$, $\vec{m}_{ij} = \vec{m}_i + \vec{m}_j$, $\vec{n}_{123} = \vec{n}_1 + \vec{n}_2 + \vec{n}_3$, and $\vec{n}_{ij} = \vec{n}_i + \vec{n}_j$. It has been demonstrated in [36] that three unbiased qubit measurements $\{N^i_\pm = (I \pm \vec{n}_i \cdot \vec{\sigma})/2\}_{i=1}^3$ are triplewise jointly measurable if and only if

$$\sum_{k=1}^{4} |\vec{p}_k - \vec{p}_F| \leqslant 4, \tag{3}$$

where $\vec{q}_1 = \vec{n}_{123}$, $\vec{q}_2 = \vec{n}_1 - \vec{n}_{23}$, $\vec{q}_3 = \vec{n}_2 - \vec{n}_{13}$, $\vec{q}_4 = \vec{n}_3 - \vec{n}_{12}$, and $\vec{q}_F$ is the Fermat-Torricelli point of $\{\vec{q}_k\}_{k=1}^4$ [37]. Minimizing $\Delta(M^1, M^2, M^3; N^1, N^2, N^3)$ under all triplewise jointly measurable measurements $\{N^i\}_{i=1}^3$ satisfying (3), we have the following theorem,

*Theorem.* The approximation error of three unbiased qubit measurements $\{M^i\}_{i=1}^3$ to triplewise jointly measurable unbiased qubit measurements $\{N^i\}_{i=1}^3$ satisfies the inequality

$$\Delta(M^1, M^2, M^3; N^1, N^2, N^3) \geqslant \tfrac{1}{2}\left(\sum_{k=1}^{4} |\vec{p}_F - \vec{p}_k| - 4\right), \tag{4}$$

where $\vec{p}_1 = \vec{m}_{123}$, $\vec{p}_2 = \vec{m}_1 - \vec{m}_{23}$, $\vec{p}_3 = \vec{m}_2 - \vec{m}_{13}$, $\vec{p}_4 = \vec{m}_3 - \vec{m}_{12}$, and $\vec{p}_F$ is the Fermat-Torricelli point of $\{\vec{p}_k\}_{k=1}^4$

*Proof.* By direct calculation we have the state-dependent approximation error

$$\sum_{i=1}^{3} d_\rho(M^i; N^i) = 2 \sum_{i=1}^{3} |\vec{r} \cdot (\vec{m}_i - \vec{n}_i)|$$
$$= 2 \times \begin{cases} |\vec{r} \cdot (\vec{m}_{123} - \vec{n}_{123})| \leqslant |\vec{m}_{123} - \vec{n}_{123}| := |\vec{g}_1| \\ \quad \text{if } [\vec{r} \cdot (\vec{m}_1 - \vec{n}_1)][\vec{r} \cdot (\vec{m}_2 - \vec{n}_2)] \geqslant 0 \wedge [\vec{r} \cdot (\vec{m}_{12} - \vec{n}_{12})][\vec{r} \cdot (\vec{m}_3 - \vec{n}_3)] \geqslant 0 \\ |\vec{r} \cdot (\vec{m}_{1-23} - \vec{n}_{1-23})| \leqslant |\vec{m}_{1-23} - \vec{n}_{1-23}| := |\vec{g}_2| \\ \quad \text{if } [\vec{r} \cdot (\vec{m}_1 - \vec{n}_1)][\vec{r} \cdot (\vec{m}_2 - \vec{n}_2)] \leqslant 0 \wedge [\vec{r} \cdot (\vec{m}_{1-2} - \vec{n}_{1-2})][\vec{r} \cdot (\vec{m}_3 - \vec{n}_3)] \leqslant 0 \\ |\vec{r} \cdot (\vec{m}_{2-13} - \vec{n}_{2-13})| \leqslant |\vec{m}_{2-13} - \vec{n}_{2-13}| := |\vec{g}_3| \\ \quad \text{if } [\vec{r} \cdot (\vec{m}_1 - \vec{n}_1)][\vec{r} \cdot (\vec{m}_2 - \vec{n}_2)] \leqslant 0 \wedge [\vec{r} \cdot (\vec{m}_{1-2} - \vec{n}_{1-2})]\vec{r} \cdot (\vec{m}_3 - \vec{n}_3)] \geqslant 0 \\ |\vec{r} \cdot (\vec{m}_{3-12} - \vec{n}_{3-12})| \leqslant |\vec{m}_{3-12} - \vec{n}_{3-12}| := |\vec{g}_4| \\ \quad \text{if } [\vec{r} \cdot (\vec{m}_1 - \vec{n}_1)][\vec{r} \cdot (\vec{m}_2 - \vec{n}_2)] \geqslant 0 \wedge [\vec{r} \cdot (\vec{m}_{1-2} - \vec{n}_{1-2})][\vec{r} \cdot (\vec{m}_3 - \vec{n}_3)] \leqslant 0. \end{cases} \tag{5}$$

We show that $\mathcal{G} := 2\max_i |\vec{g}_i|$, $i = 1, 2, 3, 4$, in (5) can be reached. Let $\rho_0$, with $\vec{r} = \vec{r}_0$, be the optimal state maximizing $\sum_{i=1}^3 d_\rho(M^i; N^i)$. Without loss of generality, we assume $\mathcal{G} = |\vec{g}_1| > 0$. Set $\vec{r}_0 = \vec{g}_1/|\vec{g}_1|$, we have

$$\begin{aligned} &[\vec{r}_0 \cdot (\vec{n}_1 - \vec{m}_1)][\vec{r}_0 \cdot (\vec{n}_2 - \vec{m}_2)] \\ &= \frac{1}{|\vec{g}_1|^2}[|\vec{n}_1 - \vec{m}_1|^2 + (\vec{n}_{23} - \vec{m}_{23}) \cdot (\vec{n}_1 - \vec{m}_1)] \\ &\times [|\vec{n}_2 - \vec{m}_2|^2 + (\vec{n}_{13} - \vec{m}_{13}) \cdot (\vec{n}_2 - \vec{m}_2)] \geqslant 0, \end{aligned} \tag{6}$$

where the inequality holds as $(\vec{n}_{23} - \vec{m}_{23}) \cdot (\vec{n}_1 - \vec{m}_1) \geqslant 0$ and $(\vec{n}_{13} - \vec{m}_{13}) \cdot (\vec{n}_2 - \vec{m}_2) \geqslant 0$, since $|\vec{g}_1| \geqslant |\vec{g}_2|$ and $|\vec{g}_1| \geqslant |\vec{g}_3|$. Similarly, from $|\vec{g}_1| \geqslant |\vec{g}_4|$ and $(\vec{n}_{12} - \vec{m}_{12}) \cdot (\vec{n}_3 - \vec{m}_3) \geqslant 0$ we have

$$\begin{aligned} &[\vec{r} \cdot (\vec{n}_{12} - \vec{m}_{12})][\vec{r} \cdot (\vec{n}_3 - \vec{m}_3)] \\ &= \frac{1}{|g_1|^2}[|\vec{n}_3 - \vec{m}_3|^2 + (\vec{n}_3 - \vec{m}_3) \cdot (\vec{n}_{12} - \vec{m}_{12})] \\ &\times [|\vec{n}_{12} - \vec{m}_{12}|^2 + (\vec{n}_{12} - \vec{m}_{12}) \cdot (\vec{n}_3 - \vec{m}_3)] \geqslant 0. \end{aligned} \tag{7}$$

Equations (6) and (7) are just the first constraints in (5). Therefore, altogether we have

$$\Delta(M^1, M^2, M^3; N^1, N^2, N^3) = 2\max\{|\vec{g}_1|, |\vec{g}_2|, |\vec{g}_3|, |\vec{g}_4|\} := 2\mathcal{G}. \tag{8}$$

Noting that $\vec{g}_i = \vec{p}_i - \vec{q}_i$ and $\sum_{k=1}^4 |\vec{q}_F - \vec{q}_k| \leqslant 4$, we have

$$\begin{aligned}\Delta(M^1, M^2, M^3; N^1, N^2, N^3) &= 2\mathcal{G}\\ &\geqslant \frac{1}{2}\sum_{k=1}^{4}|\vec{p}_k - \vec{q}_k| = \frac{1}{2}\sum_{k=1}^{4}|\vec{p}_k - \vec{q}_F + \vec{q}_F - \vec{q}_k|\\ &\geqslant \frac{1}{2}\sum_{k=1}^{4}[|\vec{p}_k - \vec{q}_F| - |\vec{q}_F - \vec{q}_k|]\\ &\geqslant \frac{1}{2}\left[\sum_{k=1}^{4}|\vec{p}_k - \vec{p}_F| - 4\right],\end{aligned} \tag{9}$$

where the second inequality is due to a triangle inequality and the third one comes from the definition of the Fermat-Torricelli point of $\{\vec{p}_k\}_{k=1}^4$ and the constraint of the triplewise joint measurability for $\{N^i\}_{i=1}^3$. ∎

Apparently, if the lower bound of (4) is zero, then $M^1$, $M^2$, and $M^3$ are triplewise jointly measurable. From the definition of $\Delta_{lb}(M^1, M^2, M^3)$ we then have $\Delta_{lb}(M^1, M^2, M^3) = 0 = \frac{1}{2}(\sum_{k=1}^4 |\vec{p}_F - \vec{p}_k| - 4)$. This means that the inequality (4) is tight in the sense that the minimal value of $\Delta(M^1, M^2, M^3; N^1, N^2, N^3)$ is achieved by the lower bound. In this case the degree of incompatibility of the given triple measurement is 0. In the following we call a triple measurement $\{M^1, M^2, M^3\}$ a genuinely incompatible triple measurement if the lower bound of (4) is strictly greater than zero.

Let us consider three sharp unbiased qubit measurements associated with the Pauli operators $\sigma_i$, $i = 1, 2, 3$. Set $\vec{m}_1 = (1, 0, 0)$, $\vec{m}_2 = (0, 1, 0)$, and $\vec{m}_3 = (0, 0, 1)$. Then the three positive-operator-valued measures $M^1$, $M^2$, and $M^3$ are just the projective measurements with respect to the eigenvectors of the three Pauli matrices, respectively. We have $\vec{p}_1 = (1, 1, 1)$, $\vec{p}_2 = (1, -1, -1)$, $\vec{p}_3 = (-1, 1, -1)$, and $\vec{p}_4 = (-1, -1, 1)$, which constitute a regular tetrahedron. In addition, the Fermat-Torricelli point is exactly the origin, $\vec{p}_F = 0$. One can verify that the optimal approximation of triplewise jointly measurable $\{N^i\}_{i=1}^3$ is given by $\vec{n}_i = \frac{1}{\sqrt{3}}\vec{m}_i$, as shown in Fig. 2.

The minimal value of $\Delta(M^1, M^2, M^3; N^1, N^2, N^3)$ is actually the lower bound of (4), i.e.,

$$\Delta_{lb}(M^1, M^2, M^3) = \frac{1}{2}\left(\sum_{k=1}^{4}|\vec{p}_k| - 4\right) = 2\sqrt{3} - 2. \tag{10}$$

Therefore, the uncertainty inequality (4) is tight not only in trivial case but also in this case. Thus the triple measurement $\{M^1, M^2, M^3\}$ is a genuinely incompatible triple measurement and its degree of incompatibility is $2\sqrt{3} - 2$.

## III. UNCERTAINTY: TRIPLEWISE VERSUS PAIRWISE JOINT MEASUREMENT APPROXIMATION

We next investigate the difference between measurement uncertainty relations which are obtained by minimizing $\Delta(M^1, M^2, M^3; N^1, N^2, N^3)$ over pairwise and triplewise jointly measurable measurements. In [32–34] this kind of Heisenberg error-disturbance relation for a pair of measurements was studied. For a given pair of measurements $M^1$ and $M^2$, their approximation to a pair of jointly measurable measurements $N^1$ and $N^2$, $\Delta(M^1, M^2; N^1, N^2) := \max_\rho \sum_{i=1}^2 d_\rho(M^i; N^i)$, satisfies the relation [35]

$$\Delta(M^1, M^2; N^1, N^2) \geqslant |\vec{m}_1 + \vec{m}_2| + |\vec{m}_1 - \vec{m}_2| - 2. \tag{11}$$

From (11) one may also derive a measurement uncertainty relation which is obtained by minimizing $\Delta(M^1, M^2, M^3; N^1, N^2, N^3)$ over pairwise jointly measurable measurements

$$\begin{aligned}\Delta(M^1, M^2, M^3; N^1, N^2, N^3) &= \frac{1}{2}\sum_{i<j}^{3}\Delta(M^i, M^j; N^i, N^j)\\ &\geqslant \frac{1}{2}\left[\sum_{i<j}^{3}(|\vec{m}_i + \vec{m}_j| + |\vec{m}_i - \vec{m}_j| - 2)\right].\end{aligned} \tag{12}$$

Nevertheless, compared with the lower bound of (12), the lower bound of (4) captures better the incompatible measurement uncertainty of the triple measurements $M^1$, $M^2$, and $M^3$. Consider the case that one pair of measurements $\{M^i, M^j\}$ is jointly measurable. From the fact that

$$\begin{aligned}\sum_{k=1}^{4}|\vec{p}_F - \vec{p}_k| &\geqslant \max_{i\neq j\neq k\neq l\in\{1,2,3,4\}}(|\vec{p}_i - \vec{p}_j| + |\vec{p}_k - \vec{p}_l|)\\ &\geqslant 2\max_{i\neq j}(|\vec{m}_i + \vec{m}_j| + |\vec{m}_i - \vec{m}_j|),\end{aligned} \tag{13}$$

one easily gets that the lower bound of (4) is greater than or equal to the lower bound of (12). As an example that all pairs of measurements are not jointly measurable, we consider the measurements with respect to three Pauli operators. By direct calculation we have $\mathcal{L}_1 = 2\sqrt{3} - 2 > \mathcal{L}_2 = 3\sqrt{2} - 3$, where $\mathcal{L}_1$ and $\mathcal{L}_2$ are the lower bounds of the inequalities (4) and (12), respectively. Therefore, the uncertainties from a triple of measurements are essentially different from the ones from pairwise measurements.

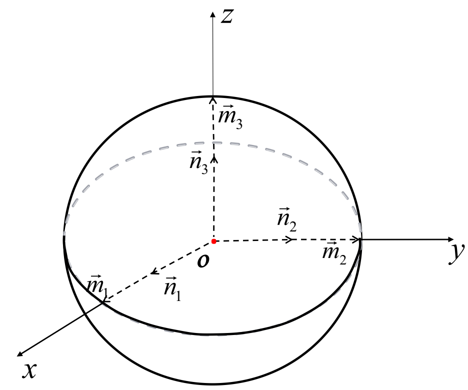


FIG. 2. Optimal approximation of $\{M^i = \sigma_i\}_{i=1}^3$ by triplewise jointly measurable $\{N^i\}_{i=1}^3$ given by $\vec{n}_i = \frac{1}{\sqrt{3}}\vec{m}_i$.

From (13) one can also analytically show that there exist triple measurements that are genuinely incompatible but pairwise jointly measurable. In particular, for three measurements $\{M_{\pm}^i = (I \pm \vec{m}_i \cdot \vec{\sigma})/2\}_{i=1}^3$, with $\vec{m}_1 = (1, 0, 0)/\sqrt{2}$, $\vec{m}_2 = (0, 1, 0)/\sqrt{2}$, and $\vec{m}_3 = (0, 0, 1)/\sqrt{2}$, which were proved to be pairwise jointly measurable, one verifies easily that the pairwise lower bounds of (11) are all zero. However, the lower bound of (4) is $\sqrt{6} - 2 > 0$.

Actually, in [23,24] Barchielli *et al.* obtained an approximation-error-based triple measurement uncertainty relation where the approximation error for probabilities of joint measurements is quantified by the sum of relative entropies. Similar to $\Delta_{lb}(M^1, M^2, M^3)$, a quantity $C_{\rm inc}(M^1, M^2, M^3)$ was introduced in [23], although it is difficult to calculate the universal and analytical lower bound of $C_{\rm inc}(M^1, M^2, M^3)$. In [24] a lower bound of $C_{\rm inc}(M^1, M^2, M^3)$ has been derived for the case of three incompatible spin-1/2 components, which is not directly related to the necessary and sufficient condition of the triplewise joint measurability of the three incompatible spin-1/2 components.

## IV. DISCUSSION AND CONCLUSION

Our approach may be generalized to the case of multiple measurements. For $n$ measurements, one has $\Delta(M^1, \ldots, M^n; N^1, \ldots, N^n) \geqslant \Delta_{lb}(M^1, \ldots, M^n)$. However, for multiple measurements the general necessary and sufficient jointly measurable conditions are still not known even for unbiased qubit measurements. Let us consider the multiplewise joint measurability for arbitrary $n$ ($n \geqslant 4$) unbiased qubit measurements. We have that the $n$ unbiased qubit measurements $\{N^i = (I \pm \vec{n}_i \cdot \vec{\sigma})/2\}_{i=1}^n$ are $n$-tuplewise jointly measurable if

$$\sum_{\mu_i=\pm 1} \left| \sum_{i=1}^n \mu_i \vec{n}_i \right| \leqslant 2^n \tag{14}$$

(see the proof in the Appendix).

Nevertheless, (14) is not both sufficient and necessary in general. Only for some special $n$ unbiased qubit measurements $M^i$ one may have the following relation from (14):

$$\Delta(M^1, \ldots, M^n; N^1, \ldots, N^n) \geqslant \left( \sum_{\mu_i=\pm 1} \left| \sum_{i=1}^n \mu_i \vec{m}_i \right| - 2^n \right) / 2^{n-2}.$$

Similar to the triple case, there would exist genuinely incompatible $n$-tuple measurements.

By approximating a given triple of unbiased qubit measurements to all possible triple measurements that are triplewise jointly measurable, we have formulated state-independent tight uncertainty inequalities satisfied by the triple of qubit measurements, with the lower bound given by the necessary and sufficient condition of the triplewise joint measurability of the given triple. These uncertainty relations can be experimentally tested, like the case of two-qubit measurements [35]. As the measurement uncertainties from a triple of measurements are essentially different from the ones from pairwise measurements, it is of significance to explore the measurement uncertainties for triple or $n$-tuple measurements by their measurement incompatibilities.

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## APPENDIX: PROOF OF THE SUFFICIENT CONDITION (14) FOR *n*-TUPLEWISE JOINT MEASURABILITY

Consider $n$ unbiased qubit measurements $\{\frac{I+\mu_i \vec{m}_i \cdot \vec{\sigma}}{2}\}_{i=1}^n$ with $\mu_i = \pm 1$. The general measurement with measurement operators $O_{\mu_1\mu_2\cdots\mu_n}$ including $\{\frac{I+\mu_i \vec{m}_i \cdot \vec{\sigma}}{2}\}_{i=1}^n$ as the marginal ones is given by

$$O_{\mu_1\mu_2\cdots\mu_n} = \frac{1}{2^n}\left\{\left[1 + \sum_{i=2}^n \sum_{\substack{j_1, j_2, \ldots, j_i \in \mathcal{I} \\ j_1 < j_2 < \cdots < j_i}} \left(\prod_{l=1}^i \mu_{j_l}\right) a^i_{j_1 j_2 \cdots j_i}\right] I + \left[\sum_{i=2}^n \sum_{\substack{j_1, j_2, \ldots, j_i \in \mathcal{I} \\ j_1 < j_2 < \cdots < j_i}} \left(\prod_{l=1}^i \mu_{j_l}\right) \vec{Z}^i_{j_1 j_2 \cdots j_i} + \sum_{i=1}^n \mu_i \vec{m}_i\right] \cdot \vec{\sigma}\right\}, \tag{A1}$$

where $a^i_{j_1 j_2 \cdots j_i}$ and $\vec{Z}^i_{j_1 j_2 \cdots j_i}$ are arbitrary parameters and vectors for all $i = 1, 2, \ldots, n$ and $\mathcal{I} = \{1, 2, \ldots, n\}$. The positivity of the operators $\{O_{\mu_1\mu_2\cdots\mu_n}\}$ implies that

$$\left| \sum_{i=2}^{n} \sum_{\substack{j_1, j_2, \ldots, j_i \in \mathcal{I} \\ j_1 < j_2 < \cdots < j_i}} \left(\prod_{l=1}^{i} \mu_{j_l}\right) \vec{Z}^i_{j_1 j_2 \cdots j_i} + \sum_{i=1}^{n} \mu_i \vec{m}_i \right| \leqslant 1 + \sum_{i=2}^{n} \sum_{\substack{j_1, j_2, \ldots, j_i \in \mathcal{I} \\ j_1 < j_2 < \cdots < j_i}} \left(\prod_{l=1}^{i} \mu_{j_l}\right) a^i_{j_1 j_2 \cdots j_i}. \tag{A2}$$

We divide the above $2^n$ inequalities into $2^{n-1}$ pairs such that in each pair the two inequalities take the opposite sign for all $\mu_i$. From each pair of such inequalities we have the inequality

$$\left| \sum_{i=2, i=2t+1}^{n} \sum_{\substack{j_1, j_2, \ldots, j_i \in \mathcal{I} \\ j_1 < j_2 < \cdots < j_i}} \left(\prod_{l=1}^{i} \mu_{j_l}\right) \vec{Z}^i_{j_1 j_2 \cdots j_i} + \sum_{i=1}^{n} \mu_i \vec{m}_i \right| \leqslant 1 + \sum_{i=2, i=2t}^{n} \sum_{\substack{j_1, j_2, \ldots, j_i \in \mathcal{I} \\ j_1 < j_2 < \cdots < j_i}} \left(\prod_{l=1}^{i} \mu_{j_l}\right) a^i_{j_1 j_2 \cdots j_i}. \tag{A3}$$

Summing up all these inequalities in (A3), we obtain

$$\sum_{\mu_i = \pm 1} \left| \sum_{i=2, i=2t+1}^{n} \sum_{\substack{j_1, j_2, \ldots, j_i \in \mathcal{I} \\ j_1 < j_2 < \cdots < j_i}} \left(\prod_{l=1}^{i} \mu_{j_l}\right) \vec{Z}^i_{j_1 j_2 \cdots j_i} + \sum_{i=1}^{n} \mu_i \vec{m}_i \right| \leqslant 2^n. \tag{A4}$$

Therefore, $n$ measurements $\{\frac{I \pm \vec{m}_i \cdot \vec{\sigma}}{2}\}_{i=1}^n$ are $n$-tuplewise jointly measurable if

$$\min_{\vec{Z}^i_{j_1 j_2 \cdots j_i}} \sum_{\mu_i = \pm 1} \left| \sum_{i=2, i=2t+1}^{n} \sum_{\substack{j_1, j_2, \ldots, j_i \in \mathcal{I} \\ j_1 < j_2 < \cdots < j_i}} \left(\prod_{l=1}^{i} \mu_{j_l}\right) \vec{Z}^i_{j_1 j_2 \cdots j_i} + \sum_{i=1}^{n} \mu_i \vec{m}_i \right| \leqslant 2^n. \tag{A5}$$

In particular, setting $\vec{Z}^i_{j_1 j_2 \cdots j_i} = 0$, the inequality (A5) reduces to $\sum_{\mu_i=\pm 1} |\sum_{i=1}^{n} \mu_i \vec{m}_i| \leqslant 2^n$, which ensures the $n$-tuplewise jointly measurability of $\{\frac{I \pm \vec{m}_i \cdot \vec{\sigma}}{2}\}_{i=1}^n$. ■


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