

**Threshold for nonthermal stabilization of open quantum systems**C. Y. Cai (蔡丞韻),<sup>1</sup> Li-Ping Yang (杨立平),<sup>1</sup> and C. P. Sun (孙昌璞)<sup>1,2,\*</sup><sup>1</sup>*State Key Laboratory of Theoretical Physics, Institute of Theoretical Physics and University of the Chinese Academy of Sciences, Beijing 100190, People's Republic of China*<sup>2</sup>*Beijing Computational Science Research Center, Beijing 100084, China*

(Received 2 April 2013; revised manuscript received 23 November 2013; published 29 January 2014)

We generally study whether or not the information of open quantum systems would be totally erased by their surrounding environments in thermalization processes. A complex system composed of a harmonic oscillator and its environment is studied. When the interaction spectral density contains zero-value regions, there is a threshold of system-bath coupling,  $\eta_c$ , above which the initial information of the system partially remains. We estimated this threshold by the properties of the system and its bath, i.e., the density of the environment states. Thus, its long-time stabilization deviates from the usual thermalization. This nonthermal stabilization happens as a non-Markovian effect.

DOI: [10.1103/PhysRevA.89.012128](https://doi.org/10.1103/PhysRevA.89.012128)

PACS number(s): 03.65.Yz, 03.67.-a, 05.70.Ln, 42.50.Lc

**I. INTRODUCTION**

Thermalization is a dynamic process of an open system reaching the thermal equilibrium at the same temperature  $T$  as its surrounding heat bath. From the point of view of information theory, thermalization is regarded as an information-erasure process [1]. The open system initially prepared in an arbitrary state will relax to a thermal state after a long-time Markovian process. This steady state is irrelevant to the system's initial state at all, while it carries partial information of the bath characterized by its temperature  $T$ . Thus, the conventional thermalization plays a necessary role in the initialization of a computation or thermodynamic cycle [2–4]. This perspective results in a comprehensive understanding of Landauer's erasure principle [1,5].

Thermalization is dynamically associated with a Markovian process [6] and can also be described by a Langevin equation under the Wigner-Weisskopf approximation [7,8]. However, it has been found that a strong system-bath coupling might result in non-Markovian processes when the interaction spectral density contains zero-value regions [9–14]. In such a process, the stabilized open system still maintains partially its initial information, which is quite different from the conventional thermalization and is referred to as nonthermal stabilization. Two questions naturally follow for further investigation: (i) To what extent does the strength of a system-bath coupling increase so that a nonthermal stabilization of the system happens? (ii) How much initial information of the system is left in the stabilized state for a nonthermal stabilization?

To answer these questions, we revisit the “standard model” of open quantum systems, a harmonic oscillator (HO) coupled to a bath of HOs. We analytically examine the mean occupation number of the system through the exact solution to the total system's Heisenberg equation. The system's mean occupation number is divided into two parts. One of them only depends on the system's initial state. The other only depends on the environment and can be neglected if the environment's temperature is small enough. A detailed asymptotic analysis

shows that the first part does not vanish even when the system is stabilized if the interaction spectrum contains some zero-value regions and the coupling strength exceeds the threshold  $\eta_c$ . The threshold depends on the structure of the interaction spectral density, which is characterized by some physical parameters, such as the density of the environment states. Thus the threshold  $\eta_c$  acts as a critical point between conventional thermalization and nonthermal stabilization. Our method can be directly generalized to deal with the fermion case. For the case of a spin interacting with a boson environment, our method is also suitable when the environment's temperature is zero or the environment represents a vacuum.

The paper is organized as follows. In Sec. II we describe the model of a quantum open system and point out that the phenomenon of nonthermal stabilization is characterized by the nonvanishing asymptotic behavior of  $u(t)$ . In Sec. III, the formal solution and the condition of the nonvanishing asymptotic behavior of  $u(t)$  are presented. In Sec. IV, the physical implication of this condition is studied, and it is found that in many situations this condition corresponds to the interaction threshold  $\eta_c$ . In Sec. V, some examples and the corresponding numerical simulations are presented to verify our theory. In Sec. VI, the effect of an environment with a nonzero temperature is discussed. Conclusions and remarks are given in Sec. VII.

**II. MODELS OF OPEN QUANTUM SYSTEMS**

We consider an open system consisting of a HO interacting with its environment (or bath). The environment is modeled as a collection of HOs with linear coupling to the system. This has been extensively studied in much of the literature on open quantum systems, since it can be universally utilized to reveal the core spirit of the quantum dissipation process according to Caldeira and Leggett [7]. We should emphasize that the method used below is also suitable for a fermion case. Another common model in the field of open quantum systems is a two-level system interacting with a boson environment. Our method can only partially deal with this system, which we will discuss later.

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The total Hamiltonian of our model reads

$$H = \Omega a^\dagger a + \sum_l \omega_l b_l^\dagger b_l + \sum_l (\eta_l a^\dagger b_l + \eta_l^* b_l^\dagger a), \quad (1)$$

where  $a$  ( $a^\dagger$ ) and  $b_l$  ( $b_l^\dagger$ ) are the annihilation (creation) operators of the system and the  $l$ th mode of the environment, respectively. The corresponding Heisenberg equation has the following formal solutions [15,16],

$$a(t) = u(t)a + \sum_l u_l(t)b_l, \quad (2)$$

$$b_l(t) = v_l(t)a + \sum_m v_{lm}(t)b_m, \quad (3)$$

where  $a$  and  $b_l$  are the abbreviations for  $a(t)|_{t=0}$  and  $b_l(t)|_{t=0}$ , respectively, and the c-number time-dependent coefficients  $u(t)$ ,  $u_l(t)$ ,  $v_l(t)$ , and  $v_{lm}(t)$  are determined by the following differential equations,

$$\frac{du(t)}{dt} = -i\Omega u(t) - i \sum_l \eta_l v_l(t), \quad (4)$$

$$\frac{du_l(t)}{dt} = -i\Omega u_l(t) - i \sum_m \eta_m v_{ml}(t), \quad (5)$$

$$\frac{dv_l(t)}{dt} = -i\omega_l v_l(t) - i\eta_l^* u(t), \quad (6)$$

$$\frac{dv_{lm}(t)}{dt} = -i\omega_l v_{lm}(t) - i\eta_l^* u_m(t), \quad (7)$$

with initial conditions  $u(t)|_{t=0} = 1$ ,  $u_l(t)|_{t=0} = v_l(t)|_{t=0} = 0$ , and  $v_{lm}(t)|_{t=0} = \delta_{lm}$ . Equations (4) and (6) form a closed differential equation system, which leads to [15]

$$\frac{du(t)}{dt} + i\Omega u(t) + \int_0^t G(t-\tau)u(\tau)d\tau = 0. \quad (8)$$

Here, the integral kernel

$$G(t) = \mathcal{F}[J(\omega)] \equiv \frac{1}{2\pi} \int_{-\infty}^{\infty} J(\omega)e^{-i\omega t} d\omega \quad (9)$$

is the time-domain representation of the system-bath interaction spectral density  $J(\omega) \equiv 2\pi \sum_l |\eta_l|^2 \delta(\omega - \omega_l)$ , which is usually taken as *a priori* microscopic knowledge.

To reveal the non-Markovian property of the open system's evolution, we take the system's mean occupation number as the starting point. When the system and the bath are initially in a direct product state  $\rho(0) = \rho_S(0) \otimes \rho_E(0)$  and we assume that the environment is initially in a thermal equilibrium state at temperature  $T$ , the system's mean occupation number is [17]

$$n(t) = |u(t)|^2 \langle a^\dagger a \rangle_S + \sum_l |u_l(t)|^2 \langle b_l^\dagger b_l \rangle_E, \quad (10)$$

where  $\langle \cdots \rangle_{S(E)} = \text{Tr}_{S(E)}[\rho_{S(E)} \cdots]$  means the average over the state  $\rho_{S(E)}$ . In fact, a weaker assumption of the environment that  $\text{tr}[b_l \rho_E(0)] = 0$  also makes the above equation valid. The mean occupation number  $n(t)$  is divided into two parts. The first part, which vanishes in a long-time Markovian process and describes the erasing of the system's initial information,

only depends on the system's initial condition. The second part, which usually leads to the thermalization of the system in the weak-coupling case [15], characterizes the contribution from the bath. If the temperature  $T$  of the bath is nearly zero, or the bath represents some kind of vacuum, the second part is negligible and one finds

$$n(t) \simeq |u(t)|^2 n(0). \quad (11)$$

Thus, the function  $u(t)$  acts as an amplitude factor in the erasing process. When the temperature is finite, the second part may be comparable to the first part. We discuss the effect of the second part later in this paper.

As mentioned before, another important case of open quantum systems is a two-level system interacting with a boson environment, whose Hamiltonian reads

$$H = \Omega |e\rangle\langle e| + \sum_l \omega_l b_l^\dagger b_l + \sum_l (\eta_l b_l |e\rangle\langle g| + \eta_l^* b_l^\dagger |g\rangle\langle e|), \quad (12)$$

where  $|e\rangle$  and  $|g\rangle$  are the excited state and the ground state of the two-level system, respectively, and  $\Omega$  is its level spacing. If the temperature of the bath is zero (as in the case to discuss spontaneous emission [15]) and the two-level system is initially in its excited state, the total system would evolve in a subspace spanned by states as

$$|\psi(t)\rangle = u(t)|e\rangle \otimes |\text{vac}\rangle + \sum_l v_l(t)|g\rangle \otimes b_l^\dagger |\text{vac}\rangle. \quad (13)$$

The Schrödinger equation

$$i \frac{d}{dt} |\psi(t)\rangle = H |\psi(t)\rangle \quad (14)$$

implies that

$$\frac{du(t)}{dt} = -i\Omega u(t) - i \sum_l \eta_l v_l(t), \quad (15)$$

$$\frac{dv_l(t)}{dt} = -i\omega_l v_l(t) - i\eta_l^* u(t). \quad (16)$$

Because the above expressions are the same as Eqs. (4) and (6),  $u(t)$  satisfies the same differential-integral equation as Eq. (8). Furthermore, the system's mean occupation number is

$$n(t) = |\langle e | \psi(t) \rangle|^2 = |u(t)|^2, \quad (17)$$

which verifies the fact that  $u(t)$  is characterized as an amplitude factor in the erasing process. For a nonzero temperature environment, the above method does not work, and how to deal with such a case still remains an open question.

As a conclusion of this section, we point out that in lots of situations, the dynamics of an open system in the erasing process can be characterized by a c-number function  $u(t)$  which satisfies the differential-integral equation, Eq. (8). A conventional thermalization process implies that  $u(t) \rightarrow 0$  as  $t \rightarrow \infty$ , while a nonvanishing  $u(t)$  at a long time corresponds to the nonthermal stabilization.

### III. FORMAL SOLUTION AND ASYMPTOTIC BEHAVIOR OF $u(t)$

In this section, we study the asymptotic behavior of  $u(t)$ . For convenience, we extend the time domain from  $[0, \infty)$  to  $(-\infty, \infty)$  by letting  $u(t) = \Theta(t)u(t)$ , where  $\Theta(t)$  is a step function. It is found that  $u(t)$  satisfies the following differential-integral equation:

$$\frac{du(t)}{dt} + i\Omega u(t) + \int_{-\infty}^t d\tau G(t-\tau)u(\tau) = \delta(t). \quad (18)$$

It is obvious that Eq. (18) is exactly equivalent to Eq. (8) in the time domain  $(0, \infty)$ . A formal solution of  $u(t)$  is obtained via the Fourier transformation as

$$u(t) = -\frac{1}{2\pi i} \int \frac{e^{-i\omega t} d\omega}{F(\omega)}, \quad (19)$$

where the denominator in the integral is

$$F(\omega) \equiv \omega - \Omega + \frac{1}{2\pi} \int \mathcal{P} \frac{J(\omega') d\omega'}{\omega' - \omega} + \frac{i}{2} J(\omega) + i\epsilon, \quad (20)$$

and  $\epsilon$  is an infinitesimal positive constant. For some special spectrum, e.g., a Lorentzian-type spectrum, the above integral can be carried out analytically [11].

We assume an asymptotic solution of Eq. (8)  $u(t) \sim A \exp(-i\omega_0 t)$  as  $t \rightarrow \infty$ , which oscillates with a single frequency  $\omega_0$  and amplitude  $A$ . Due to the linearity of Eq. (8), the superposition of several such single-mode solutions is also an asymptotic solution of Eq. (8). Therefore, we only need to investigate the existence conditions and the properties of the single-mode case. To this end, we let  $\tilde{u}(t) \equiv \exp(i\omega_0 t)u(t)$ , which satisfies an integral-differential equation similar to Eq. (8) with modified frequency  $\tilde{\Omega} \equiv \Omega - \omega_0$  and modified kernel  $\tilde{G}(t) \equiv \mathcal{F}[J(\omega + \omega_0)]$ , namely,

$$\frac{d\tilde{u}(t)}{dt} + i(\Omega - \omega_0)\tilde{u}(t) + \int_0^t \tilde{G}(t-\tau)\tilde{u}(\tau)d\tau = 0 \quad (21)$$

and

$$\tilde{G}(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} J(\omega + \omega_0) e^{-i\omega t} d\omega. \quad (22)$$

If  $u(t)$  has an asymptotic behavior such as  $u(t) \sim A \exp(-i\omega_0 t)$ , then  $\tilde{u}(t)$  would be stabilized to a constant  $\tilde{u}(t) \sim A$ , which leads to  $d\tilde{u}(t)/dt \sim 0$ . To deal with the last term of Eq. (21), note that

$$\int_0^t \tilde{G}(t-\tau)\tilde{u}(\tau)d\tau = \int_0^t \tilde{G}(\tau)\tilde{u}(t-\tau)d\tau. \quad (23)$$

Because the integrand in the right-hand side of the above equation is dominant in the range when  $\tau$  is small, the asymptotic behavior of this term is

$$A \int_0^{\infty} \tilde{G}(\tau) d\tau. \quad (24)$$

By means of the identity

$$\int_0^{\infty} \frac{d\tau}{2\pi} e^{-i\omega\tau} = \frac{1}{2\pi i} \mathcal{P} \frac{1}{\omega} + \frac{1}{2} \delta(\omega), \quad (25)$$

one finds

$$A \int_0^{\infty} \tilde{G}(\tau) d\tau = A \left( \frac{1}{2\pi i} \mathcal{P} \int_{-\infty}^{\infty} \frac{J(\omega + \omega_0)}{\omega} d\omega + \frac{1}{2} J(\omega_0) \right). \quad (26)$$

Thus, from Eq. (21), we find the relationship between the frequency  $\omega_0$  and the amplitude  $A$  of the single-mode asymptotic behavior  $u(t) \sim A \exp(-i\omega_0 t)$ :

$$\left[ i(\Omega - \omega_0) + \frac{1}{2\pi i} \mathcal{P} \int_{-\infty}^{\infty} \frac{J(\omega)}{\omega - \omega_0} d\omega + \frac{1}{2} J(\omega_0) \right] A = 0. \quad (27)$$

If  $A \neq 0$ , then the first factor in the left-hand side of Eq. (27) vanishes, and its real part and its imaginary part read

$$J(\omega_0) = 0, \quad (28a)$$

$$\Omega - \omega_0 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{J(\omega)}{\omega - \omega_0} d\omega. \quad (28b)$$

The above system of equations thus constitutes the criteria for the existence of the nonvanishing asymptotic solution of Eq. (8) around a real oscillating frequency  $\omega_0$ . Generally, the asymptotic behavior of  $u(t)$  may contain more than one mode. Thus, the condition of a nonvanishing asymptotic solution of  $u(t)$  is that the criteria, Eqs. (28a) and (28b), considered as equations of  $\omega_0$ , do have solution(s).

As mentioned in the previous section, a nonvanishing asymptotic solution of  $u(t)$  implies nonthermal stabilization of the open system. Thus, we conclude that Eqs. (28a) and (28b) are criteria of nonthermal stabilization. In next section, we consider the physical implication of these criteria.

### IV. CRITERIA FOR NONTHERMAL STABILIZATION

During a conventional thermalization process,  $u(t)$  decays to 0 as  $t \rightarrow \infty$ . This effect implies that the system's initial information will be totally erased by its environment. However, there exist some clues reminding us that  $u(t)$  may not vanish at a long time [9–11]. We have explicitly presented the criteria for the occurrence of such nonthermal stabilization by Eqs. (28a) and (28b). Now, we study the physical implication of these criteria.

According to Ref. [11], the nonthermal stabilization first requires the spectrum  $J(\omega)$  to have at least one zero-value region. This is true because Eq. (28a) holds. Thus, the nonthermal stabilization would never happen if the spectrum were of the Lorentzian type. However, in practice, the spectrum one meets always has a cutoff, e.g., a cutoff Lorentzian-type spectrum with  $J(\omega < 0) = 0$  instead of a pure Lorentzian-type spectrum. Theoretically, the interaction spectrum of a boson bath must be zero when  $\omega < 0$ , otherwise the total Hamiltonian would have no lower bound. Thus, it loses no generality to study a spectrum that satisfies  $J(\omega)|_{\omega < 0} = 0$ , called a half-side spectrum [see Fig. 1(a)].

For a half-side spectrum, we consider whether there exists a solution  $\omega_0$  ( $\omega_0 < 0$ ) satisfying Eq. (28b). The left-hand side of Eq. (28b) is a monotonically increasing function of  $-\omega_0$  and has no upper limit, while the right-hand side is a monotonically decreasing function of  $-\omega_0$  (see Fig. 2). Thus, Eq. (28b) holds

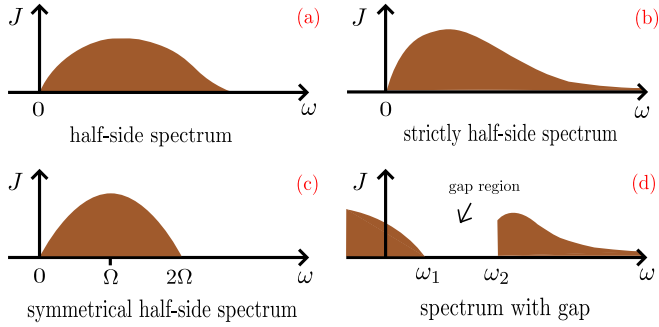


FIG. 1. (Color online) The four classes of spectrums studied in this article: (a) half-side spectrum, (b) strictly half-side spectrum, (c) symmetrical half-side spectrum, and (d) gapped spectrum.

if and only if

$$\frac{1}{2\pi} \int_0^{\infty} \frac{J(\omega)}{\omega} d\omega \geq \Omega. \quad (29)$$

If the above condition holds, there is only one solution of  $\omega_0$  in the region  $(-\infty, 0)$ . To see the physical significance of this condition, we rewrite the spectral density as  $J(\omega) = \eta J_0(\omega)$ , where  $\eta$  characterizes the system-bath interaction strength and  $J_0(\omega)$  describes the structure of the spectrum. Then, the above condition (29) becomes  $\eta \geq \eta_c$ , where the threshold strength  $\eta_c$  is

$$\eta_c = 2\pi\Omega \left( \int_0^{\infty} \frac{J_0(\omega)}{\omega} d\omega \right)^{-1}. \quad (30)$$

The above arguments show that, if the coupling strength  $\eta < \eta_c$ ,  $u(t)$  asymptotically vanishes as  $t \rightarrow \infty$ . Thus, we have found the quantitative meaning of the sentence ‘‘coupling is weak enough’’ in conventional text. When the coupling strength exceeds the threshold  $\eta_c$ , the asymptotic value of  $|u(t)| \neq 0$  and then the initial information of the system will not be totally erased even at a long time. Consequently, the Markovian approximation does not work when  $\eta > \eta_c$ . When the half-side spectral density satisfies  $\int_0^{\infty} [J_0(\omega)/\omega] d\omega = \infty$ , the critical coupling strength becomes zero according to Eq. (30) [see Fig. 2(b)]. Thus, no matter how weak the

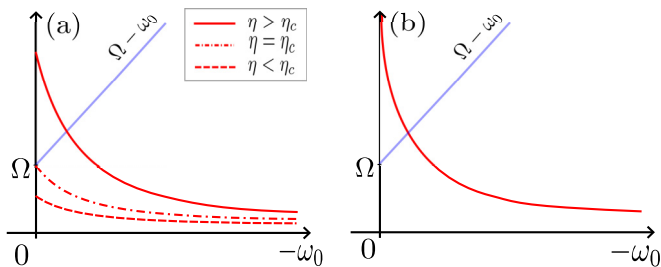


FIG. 2. (Color online) Schematic for the criterion (28b) for a half-side spectrum. The blue lines represent the left-hand side of Eq. (28b) as a function of  $-\omega_0$ . The red lines represent the right-hand side of Eq. (28b). The position where the red line crosses the y axis is mainly determined by  $\eta$ . (a)  $\eta_c > 0$ . In this case, there exists one solution for Eq. (28b) if  $\eta \geq \eta_c$ , while there is no solution if  $\eta < \eta_c$ . (b)  $\eta_c = 0$ . In this case, there always exists a solution as long as the system couples to the bath.

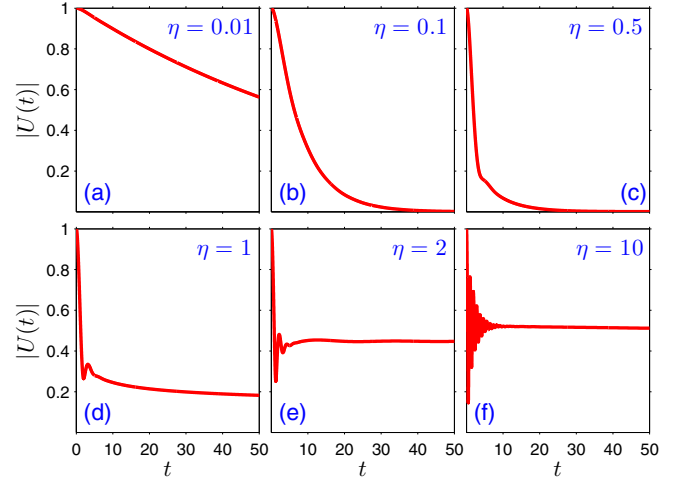


FIG. 3. (Color online)  $|u(t)|$  as a function of  $t$  for ohmic spectrum with coupling strengths (a)  $\eta = 0.01$  Hz, (b)  $\eta = 0.1$  Hz, (c)  $\eta = 0.5$  Hz, (d)  $\eta = 1$  Hz, (e)  $\eta = 2$  Hz, and (f)  $\eta = 10$  Hz. The system’s frequency is  $\Omega = 1$  Hz and the spectrum cutoff frequency is  $\Omega_c = 1$  Hz, which implies the critical coupling strength  $\eta_c = 1$  Hz.

system-bath interaction is, the stabilization is nonthermal and the Markovian approximation or the Wigner-Weisskopf approximation is not valid. In other words, such a spectrum is born to be non-Markovian, e.g., the square spectrum.

Next, let us consider what happens when the spectral density has a gap. Suppose the spectrum  $J(\omega) = \eta J_0(\omega)$  to be studied contains a gap between  $\omega_1$  and  $\omega_2$  as shown in Fig. 3(d). In the gap region,  $\omega_0 \in (\omega_1, \omega_2)$ , Eq. (28b) reads

$$\Omega - \omega_0 = \frac{1}{2\pi} \int_{\omega_2}^{\infty} \frac{\eta J_0(\omega)}{\omega - \omega_0} d\omega - \frac{1}{2\pi} \int_{-\infty}^{\omega_1} \frac{\eta J_0(\omega)}{\omega_0 - \omega} d\omega. \quad (31)$$

The left-hand side of the above equation is a monotonically decreasing function of  $\omega_0$  while the right-hand side is a monotonically increasing function of  $\omega_0$ . Thus, Eq. (31) has one solution of  $\omega_0$  in the region  $[\omega_1, \omega_2]$  if and only if

$$\Omega - \omega_1 \geq \eta A_1, \quad (32)$$

$$\Omega - \omega_2 \leq \eta A_2, \quad (33)$$

where

$$A_1 = \frac{1}{2\pi} \int_{\omega_2}^{\infty} \frac{J_0(\omega)}{\omega - \omega_1} d\omega - \frac{1}{2\pi} \int_{-\infty}^{\omega_1} \frac{J_0(\omega)}{\omega_1 - \omega} d\omega, \quad (34)$$

$$A_2 = \frac{1}{2\pi} \int_{\omega_2}^{\infty} \frac{J_0(\omega)}{\omega - \omega_2} d\omega - \frac{1}{2\pi} \int_{-\infty}^{\omega_1} \frac{J_0(\omega)}{\omega_2 - \omega} d\omega. \quad (35)$$

There are three situations: (i)  $0 < A_1 < A_2$ , (ii)  $A_1 < A_2 < 0$ , and (iii)  $A_1 < 0 < A_2$ . For the first situation, the condition becomes

$$\frac{\Omega - \omega_2}{A_2} \leq \eta \leq \frac{\Omega - \omega_1}{A_1}, \quad (36)$$

and the second situation leads to

$$\frac{\Omega - \omega_1}{A_1} \leq \eta \leq \frac{\Omega - \omega_2}{A_2}. \quad (37)$$

In both case, the existence condition of a nonvanishing mode of  $u(t)$  with frequency  $\omega_0 \in (\omega_1, \omega_2)$  has the form of

$$\eta_{\min} \leq \eta \leq \eta_{\max}. \quad (38)$$

One notices that if the open system's eigenfrequency  $\Omega \in (\omega_1, \omega_2)$ ,  $\eta_{\min} < 0$  and  $\eta_{\max} > 0$ . Thus, the condition holds even when  $\eta \rightarrow 0$ . This gives another case where the Markovian approximation fails even when the interaction strength is small enough. For the third situation, the condition reads

$$\eta \geq \eta_0, \quad (39)$$

where

$$\eta_0 = \max \left\{ \frac{\Omega - \omega_1}{A_1}, \frac{\Omega - \omega_2}{A_2} \right\}. \quad (40)$$

In the case of a spectrum whose density is discontinuous at  $\omega_1$  and  $\omega_2$ , one finds  $A_1 = -\infty$  and  $A_2 = +\infty$ , which implies that the condition, Eq. (39), is  $\eta \geq 0$ . Again, we see that the structure of the spectrum plays an important role in nonthermal stabilization.

So far, we have discussed two basic cases where a spectrum has a zero region, a cutoff or a gap. For a spectrum that contains more than one cutoff or gap, each cutoff or gap contributes a corresponding condition of the form  $\eta \geq \eta_0$  or  $\eta_{\min} \leq \eta \leq \eta_{\max}$ . If the condition related to a particular zero region holds, the asymptotic behavior of  $u(t)$  would have one and only one nonvanishing mode with a frequency in this region.

Finally, we show how to estimate the amplitude  $A$  of the single-mode asymptotic behavior  $u(t) \sim A \exp(-i\omega_0 t)$ . For a given solution  $\omega_0$  of the Eqs. (28a) and (28b),  $F(\omega_0)$  vanishes. Therefore, the integral around  $\omega_0$  contributes most to the integration in Eq. (19) and  $F(\omega)$  can be approximately replaced by  $F'(\omega_0)(\omega - \omega_0)$ . According to the residue theorem, we have  $u(t) \simeq \exp(-i\omega_0 t)/F'(\omega_0)$ . Then, the amplitude  $A$  is approximated as  $1/F'(\omega_0)$ , i.e.,

$$A \simeq \left( 1 + \frac{1}{2\pi} \int \mathcal{P} \frac{J(\omega) d\omega}{(\omega - \omega_0)^2} \right)^{-1}. \quad (41)$$

## V. EXAMPLE OF NONTHERMAL STABILIZATIONS

### A. Symmetrical spectrums

The first example of the nonthermal stabilization is the case with a symmetrical half-side spectrum that satisfies  $J(\Omega - \omega) = J(\Omega + \omega)$  with respect to the resonance point  $\omega = \Omega$  and  $J(\omega)$  does not vanish if and only if  $\omega \in (0, 2\Omega)$  [see Fig. 1(c)]. Such a spectrum has two cutoffs. Due to the symmetry of the spectrum, both cutoffs relate to the same critical coupling strength  $\eta_c$ , which is determined by Eq. (30),

$$\eta_c = 2\pi\Omega \left( \int_{-\Omega}^{\Omega} \frac{J_0(\omega + \Omega)}{\omega + \Omega} d\omega \right)^{-1}. \quad (42)$$

When the coupling strength  $\eta < \eta_c$ ,  $u(t)$  vanishes as  $t \rightarrow \infty$ . On the contrary, in the nonthermal stabilization region  $\eta \geq \eta_c$ ,  $u(t)$  has two single-mode solutions. One is  $A \exp(-i\omega_0 t)$  and another single-mode solution possesses the same amplitude  $A$

TABLE I. Contrast between analytical results  $O^{(a)}$  and numerical results  $O^{(n)}$  for a triangle spectrum with  $\Omega = 1$  and (i)  $\eta = 0.7321$  and (ii)  $\eta = 7.9577$ .

$\eta$	$\omega_0^{(a)}$	$\omega_0^{(n)}$	$2A^{(a)}$	$2A^{(b)}$
0.7321	-0.0027	-0.0026	0.4141	$\sim 0.4$
7.9577	-1.8512	-1.8501	0.9782	$\sim 1.0$

and the antipodal frequency  $2\Omega - \omega_0$ . Thus, the asymptotic behavior of  $u(t)$  is described by the superposition of the two single-mode solutions,

$$u(t) \sim 2A e^{-i\Omega t} \cos(\Omega - \omega_0)t. \quad (43)$$

Strictly speaking, it is not a stabilization because of the oscillation behavior of  $|u(t)|^2$ .

We use these results to examine two kinds of spectra as examples, the triangle spectrum,

$$J_1(\omega) = \begin{cases} 2\pi\eta \frac{\omega}{\Omega} & 0 \leq \omega \leq \Omega, \\ 2\pi\eta \frac{2\Omega - \omega}{\Omega} & \Omega \leq \omega \leq 2\Omega, \\ 0 & \text{otherwise,} \end{cases} \quad (44)$$

and the square spectrum,

$$J_2(\omega) = \begin{cases} 2\pi\eta & 0 < \omega < 2\Omega, \\ 0 & \text{otherwise.} \end{cases} \quad (45)$$

According to Eq. (42), the critical strength for the triangle spectrum is  $\eta_c = \Omega/(2 \ln 2) = 0.7213\Omega$ , while the critical strength for the square spectrum is zero. The contrasts between analytical calculations [starting from Eq. (28b) to determine  $\omega_0$  and Eq. (41) to determine  $A$ ] and numerical results [obtained by the numerical solution of Eq. (8)] are listed in Tables I and II. By measuring the oscillation period of  $u_0(t)$ , one finds out the oscillation frequency  $\Omega - \omega_0$ . The agreement between the analytical calculations and numerical results implies that the criteria equation, Eq. (28b), and the estimation of the amplitude, Eq. (41), indeed work well.

### B. Ohmic spectrum

The second example is a more realistic one—the ohmic spectrum with its density distribution reads

$$J(\omega) = 2\pi\eta\theta(\omega)\omega e^{-\omega/\Omega_c}. \quad (46)$$

Here,  $\eta$  characterizes the coupling strength and  $\Omega_c$  is the cutoff frequency. This spectrum is widely applied in open systems

TABLE II. Contrast between analytical results  $O^{(a)}$  and numerical results  $O^{(n)}$  for a square spectrum with  $\Omega = 1$  and (i)  $\eta = 0.1$ , (ii)  $\eta = 0.5$ , and (iii)  $\eta = 50$ . One should note that the actual observable in the numerical test is  $\pi/(1 - \omega_0)$  instead of  $\omega_0$ , thus  $\omega_0^{(a)}$  is indeed very close to  $\omega_0^{(n)}$ .

$\eta$	$\omega_0^{(a)}$	$\omega_0^{(n)}$	$2A^{(a)}$	$2A^{(b)}$
0.1	-9.0722e-5	-5.0721e-4	0.0018	About $1e-3$
0.5	-0.1997	-0.2006	0.6104	About 0.6
50	-9.0167	-9.0264	0.9967	About 1

[18,19]. There exists a critical coupling strength,  $\eta_c = \Omega/\Omega_c$ , according to Eq. (30). In the nonthermal region, the oscillating frequency is determined by the criteria equation (28b),

$$\Omega - \omega_0 = \eta \int_0^\infty \frac{\omega}{\omega - \omega_0} e^{-\omega/\Omega_c} d\omega, \quad (47)$$

and its amplitude is

$$A \simeq \left( \frac{\Omega - \eta\Omega_c}{\omega_0} - \frac{\Omega - \omega_0}{\Omega_c} \right)^{-1}. \quad (48)$$

As shown by the numerical simulation of  $u(t)$  in Figs. 3(a)–3(c), when  $\eta < \eta_c$ ,  $u(t)$  decays exponentially, and the decay rate increases with  $\eta$  increases. When  $\eta \geq \eta_c$ ,  $|u(t)|$  has a nonvanishing asymptotic value  $|A|$ , which increases as the coupling strength  $\eta$  increases. The numerical results also confirm the former qualitative analysis from Eqs. (28b) and (41). Sometimes, the spectrum that one meets in the experiment may be a modified one, such as subohmic or superohmic spectra. Our method can be applied to these cases straightforwardly [11].

## VI. THE INFORMATION FROM THE BATH INHERITED BY THE SYSTEM

We have described how the first part of the system's mean occupation number can represent the residual information of the system's initial state. Now, we turn our attention to the second part, which depends on the population distribution of the bath.

The second part of the system's mean occupation number in Eq. (10) is rewritten as

$$\sum_l |u_l(t)|^2 \langle b_l^\dagger b_l \rangle = \int p(\omega) f_\beta(\omega) d\omega, \quad (49)$$

where  $p(\omega) = \sum_l |u_l(t)|^2 \delta(\omega - \omega_l)$  is a distribution function (which need not be normalized) and  $f_\beta(\omega) = 1/[\exp(\beta\omega) - 1]$  is the average occupation number of the environment mode with frequency  $\omega$  at temperature  $T = 1/(k_B\beta)$ .

To calculate  $p(\omega)$ , we first analyze the dynamic behavior of  $u_l(t)$ . Following from Eqs. (4) and (6), we find that  $u_l(t)$  obeys the differential-integral equation

$$\frac{du_l(t)}{dt} + i\Omega u_l(t) + \int_0^t G(t - \tau) u_l(\tau) d\tau = -i\eta_l e^{-i\omega_l t}, \quad (50)$$

with the initial condition  $u_l(0) = 0$ . Comparing this equation with Eq. (8), we express  $u_l(t)$  in terms of  $u(t)$  as [10]

$$u_l(t) = -i\eta_l \int_0^t u(t - \tau) e^{-i\omega_l \tau} d\tau. \quad (51)$$

Thus  $u_l(t)$  is determined by  $u(t)$  over the time domain  $[0, t]$ . As shown in Fig. 3,  $u(t)$  decays exponentially in a short time and relaxes to an asymptotic form as  $A \exp(-i\omega_0 t)$  at a long time. For simplicity, we focus on the single-mode case here and assume  $u(t)$  to be of the form

$$u(t) \simeq \begin{cases} e^{-i\Omega' t - \gamma t}, & t < t_1, \\ A e^{-i\omega_0 t}, & t \geq t_1. \end{cases} \quad (52)$$

Though it seems a rough approximation, it seizes the essence. How to choose a suitable  $t_1$  will be pointed out later. Then we obtain  $u_l(t)$  as

$$u_l(t) = -i\eta_l e^{-i\omega_l t} \frac{\exp[-i(\Omega' - \omega_l)t_1 - \gamma t_1] - 1}{-i(\Omega' - \omega_l) - \gamma} - i\eta_l A e^{-i\omega_l t} \frac{\exp[-i(\omega_0 - \omega_l)t] - \exp[-i(\omega_0 - \omega_l)t_1]}{-i(\omega_0 - \omega_l)}. \quad (53)$$

Now, we consider two special cases: (i)  $A = 0$  for small  $\eta$ , and (ii)  $A \neq 0$  for large  $\eta$ . In the first case, the second term in the right-hand side of Eq. (53) vanishes, which is equivalent to choosing  $t_1 = \infty$  in Eq. (52). Then, the distribution of the written information is

$$p(\omega) = \frac{1}{2\pi} \frac{J(\omega)}{(\omega - \Omega')^2 + \gamma^2}. \quad (54)$$

The two parameters  $\Omega'$  and  $\gamma$  can be estimated by the Wigner-Weisskopf approximation:

$$\Omega' = \Omega - \frac{1}{2\pi} \int \mathcal{P} \frac{J(\omega') d\omega'}{\omega' - \Omega}, \quad (55)$$

$$\gamma = J(\Omega)/2. \quad (56)$$

It is discovered that  $p(\omega)$  is a sharp distribution centered at  $\Omega'$  with width  $\gamma$ :

$$p(\omega) \simeq \frac{1}{2\pi} \frac{2\gamma}{(\omega - \Omega')^2 + \gamma^2}. \quad (57)$$

Since  $A = 0$ , the first part of the system's mean occupation number vanishes, thus the mean occupation number of the system reads

$$n(\gamma, \Omega') = \int \frac{1}{2\pi} \frac{2\gamma}{(\omega - \Omega')^2 + \gamma^2} f_\beta(\omega) d\omega. \quad (58)$$

In the weak coupling strength limit  $\eta \rightarrow 0$  ( $\gamma \rightarrow 0$ ),  $p(\omega) \rightarrow \delta(\omega - \Omega')$ , which leads to

$$n(\gamma \rightarrow 0, \Omega') = \int \delta(\omega - \Omega') f_\beta(\omega) d\omega = f_\beta(\Omega'). \quad (59)$$

This implies that the system's mean occupation number actually inherits the population of the environment mode with the renormalized mode frequency  $\Omega'$ , which actually corresponds to a conventional thermalization process [15]. If the coupling strength is small but finite,  $p(\omega)$  becomes a Lorentzian-type distribution with a broadened width and a translated center. The center translation effect is characterized by

$$\Delta n = n(\gamma \rightarrow 0, \Omega') - n(\gamma \rightarrow 0, \Omega) \simeq \frac{1}{2\pi} f'_\beta(\Omega) \int \mathcal{P} \frac{J(\omega) d\omega}{\Omega - \omega}, \quad (60)$$

which is proportional to the coupling strength. The broadening width effect is denoted by

$$\begin{aligned}\Delta n &= n(\gamma, \Omega) - n(\gamma \rightarrow 0, \Omega) \\ &\simeq \frac{1}{2} f_{\beta}''(\Omega) \langle (\Delta\omega)^2 \rangle,\end{aligned}\quad (61)$$

which is proportional to  $\langle (\Delta\omega)^2 \rangle \propto \gamma^2$ .

In the second case,  $\gamma$  is very large ( $\gamma \propto \eta$ ), then  $u_l(t)$  is dominated by the second term in the right-hand side of Eq. (53):

$$|u_l(t)|^2 = A^2 |\eta_l|^2 \frac{2 - 2 \cos(\omega_0 - \omega_l)(t - t_1)}{(\omega_0 - \omega_l)^2}.\quad (62)$$

Since  $p(\omega)$  appears in the integral over  $\omega$ , the oscillation term  $\cos(\omega_0 - \omega_l)(t - t_1)$  can be omitted. Then, the distribution of the written information is approximated as

$$p(\omega) = \frac{A^2}{2\pi} \frac{2J(\omega)}{(\omega - \omega_0)^2}.\quad (63)$$

This distribution is totally different from that in the weak coupling case in the two following ways. First, it is no longer normalized to unity. This is a natural result since the system's mean occupation number now depends on both the bath and its own initial condition. Second, it is a widespread distribution instead of a sharp one, which implies that the information written by the environment becomes more complicated. However, it should be emphasized that when the temperature is low enough [ $f_{\beta}(\omega) \rightarrow 0$ ], the second term in Eq. (10) will be small compared to the first term. Thus, in this situation, one may physically observe the nonthermal stabilization effect by measuring the system's mean occupation number [20].

## VII. CONCLUSIONS AND DISCUSSIONS

We have studied the nonthermal stabilization phenomenon by calculating the open system's mean occupation number. The

criteria for this non-Markovian effect have been presented. For a spectrum with one cutoff, the criteria correspond to the quantitative threshold  $\eta_c$ . In the nonthermal region,  $\eta \geq \eta_c$ , the system's initial information is not erased totally when stabilized.

Actually,  $\eta_c$  explicitly provides an upper limit of the region where the Markovian approximation is valid. Our investigation has undoubtedly clarified the misunderstanding that the Markovian approximation is valid only when the coupling strength is small enough. Furthermore, it has been found that the nonthermal phenomenon is closely related to the structure of the system-bath interaction spectral density. In this sense the nonthermal stabilization effect due to non-Markovian process above the threshold  $\eta_c$  provides us with a new way to understand the information lost in open systems.

Apparently, our approach is universal and can be applied to the fermion case. For a fermionlike system, such as a two-level atom coupling to a boson bath, our method is workable only when the bath is at zero temperature. As all the results and criteria are temperature independent, one might guess that our result may be generalized to low-temperature cases. However, this has not been proven yet and remains an open question. It is also worth discussing the impact of the nonthermal stabilization on the entanglement evolution [21–23].

*Note added in proof.* Recently, we became aware of another work dealing with a similar topic of non-Markovian dynamics [24]. Starting from the same models as ours, they focus on another aspect of non-Markovian effects, i.e., the form of the system's steady state.

## ACKNOWLEDGMENTS

This work was supported by the NSFC through Grants No. 10974209 and No. 10935010 and by the National 973 Program (Grant No. 2012CB922104).

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