Microwave degenerate parametric down-conversion with a single cyclic three-level system in a circuit-QED setup

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With the assistance of a cyclic three-level artificial atom of a superconducting flux quantum circuit interacting with a two-mode superconducting transmission-line resonator, we study theoretically the degenerate microwave parametric down-conversion (PDC) in a circuit-QED system. By adiabatically eliminating the excited states of the three-level artificial atom, we obtain an effective microwave PDC Hamiltonian for the two resonator modes (i.e., the fundamental and second-harmonic modes). The corresponding PDC efficiency in our model can be much larger than that in the similar circuit-QED system based on a single two-level superconducting qubit [K. Moon and S. M. Girvin, Phys. Rev. Lett. **95**, 140504 (2005)]. Furthermore, we investigate the squeezing and bunching behavior of the fundamental resonator mode resulting from the coherent driving to the second-harmonic one.

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I. INTRODUCTION

Photonic parametric down-conversion (PDC) is the coherent generation of a pair of photons with lower frequency via injecting a higher-frequency photonic field into a nonlinear medium [1] through a three-wave mixing process. The PDC together with three-wave mixing in an atomic medium has been widely studied both theoretically [2–5] and experimentally [6–8]. For instance, the PDC process can be used to generate a squeezed state [9,10] in which the fluctuation of one quadrature is suppressed while the fluctuation of the other one is increased and can be used for precise measurement [11].

In the early days, the degenerate [2] and nondegenerate [3] PDC processes were studied in a cyclic three-level atomic system in which any two of the three levels can be coupled via electric dipole transition. In general, the cyclic three-level structure does not exist in natural atoms due to the rules of electric dipole transitions. The key point to form a cyclic atomic structure in Refs. [2,3] is to use a sufficiently strong external field to break the symmetry of the atomic system.

Recently, it has been found that such an electrical-dipole-transition-based cyclic three-level (also called Δ type) structure can be formed in the system of a superconducting flux quantum circuit (SFQC) [12] by adjusting the bias magnetic flux threaded through the loop composed of three Josephson junctions. In addition, the cyclic three-level structures also exist in chiral molecular systems [13,14] and are used to separate the chiral molecules with different chiralities using the generalized Stern-Gerlach effect [15]. Based on the cyclic transitions, it is convenient to use cyclic three-level systems to generate single microwave photons [16,17], to produce microwave amplification without population inversion [18,19], and to serve as a single-photon quantum router [20].

On the other hand, there has been great progress in simulating a quantum optics phenomenon in a circuit-QED system [21–23], in which the superconducting qubit (e.g., charge, phase, or flux qubit) or qutrit serves as a two-level or three-level "artificial atom" interacting with the microwave superconducting transmission-line resonator. In the circuit-QED system, the energy structure of the "atom" can easily be controlled by tuning the external conditions such as currents, voltages, and electromagnetic fields. The strong couplings between the artificial atoms and superconducting resonators have also been realized experimentally [24–26]. Recently, researchers have proposed realizing the PDC process and generating a squeezing state in the circuit-QED system [27–33].

Based on the above achievements, we consider in this paper the microwave degenerate PDC and generation of the squeezed state in a circuit-QED system consisting of a cyclic three-level artificial atom of SFQC and a two-mode transmission-line resonator. In the case where the detunings between the three-level artificial atom and the resonator modes are much larger than their coupling strengths, we can eliminate adiabatically the degrees of freedom of the artificial atom and derive the effective coupling between the two modes in the resonator by Frölich-Nakajima transformation [34–38] (also called Schrieffer-Wolff transformation [39,40]). The effective Hamiltonian has a form similar to that of the degenerate parametric oscillator [41–43], which provides the PDC process and the generation of the squeezed field.

In our proposal, the efficiency of the microwave PDC is inversely proportional to the detuning between the fundamental mode and the three-level artificial atom. It is much larger than that in the system of a superconducting two-level qubit interacting with the superconducting transmission-line resonator, in which the efficiency is inversely proportional to the frequency of the qubit's transition [27]. We further discuss the squeezing and bunching behavior of the fundamental mode by means of the mean-field approach [44,45] when the higher-frequency second-harmonic mode is resonantly driven.

The rest of this paper is organized as follows. In Sec. II, we illustrate our model and derive the effective PDC Hamiltonian by adiabatically eliminating the degrees of freedom of the cyclic three-level system. In Sec. III, we investigate the squeezing and bunching behavior of the fundamental mode in a superconducting transmission-line resonator by means

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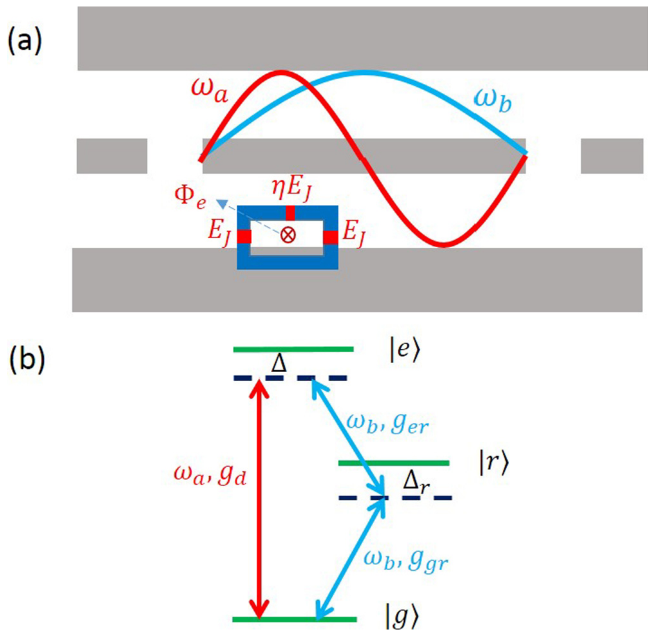

FIG. 1. (Color online) (a) Schematic diagram of the circuit-QED setup consisting of a superconducting qutrit and a two-mode microwave superconducting transmission-line resonator. (b) The cyclic three-level structure of the qutrit interacting with the two modes, e.g., the fundamental and second-harmonic modes, of the microwave resonator.

of the Langevin equations. In Sec. IV, we give some brief conclusions.

II. THE MODEL AND PARAMETRIC DOWN-CONVERSION

As shown in Fig. 1(a), we consider a circuit-QED system with a SFQC interacting with a two-mode superconducting transmission-line resonator. The SFQC is composed of a superconducting loop with three Josephson junctions. Two of the junctions have equal Josephson energies E_J and capacitances C_J, while the third one has ηE_J and ηC_J ($1/2 < \eta < 1$). The loop is threaded by an external magnetic flux Φ_e. The Hamiltonian is written as [12,46]

$$H_q = \frac{P_p^2}{2M_p} + \frac{P_m^2}{2M_m} + 2E_J(1 - \cos\varphi_p \cos\varphi_m) + \eta E_J[1 - \cos(2\pi f + 2\varphi_m)], \tag{1}$$

where $P_j = -i\partial/\partial\varphi_j$ for $j = m,p$, $M_p = 2C_J(\Phi_0/2\pi)^2$, and $M_m = M_p(1+2\eta)$, with $\Phi_0 = \pi\hbar/e$ being the flux quanta, $\hbar$ being the reduced Plank constant, e being the electronic charge. Here $\varphi_p = (\varphi_1 + \varphi_2)/2, \varphi_m = (\varphi_1 - \varphi_2)/2$, with φ_1,φ_2 being the phase drops across the two lager junctions and $f = \Phi_e/\Phi_0$ being the reduced magnetic flux. Here we just focus on the transitions among the lowest three energy levels, whose related eigenenergies and wave functions can be obtained numerically [12].

When $f = 0.5$, the potential and the energy eigenstates of the qutrit (that is, the three-level superconducting artificial atom) have fixed parities so that the electrical-dipole transitions between the states with same parity are forbidden. In this case, the three states of the qutrit can just form a cascade three-level structure. However, when $f \neq 0.5$, the symmetry of the potential is broken, and the transitions between arbitrary two states are permissible. Thus the qutrit can form a cyclic Δ-type energy-level configuration [12]. Such a symmetry-breaking phenomenon which allows for the coexistence of one- and two-photon processes has been observed experimentally in the superconducting quantum flux qubit circuit [47,48].

We consider the case where two of the modes (e.g., the fundamental and second-harmonic modes) in the superconducting transmission-line resonator are involved. For such a circuit-QED setup, the Hamiltonian is written as $H = H_0 + H_I$ (hereafter, we set $\hbar = 1$), where

$$H_0 = \omega_g|g\rangle\langle g| + \omega_r|r\rangle\langle r| + \omega_e|e\rangle\langle e| + \omega_a a^\dagger a + \omega_b b^\dagger b \tag{2}$$

is the free Hamiltonian of the qutrit and the microwave modes in the resonator. Here $b^\dagger$ ($a^\dagger$) is the creation operator for the fundamental (second-harmonic) mode with eigenfrequency ω_b ($\omega_a \equiv 2\omega_b$). ω_g, ω_r, ω_e are, respectively, the eigenfrequencies of the ground state $|g\rangle$, the first excited state $|r\rangle$, and the second excited state $|e\rangle$. As a reference, we will set $\omega_g = 0$ in what follows.

The interactions between the resonator modes and the qutrit are described by

$$H_I = g_{\rm eg}a|e\rangle\langle g| + g_{\rm rg}b|r\rangle\langle g| + g_{\rm er}b|e\rangle\langle r| + \text{H.c.}, \tag{3}$$

where g_{ij} $(i,j = g,e,r)$ are the complex coupling strengths with the amplitudes [12,49]

$$|g_{\rm eg}| = \frac{2\pi\eta E_J}{\Phi_0}\sqrt{\frac{\omega_a}{C}}\left|\cos\left(\frac{\pi x}{L}\right)\langle e|\sin(2\pi f + 2\varphi_m)|g\rangle\right|, \tag{4a}$$

$$|g_{\rm er}| = \frac{2\pi\eta E_J}{\Phi_0}\sqrt{\frac{\omega_b}{C}}\left|\cos\left(\frac{2\pi x}{L}\right)\langle e|\sin(2\pi f + 2\varphi_m)|r\rangle\right|, \tag{4b}$$

$$|g_{\rm rg}| = \frac{2\pi\eta E_J}{\Phi_0}\sqrt{\frac{\omega_b}{C}}\left|\cos\left(\frac{2\pi x}{L}\right)\langle r|\sin(2\pi f + 2\varphi_m)|g\rangle\right|. \tag{4c}$$

Here C and L are, respectively, the capacitance and the length of the transmission-line resonator, and x is the position of the qutrit in the transmission-line resonator.

In Eq. (3), we have used the rotating-wave approximation, which requires the coupling strengths and the detunings be much smaller than the transition frequencies of the qutrit and eigenfrequency of the resonator modes: $\{|g_{\rm eg}|,|g_{\rm rg}|,|g_{\rm er}|\} \ll \{\omega_a,\omega_b,\omega_r,\omega_e\}$ and $\{|\Delta|,|\Delta_r|\} \ll \{\omega_a,\omega_b,\omega_r,\omega_e\}$. Here $\Delta := \omega_e - \omega_a = \omega_e - 2\omega_b$ and $\Delta_r := \omega_r - \omega_b$ are the related detunings.

In order to obtain the effective coupling between the two resonator modes, we should eliminate the degrees of freedom of the qutrit. Here we adopt the Frölich-Nakajima transformation, which is a canonical transformation widely used in condensed-matter physics [34] and quantum optics [35–38], to eliminate the variables of the qutrit.

In the regime of large detunings

$$|\Delta| \gg |g_{\text{eg}}|, |\Delta_r| \gg |g_{\text{rg}}|, |\Delta - \Delta_r| \gg |g_{\text{er}}|, \tag{5}$$

the effective coupling between the two modes in the resonator can be obtained by introducing a unitary transformation $\mathcal{H} = \exp(-\lambda S)H\exp(\lambda S)$, where S is an anti-Hermitian operator. According to the Baker-Hausdorff formula, one has

$$\begin{aligned}\mathcal{H} &= \exp(-\lambda S)(H_0 + \lambda H_I)\exp(\lambda S)\\ &= H_0 + \lambda H_I + \lambda[H_0,S] + \lambda^2[H_I,S] + \frac{\lambda^2}{2}[S,[S,H_0]]\\ &\quad + \frac{\lambda^3}{2}[S,[S,H_I]] - \frac{\lambda^3}{3!}[S,[S,[S,H_0]]] + O(\lambda^4),\end{aligned} \tag{6}$$

where λ is introduced to stress that the interaction Hamiltonian can be treated as perturbation compared with the free Hamiltonian and should be set to 1 after the calculations. Here we choose

$$S = \frac{g_{\text{eg}}^*}{\Delta}a^\dagger|g\rangle\langle e| + \frac{g_{\text{er}}^*}{\Delta - \Delta_r}b^\dagger|r\rangle\langle e| + \frac{g_{\text{rg}}^*}{\Delta_r}b^\dagger|g\rangle\langle r| - \text{H.c.}, \tag{7}$$

so that it satisfies $H_I + [H_0,S] = 0$. Note that in Eq. (6) we keep to the third order (instead of to the usual second order) in the expansion to obtain the desired PDC process of three-wave mixing.

Since we focus on the situation of large detunings, the qutrit populated in the initial ground state $|g\rangle$ will remain mostly in the ground state. Neglecting the high-frequency terms and the virtual photon-induced modification to the energy of the qutrit's excited states, one can obtain the effective Hamiltonian in the form of

$$\mathcal{H} = H_{\text{eff}} \otimes |g\rangle\langle g|. \tag{8}$$

Here, up to the third order of the interaction, the effective Hamiltonian for the two resonator modes is given as

$$\begin{aligned}H_{\text{eff}} &= \langle g|\mathcal{H}|g\rangle\\ &\approx \langle g|\left(H_0 + \frac{1}{2}[H_I,S] + \frac{1}{3}[[H_I,S],S]\right)|g\rangle\\ &= \left(\omega_a - \frac{|g_{\text{eg}}|^2}{\Delta}\right)a^\dagger a + \left(\omega_b - \frac{|g_{\text{rg}}|^2}{\Delta_r}\right)b^\dagger b\\ &\quad + \left(\frac{g_{\text{eg}}^* g_{\text{er}} g_{\text{rg}}}{\Delta_r\Delta}a^\dagger b^2 + \text{H.c.}\right),\end{aligned} \tag{9}$$

where $-|g_{\text{eg}}|^2/\Delta$ and $-|g_{\text{rg}}|^2/\Delta_r$ are the frequency shifts of the second-harmonic and fundamental modes due to their largely detuned couplings to the qutrit, respectively. Usually, these shifts are negligibly small compared with the corresponding resonant frequencies.

In the case of [50]

$$\omega_a - \frac{|g_{\text{eg}}|^2}{\Delta} = 2\left(\omega_b - \frac{|g_{\text{rg}}|^2}{\Delta_r}\right) =: \omega_{\text{eff}} \approx \omega_a, \tag{10}$$

the effective Hamiltonian becomes

$$H_{\text{eff}} = \omega_{\text{eff}}a^\dagger a + \frac{\omega_{\text{eff}}}{2}b^\dagger b + \frac{\chi}{2}(e^{i\varphi}a^\dagger b^2 + e^{-i\varphi}ab^{\dagger 2}), \tag{11}$$

where

$$\chi = \frac{2|g_{\text{eg}}^* g_{\text{er}} g_{\text{rg}}|}{\Delta_r\Delta} \tag{12}$$

is the effective coupling strength between the two modes in the resonator and φ is the global phase contributed from the three couplings between the resonator and the qutrit. Note that here the individual phases of the couplings g_{ij} $(i,j = e,g,r)$ do not matter. In the PDC process implied by Eqs. (8) and (11), the three-level system evolves along $|g\rangle \to |e\rangle \to |r\rangle \to |g\rangle$ by annihilating one photon of the a mode and creating two photons of the b mode.

Note that the condition for the Frölich-Nakajima transformation in our model has been given in Eq. (5). Strictly speaking, when the average photon numbers of the resonator modes are larger than 1, e.g., $n_{a,b} > 1$, the effect of large $n_{a,b}$ should be taken into consideration, and we should modify the condition for the Frölich-Nakajima transformation as

$$n_a \ll \Delta^2/|g_{\text{eg}}|^2, \quad n_b \ll \{(\Delta - \Delta_r)^2/|g_{\text{er}}|^2, \Delta_r^2/|g_{\text{rg}}|^2\}, \tag{13}$$

which is similar to that in Ref. [37].

Governed by the effective Hamiltonian (11), the time evolution of the system can be obtained analytically when the dissipation is neglected. Assuming that the system is initially prepared in the state $|\psi(0)\rangle = |g;1;0\rangle \equiv |g\rangle_q \otimes |1\rangle_a \otimes |0\rangle_b$, which means that the qutrit is in its ground state, and the second-harmonic (fundamental) mode in the resonator is in its Fock state $|1\rangle$ ($|0\rangle$), the evolved state is expressed as

$$\begin{aligned}|\psi(t)\rangle = e^{-i\omega_{\text{eff}}t}\Bigg(&\cos\frac{\sqrt{2}\chi t}{2}|g;1;0\rangle\\ &- ie^{-i\varphi}\sin\frac{\sqrt{2}\chi t}{2}|g;0;2\rangle\Bigg).\end{aligned} \tag{14}$$

In what follows, we will choose the parameters in the same order as those in Ref. [27]: $\omega_a/2\pi = 2\omega_b/2\pi = 5.5$ GHz, $\Delta = 2\Delta_r = \omega_a/10$, and $|g_{\text{eg}}|/2\pi = 20$ MHz, $|g_{\text{rg}}|/2\pi = |g_{\text{er}}|/2\pi = 10$ MHz. Under these parameters, the conditions given by Eq. (5) are fulfilled, and the effective coupling strength is $\chi/2\pi \approx 26$ kHz; the effective frequency of the second-harmonic mode is obtained as $\omega_{\text{eff}}/2\pi \approx 5.5$ GHz.

In order to verify the validity of the adiabatic elimination with the Frölich-Nakajima transformation, we illustrate the time evolution of the system in Fig. 2. In Fig. 2(a), we plot the probability of the qutrit remaining in the initial ground state $|g\rangle$ during the time evolution governed by the Hamiltonians (2) and (3) without considering the dissipation of the resonator modes and the qutrit. The fact that the probability even surpasses 0.99 means it is reasonable to assume that the qutrit is always populated in the ground state. Moreover, it shows obvious Rabi oscillation between states $|g;1;0\rangle$ and $|g;0;2\rangle$ in Fig. 2(b), where we plot the average photon numbers as functions of the evolution time t. We also observe from Fig. 2(b) that our results based on Eq. (14) (represented by the dashed and solid lines) coincide with the direct numerical results (represented by the empty rectangles and circles) based on the original Hamiltonian in Eqs. (2) and (3) very well. This further confirms the validity of the approach of adiabatic elimination we used here.

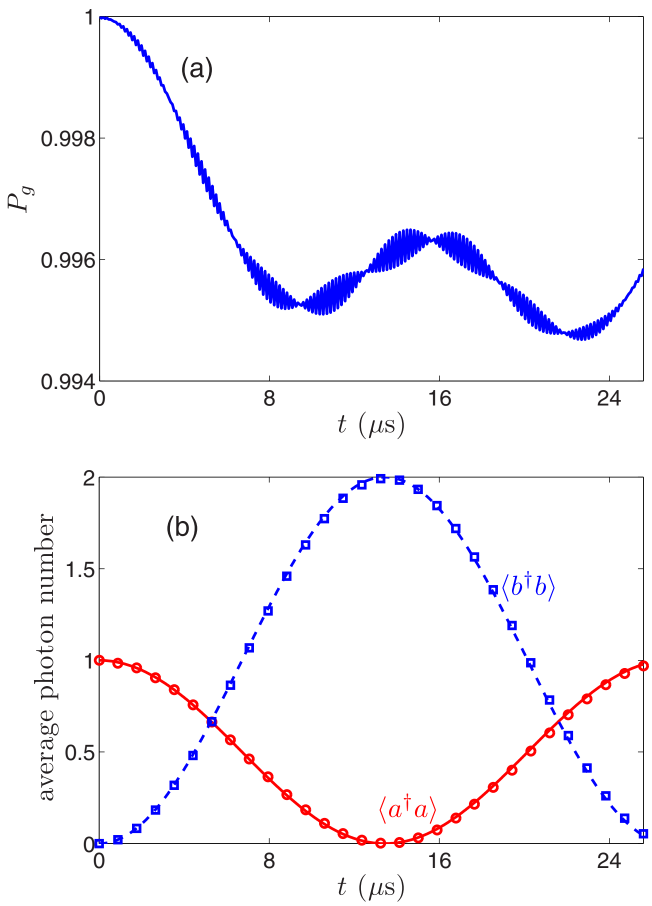

FIG. 2. (Color online) (a) The probability of the qutrit in the ground state. (b) The average photon numbers of the second-harmonic mode (red solid line) and fundamental mode (blue dashed line) in the resonator. The empty circles and rectangles are the corresponding numerical results. The parameters are set as $\omega_a/2\pi = 2\omega_b/2\pi = 5.5$ GHz, $\Delta = 2\Delta_r = \omega_a/10$, and $|g_{\rm eg}|/2\pi = 20$ MHz, $|g_{\rm gr}|/2\pi = |g_{\rm er}|/2\pi = 10$ MHz. Under these parameters, the effective coupling strength is $\chi/2\pi \approx 26$ kHz, and the effective frequency of the second-harmonic mode is $\omega_{\rm eff}/2\pi \approx 5.5$ GHz. We assume the system is prepared in the state $|\psi(0)\rangle = |g;1;0\rangle$ initially.

The effective Hamiltonian (11) demonstrates the degenerate PDC mechanism via nonlinear three-wave mixing. Actually, a similar process has also been investigated in the system of a two-level superconducting qubit interacting with a two-mode superconducting transmission-line resonator in Ref. [27]. The difference is that the PDC efficiency χ is inversely proportional to Δ_r in our result, instead of ω_a in Ref. [27]. In experiment, ω_a is always of the order of gigahertz [21,51], and Δ_r can be taken to be about 100 MHz, so the PDC efficiency can be enlarged by one or nearly two orders in our scheme.

III. SQUEEZING AND PHOTONIC CORRELATION

Now, we will study the optical character of the effective PDC system in the steady state in the presence of the optical driving and dissipation simultaneously. Here we focus on the situation where the second-harmonic mode is resonantly driven by an external field. The action of the driving field is described by

$$H_{\rm drive} = i\epsilon(a^\dagger e^{-i\omega_{\rm eff}t} - a e^{i\omega_{\rm eff}t}), \tag{15}$$

where ϵ is the strength of the driving field and is assumed to be positive.

In the rotating frame with respect to $U(t) = \exp[i\omega_{\rm eff}(a^\dagger a + b^\dagger b/2)t]$, the Hamiltonian of the system under the driving field becomes

$$\begin{aligned}\tilde{H} &= U(t)(H_{\rm eff} + H_{\rm drive})U^\dagger(t) + i\frac{\partial U(t)}{\partial t}U^\dagger(t)\\ &= i\epsilon(a^\dagger - a) + \frac{i\chi}{2}(ab^{\dagger 2} - a^\dagger b^2),\end{aligned} \tag{16}$$

where we have set the global phase as $\varphi = -\pi/2$.

Based on the above standard PDC Hamiltonian (16), the Langevin equations for the operators a and b are

$$\frac{d}{dt}a = \epsilon - \frac{\chi}{2}b^2 - \gamma_a a + \sqrt{2\gamma_a}a_{in}, \tag{17}$$

$$\frac{d}{dt}b = \chi a b^\dagger - \gamma_b b + \sqrt{2\gamma_b}b_{in}, \tag{18}$$

where the noise operators satisfy $\langle a_{in}(t)a_{in}^\dagger(t')\rangle = \langle b_{in}(t)b_{in}^\dagger(t')\rangle = \delta(t-t')$ since we have restricted our consideration to very low temperature such that the corresponding thermal photon numbers for the superconducting resonator modes are close to zero. γ_a and γ_b are, respectively, the decay rates of the second-harmonic and fundamental modes.

In order to linearize the above equations, we define the operators as $a = \alpha + \delta a$, $b = \beta + \delta b$, where α (β) is the average value of operator a (b) in the steady state and δa (δb) is its fluctuation. The average values can be readily given by [44,52]

$$\begin{aligned}&\alpha = \epsilon/\gamma_a, \quad \beta = 0, \quad \epsilon \leqslant \epsilon_c,\\ &\alpha = \gamma_b/\chi, \quad \beta = \pm\sqrt{\frac{2}{\chi}(\epsilon - \epsilon_c)}, \quad \epsilon > \epsilon_c.\end{aligned} \tag{19}$$

Obviously, there is a phase transition at the critical driving strength (threshold) $\epsilon_c = \gamma_a\gamma_b/\chi$, which is inversely proportional to χ. In what follows, we consider only the positive branch for β when the driving strength is above the threshold.

After neglecting the high-order terms of the fluctuations, we obtain the linearized quantum Langevin equations for the fluctuation operators δa and δb as

$$\frac{d}{dt}\delta a = -\chi\beta\delta b - \gamma_a\delta a + \sqrt{2\gamma_a}a_{in}, \tag{20}$$

$$\frac{d}{dt}\delta b = \chi\alpha\delta b^\dagger + \chi\beta^*\delta a - \gamma_b\delta b + \sqrt{2\gamma_b}b_{in}, \tag{21}$$

which can be solved analytically by means of Fourier transformation. Let us define the quadratures $\delta x(t) = \delta b(t) + \delta b^\dagger(t)$ and $\delta y(t) = -i[\delta b(t) - \delta b^\dagger(t)]$; the expressions of their variances are obtained as

$$\langle \delta x^2\rangle = \begin{cases}\frac{\gamma_a\gamma_b}{\gamma_a\gamma_b - \epsilon\chi}, & \epsilon \leqslant \epsilon_c,\\ 1 + \frac{\gamma_b}{\gamma_a} - \frac{\gamma_a\gamma_b}{2(\gamma_a\gamma_b - \epsilon\chi)}, & \epsilon > \epsilon_c,\end{cases} \tag{22}$$

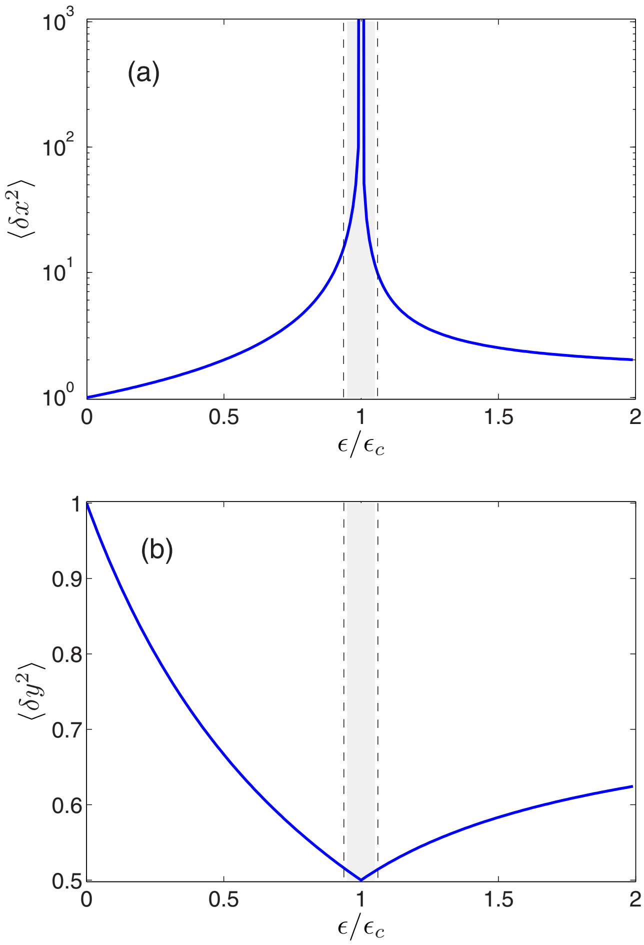

FIG. 3. (Color online) Variances of the quadratures (a) $\langle \delta x^2 \rangle$ and (b) $\langle \delta y^2 \rangle$ for the fundamental mode as a function of the driving strength. The parameters are set as $\gamma_a/2\pi = 0.6$ MHz and $\gamma_b/2\pi = 0.3$ MHz. For the other parameters, see Fig. 2. The shading implies that in the regime close to the threshold the linearization process is not reasonable.

and

$$\langle \delta y^2 \rangle = \begin{cases} \frac{\gamma_a \gamma_b}{\gamma_a \gamma_b + \epsilon \chi}, & \epsilon \leqslant \epsilon_c, \\ 1 - \frac{\gamma_a^2 \gamma_b}{(\gamma_a + 2\gamma_b)\epsilon \chi}, & \epsilon > \epsilon_c. \end{cases} \tag{23}$$

We plot the variances as functions of the driving strength in Fig. 3. It can be observed from Eqs. (22) and (23) and Fig. 3 that $\langle \delta x^2 \rangle$ is always larger than 1 and $\langle \delta y^2 \rangle$ is always smaller than 1. In other words, when the second-harmonic mode is coherently driven resonantly, the fundamental mode exhibits a squeezing effect. It can be observed in Fig. 3(a) that $\langle \delta x^2 \rangle$ diverges when the driving strength is close to the threshold. This implies that the linearization does not work well in this regime [as shown in the shaded regime in Figs. 3(a) and 3(b)]. Recently, the modification near the threshold has been made by the regularized linearization approach [45,53], in which the steady values α and β are determined self-consistently. However, the results from the regularized linearization approach coincide with ours in the regime deviating from the threshold.

Furthermore, we can also investigate the statistical properties of the fundamental mode by calculating its equal-time second-order correlation. The second-order correlation of the fundamental mode is defined by

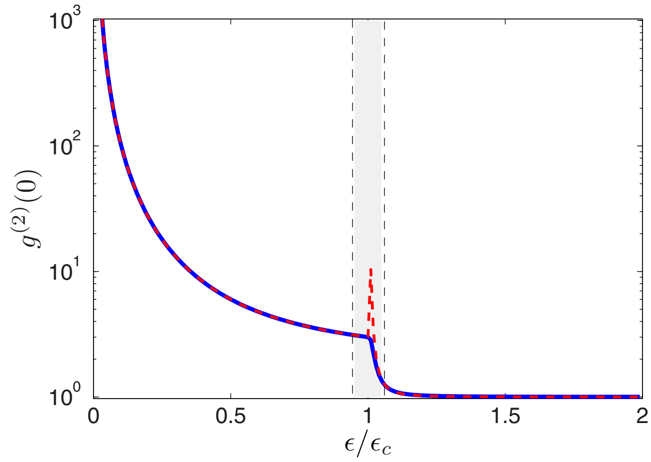

FIG. 4. (Color online) The equal-time second-order correlation $g^{(2)}(0)$ as a function of the driving strength. The parameters are the same as those in Fig. 3. The blue solid line represents our result, and the red dashed line represents the results according to those given in Ref. [54] (for the details, see the context). The shading implies that in the regime close to the threshold the linearization process is not reasonable.

$$g^{(2)}(0) = \frac{\langle b^\dagger b^\dagger b b \rangle}{\langle b^\dagger b \rangle^2}, \tag{24}$$

where $\langle \cdot \rangle$ is the average over the steady state. As mentioned in Ref. [54], the behavior of the correlation is related to the fluctuation of the photon number. Actually, the second-order correlation can be also written as [54] $g^{(2)}(0) = 1 + \langle :(\Delta \hat{n})^2: \rangle / \langle \hat{n} \rangle^2$, where $\hat{n} = b^\dagger b$ is the photon number operator and $\Delta \hat{n} = \hat{n} - \langle \hat{n} \rangle$ is its fluctuation, : : indicates the normal order.

The numerical results of second-order correlation in Eq. (24) are shown in Fig. 4 (blue solid line). It is observed that $g^{(2)}(0) \gg 1$ when the driving strength is below the threshold, which implies a strong bunching character. With the increase of the driving strength, the correlation $g^{(2)}(0)$ decreases and approaches 1 when $\epsilon > \epsilon_c$. Below the threshold, the photon number has a small average value but large fluctuation, leading to a strong bunching [54]. When the driving strength is above the threshold, the photon number fluctuation decreases dramatically compared with the average photon number, and the bunching effect decreases. We would like to point out that except in the regime close to the threshold (see the shaded regime in Fig. 4, where the linearization process is not valid), our results for $g^{(2)}(0)$ agree well with the asymptotic ones given in Ref. [54], wherein the second-order correlation is given as $g^{(2)}(0) \rightarrow 2 + \epsilon_c^2/\epsilon^2$ when the driving strength is below the threshold and $g^{(2)}(0) \rightarrow 1 + \chi \epsilon_c / [4(\epsilon - \epsilon_c)^2]$ for the driving strength is above the threshold (see the red dashed line in Fig. 4) [55]. It can be seen in Fig. 4 that $g^{(2)}(0)$ experiences a sudden change at the threshold. This discontinuity can also be removed by the regularized linearization approach [45,53], which is beyond what we focus on in this paper.

IV. CONCLUSION

In summary, we have studied the microwave degenerate PDC in the circuit-QED system where a single cyclic three-level superconducting qutrit couples to the fundamental and second-harmonic modes in a superconducting transmission-line resonator simultaneously. In the situation of large detunings and weak couplings, we adiabatically eliminate the degree of freedom of the qutrit (that is, keep the qutrit in the ground state) and obtain the effective PDC Hamiltonian for the two microwave resonator modes. Within the available experimental parameters, we show that the method of the adiabatic elimination is reasonable by comparing the corresponding approximate analytical results with the direct numerical calculations. Compared with the scheme in which the two-mode resonator couples to a single two-level qubit [27], the PDC efficiency in our model is dramatically enhanced with a single cyclic three-level system, which can be realized and tuned in the realistic system of a superconducting circuit. Based on the obtained effective Hamiltonian, we show that a coherent driving of the second-harmonic mode will result in the squeezing and bunching effect of the fundamental mode. We hope that our proposal will open a way to generate the high-efficiency microwave PDC process in the system of circuit QED.

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