Disentanglement in a quantum-critical environment

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We study the dynamical process of disentanglement of two qubits and two qutrits coupled to an Ising spin chain in a transverse field, which exhibits a quantum phase transition. We use the concurrence and negativity to quantify entanglement of two qubits and two qutrits, respectively. Explicit connections between the concurrence (negativity) and the decoherence factors are given for two initial states, the pure maximally entangled state and the mixed Werner state. We find that the concurrence and negativity decay exponentially with fourth power of time in the vicinity of the critical point of the environmental system.

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I. INTRODUCTION

Entanglement is one of the most essential features in quantum mechanics [1] and in recent decades has been the focus for people in many fields of physics. Motivated by the progress of quantum information, entanglement has become a basic resource in quantum technologies such as quantum teleportation and quantum cryptography [2–4]. On the other hand, generally a realistic system is surrounded by an environment. The coupling between a quantum system and its environment leads to decoherence of the system. Thus, it is natural for us to consider the process of degradation of entanglement due to decoherence. More recently, Yu and Eberly [5] showed that two entangled qubits become completely disentangled in a finite time under the influence of pure vacuum noise. Surprisingly, they found that the behaviors of local decoherence is different from spontaneous disentanglement. The decoherence effects take an infinite time evolution under the influence of vacuum while the entanglement vanishes suddenly in finite time. Some other researchers have investigated the process of disentanglement in the open quantum systems [6–9]. The problem of decoherence from spin environments was studied by Cucchietti *et al.* [10], who considered spin environments consisting of N independent spins.

In most of the previous studies, uncorrelated environments were usually considered, and modeled by a reservoir consisting of harmonic oscillators. Although a collection of harmonic oscillators is a well approximated model to represent the environment weakly coupled to a system, however, in the practical situation, particles in the environment may have interactions with each other. Consequently, a problem comes out: How does the entanglement evolve in a correlated environment? In this paper, we consider this problem and choose a correlated spin chain, the Ising model in a transverse field, as the surrounding system. Moreover, this surrounding system displays quantum phase transition (QPT) at some critical point and thus possesses dynamic hypersensitivity with respect to the perturbation even induced by a single qubit [11].

As a quantum critical phenomenon, QPT happens at zero temperature, at which the thermal fluctuations vanish. Thus, QPT is driven only by quantum fluctuation. At the critical point there usually exists degeneracy between the energy levels of the systems when QPT happens. Therefore, it can be excepted that, when we study the dynamic evolution of the system coupled to a environment with QPT, some special dynamic features will appear at the critical point. Quan *et al.* [11] have studied the decoherence induced by the correlated environment. It was shown that at the critical point of a QPT the decoherence is enhanced. Following this work, Cucchietti *et al.* [12] discovered that the decoherence induced by the critical environment possesses some universality with the Boson-Hubbard model as an illustration.

Now, we consider two spins coupled to the Ising spin chain in a transverse field to reveal the effect of the correlated environment on the dynamic evolution of the two-spin entanglement. We will study different cases including two qubits and qutrits. Moreover, we will consider cases where the two spins initially start from a pure maximally entangled state and a mixed Werner state [13]. The "sudden death" of entanglement is found to be a quite common phenomenon.

This paper is organized as follows. In Sec. II, we introduce the model of two-spin system coupled to Ising spin chain with a transverse field. By exactly diagonalizing the Hamiltonian, we give an expression of the time evolution operator. In Sec. III, the analytical results of the concurrence [14] of the two qubits are calculated to show the dynamics of entanglement. Numerical results are also given to illustrate the details of the dynamical behaviors of entanglement. In Sec. IV, two qutrits are coupled to the Ising spin chain. The analytical and numerical results of the negativity [15,16] are given. We give conclusions in Sec. V.

II. MODEL HAMILTONIAN AND EVOLUTION OPERATOR

We choose the engineered environment system to be an Ising spin chain in a transverse field which displays a QPT.

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Two spins are transversely coupled to the chain. The corresponding Hamiltonian reads

$$H = \sum_{l=-M}^{M} \sigma_l^x \sigma_{l+1}^x + \left[\lambda + \frac{g}{2}(s_{1z} + s_{2z})\right] \sum_{l=-M}^{M} \frac{\sigma_l^z}{2}, \quad (1)$$

where λ characterizes the strength of the transverse field, g denotes the coupling strength between the Ising chain and the two spins, s_1 and s_2, σ_l^α $(\alpha=x,y,z)$ are the Pauli operators defined on the lth site, and the total number of spins in the Ising chain is $L=2M+1$. The Ising model is the simplest model which exhibits a QPT, and can be exactly calculated.

In order to diagonalize the Hamiltonian, we first notice that $[s_{1z}+s_{2z},\sigma_l^\alpha]=0$, thus it is convenient to define an operator-valued parameter

$$\hat{\Lambda} = \lambda + \frac{g}{2}(s_{1z} + s_{2z}), \quad (2)$$

which is a conserved quantity. When we diagonalize the Ising spin chain, the parameter $\hat{\Lambda}$ can be treated as a c number with different values corresponding to the eigenvalues of $s_{1z}+s_{2z}$ in the two-spin subspace.

By combining Jordan-Wigner and Fourier transformations with the momentum space [17], the Hamiltonian can be written as [18]

$$H = \sum_{k>0} e^{i(\theta_k/2)\sigma_{kx}}(\Omega_k \sigma_{kz}) e^{-i(\theta_k/2)\sigma_{kx}} + \left(-\frac{\hat{\Lambda}}{2} + 1\right)\sigma_{0z}, \quad (3)$$

where we have used the following pseudospin operators $\sigma_{k\alpha}$ $(\alpha=x,y,z)$: [18]

$$\sigma_{kx} = d_k^\dagger d_{-k}^\dagger + d_{-k} d_k \quad (k = 1,2,\ldots M),$$

$$\sigma_{ky} = -i d_k^\dagger d_{-k}^\dagger + i d_{-k} d_k,$$

$$\sigma_{kz} = d_k^\dagger d_k + d_{-k}^\dagger d_{-k} - 1,$$

$$\sigma_{0z} = 2 d_0^\dagger d_0 - 1, \quad (4)$$

and $d_k^\dagger, d_k\{k=0,1,2,\ldots\}$ denote the fermionic creation and annihilation operators in the momentum space, respectively. Here,

$$\Omega_k = \sqrt{[-\hat{\Lambda} + 2\cos(2\pi k/L)]^2 + 4\sin^2(2\pi k/L)}, \quad (5)$$

$$\theta_k = \arcsin\left[\frac{-2\sin\left(\frac{2\pi k}{L}\right)}{\Omega_k}\right]. \quad (6)$$

From Eq. (3) and the units where $\hbar=1$, the time evolution operator is obtained as

$$U(t) = e^{-i(-\hat{\Lambda}/2+1)\sigma_{0z}t} \prod_{k>0} e^{i(\theta_k/2)\sigma_{kx}} e^{-it\Omega_k\sigma_{kz}} e^{-i(\theta_k/2)\sigma_{kx}}. \quad (7)$$

Explicitly knowning the evolution operator, we now consider the entanglement dynamics of the two qubits and two qutrits.

III. DYNAMICAL DISENTANGLEMENT OF TWO QUBITS

A. The case with initial pure entangling state

We investigate the dynamic evolution of two-qubit entanglement and assume that the two qubits initially start from a maximally entangled state

$$|\Phi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle). \quad (8)$$

Here, $|0\rangle$ and $|1\rangle$ denote the spin up and down, respectively. The initial state of environment is assumed to be the vacuum state in the momentum space, namely, $|\psi_E\rangle = |0\rangle_{k=0} \otimes_{k>0} |0\rangle_k |0\rangle_{-k}$, and the vacuum state $|0\rangle_k$ satisfies $d_k|0\rangle_k=0$. We may write a more general initial state of this composite system as

$$|\Psi(0)\rangle = (a|00\rangle + b|11\rangle) \otimes |\psi_E\rangle. \quad (9)$$

From the evolution operator (7), the state vector at time t is given by

$$|\Psi(t)\rangle = a|00\rangle \otimes U_0|\psi_E\rangle + b|11\rangle \otimes U_1|\psi_E\rangle, \quad (10)$$

where the unitary operator U_0 and U_1 can be obtained from the unitary operator $U(t)$ by replacing operator $\hat{\Lambda}$ with number $\lambda+g/2$ and $\lambda-g/2$, respectively.

Tracing out the environment, in the basis spanned by $\{|00\rangle,|11\rangle,|01\rangle,|10\rangle\}$, the reduced density matrix of the two-spin system is obtained as

$$\rho_{1,2} = \begin{pmatrix} |a|^2 & ab^* F(t) \\ a^* b F^*(t) & |b|^2 \end{pmatrix} \oplus Z_{2\times 2}, \quad (11)$$

where $F(t)=\langle\psi_E|U_1^\dagger U_0|\psi_E\rangle$ is the decoherence factor, and $Z_{2\times 2}$ denotes the 2×2 zero matrix. Now, the concurrence [14] of the reduced density matrix can be readily given by

$$C = 2|ab^* F(t)| = C_0|F(t)|, \quad (12)$$

where C_0 is the concurrence of the initial state. We see that the concurrence is proportional to the norm of the decoherence factor, and when the initial state is in a maximally entangled state (8), $C=|F(t)|$, namely, the concurrence is equal to the norm of the decoherence factor.

Let us consider the decoherence factor

$$F(t) = \langle\psi_E|U_1^\dagger U_0|\psi_E\rangle = \prod_{k>0} F_k, \quad (13)$$

where U_n $(n=0,1)$ is generated from Hamiltonian H_n with $\hat{\Lambda}=\Lambda_n$ (a number). From the unitary operator (7) and the initial vacuum state, we obtain

$$\begin{aligned} |F(t)| = \prod_{k>0} \{1 &- [\sin(\Omega_k^{(0)}t)\cos(\Omega_k^{(1)}t)\sin\theta_k^{(0)} \\ &- \cos(\Omega_k^{(0)}t)\sin(\Omega_k^{(1)}t)\sin\theta_k^{(1)}]^2 \\ &- \sin^2(\Omega_k^{(0)}t)\sin^2(\Omega_k^{(1)}t)\sin^2(\theta_k^{(0)} - \theta_k^{(1)})\}^{1/2}, \end{aligned} \quad (14)$$

where $\Omega_k^{(n)}$ and $\theta_k^{(n)}$ are obtained by replacing $\hat{\Lambda}$ with Λ_n in Eqs. (5) and (6), respectively. Here, $\Lambda_0=\lambda+g/2$ and $\Lambda_1=\lambda-g/2$. This is one of our main results. We see that the zero mode $(k=0)$ has no contribution to the decoherence

factor. Clearly, every factor F_k is less than unit. So it can be expected that in the large L limit, $|F(t)|$ will go to zero under some reasonable conditions.

By carrying out an analysis similar to Ref. [11], we introduce a cutoff number K_c and define the partial product for the decoherence factor

$$|F(t)|_c = \prod_{k>0}^{K_c} F_k \geqslant |F(t)|, \tag{15}$$

from which the corresponding partial sum

$$S(t) = \ln |F(t)|_c \equiv -\sum_{k>0}^{K_c} |\ln F_k|. \tag{16}$$

For the case of small k and large L, we have $\Omega_k^{(n)} \approx |2-\Lambda_n|$, consequently

$$\sin^2(\theta_k^{(0)} - \theta_k^{(1)}) \approx \frac{16k^2\pi^2(\Lambda_0-\Lambda_1)^2}{L^2(2-\Lambda_0)^2(2-\Lambda_1)^2}. \tag{17}$$

As a result, if L is large enough and $\Lambda_0-\Lambda_1$ is very small perturbation the approximation of S can be obtained as

$$\begin{aligned} S(t) \approx & -2E(K_c)(2-\Lambda_0)^{-2}(2-\Lambda_1)^{-2} \\ & \times\{(\Lambda_0-\Lambda_1)^2 \sin^2(|2-\Lambda_0|t)\sin^2(|2-\Lambda_1|t) \\ & + [\sin(|2-\Lambda_0|t)\cos(|2-\Lambda_1|t)|2-\Lambda_1| \\ & - \sin(|2-\Lambda_1|t)\cos(|2-\Lambda_0|t)|2-\Lambda_0|]^2\}, \end{aligned} \tag{18}$$

where

$$E(K_c) = 4\pi^2 K_c(K_c+1)(2K_c+1)/(6L^2). \tag{19}$$

In the derivation of the above equation, we have used $\ln(1-x) \approx -x$ for small x and $\Sigma_{k=1}^n k^2 = n(n+1)(2n+1)/6$.

For our two-qubit case, $\Lambda_0 = \lambda + g/2$, $\Lambda_1 = \lambda - g/2$. When $\lambda \to 2$, and with a proper small g we have

$$|F(t)|_c \approx e^{-\gamma t^4} \tag{20}$$

with $\gamma = 2E(K_c)g^2$. Notice that $|F(t)|_c$ is larger than $|F(t)| = C$. Therefore, from the above heuristic analysis we may expect that when the parameter λ is adjusted to the vicinity of the critical point $\lambda_c = 2$, the concurrence (or the decoherence factor) will exponentially decay with the fourth power of time. Moreover, for short times, from Eq. (14), the concurrence becomes

$$C \approx e^{-\Gamma t^4} \tag{21}$$

with $\Gamma = 1/2\Sigma_{k>0}\sin^2(\theta_k^{(0)} - \theta_k^{(1)})(\Omega_k^{(0)})^2(\Omega_k^{(1)})^2$.

Now we resort to a numerical analysis of the dynamical sensitivity and the concurrence decay. In Figs. 1(a) and 1(b), we plot the concurrence versus time for different λ. We find that in the vicinity of the critical point about $\lambda \in [2-0.3, 2+0.3]$, concurrence decays monotonously with time. Extending the time range, however, does not affect the concurrence. Figure 1(a) shows the cases of $\lambda \leqslant 2$. We can see that concurrence for the case $\lambda = 2$ decays more rapidly than other cases. It should be noted that the dynamics of the two-qubit entanglement in Eq. (12) is absolutely determined by the decoherence factor in Eq. (14). Thus from a theoretical point of view, the complete disentanglement cannot be realized in a finite time. When parameter λ becomes larger than λ_c, ($g=3$, 4 and 5), the numerical results of the concurrence are shown in Fig. 1(b). The concurrence oscillates with time, and collapses and revivals are observed. This is in contrast to the case of small λ, where no revivals are found.

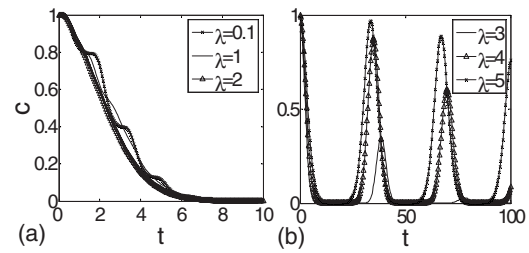

FIG. 1. (a) Concurrence versus time t with different λ in the case of weak coupling strength $g=0.1$. The size of the environment is $L=300$. (b) shows the cases of larger λ.

The surrounding system displays a QPT near the critical point and a competition exists between different order tendencies [17]. From another point of view, near the critical point quantum chaotic behaviors may emerge [19]. For a system with quantum chaos, though it is prepared in an identical initial state, two slightly different interactions can lead to two quite different quantum evolutions. In our system the decoherence factor can act as a fidelity and quantify the difference between the two states which are produced through two different evolutions. Decay of the fidelity can indicate the presence of quantum chaos [20], and the monotonous decay of the decoherence factor (concurrence) at the critical point may be considered as a signature of quantum chaos.

In Fig. 2, for weak coupling $g=0.1$ and $\lambda=4$, the oscillation of concurrence is suppressed by enlarging the size of the environment. The larger environment prevents the revival of entanglement. In the short-time region, we can see that the larger size of the environment will accelerate the monotonous decay of concurrence. From Eq. (14), each factor F_k is smaller than 1, thus it is reasonable to conclude that the large size of the environment will more effectively suppress the factor $F(t)$, and consequently suppress the concurrence.

In Fig. 3, we consider the effects of coupling g on the dynamics of entanglement. At the critical point $\lambda=2$, we adjust g from a small one $g=0.1$ to a strong one $g=100$. It can

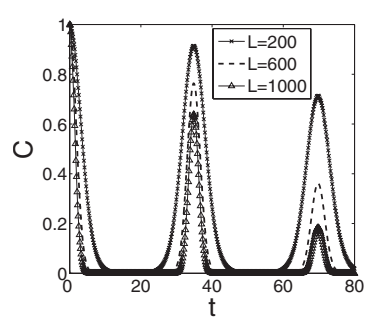

FIG. 2. Concurrence versus time with different environment size $L=200$, 600, and 1000. The transverse field $\lambda=4$, and the coupling strength $g=0.1$.

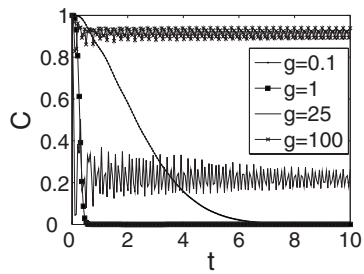

FIG. 3. Concurrence versus time at the critical point $\lambda=2$ with different coupling strength g.

be found that when we properly enlarge the coupling, e.g., $g=1$, the concurrence decays more sharply than the case $g=0.1$. However, when we continue enlarging the coupling to about $g>10$, e.g., $g=25$, concurrence will oscillate quickly and no longer decays monotonously to zero. For the case of very large coupling $g=100$, concurrence behaves as a weak oscillation near the initial value of $C=1$. It can be expected that up to the strong coupling limit of g, the concurrence will stay at $C=1$ without changing with time. The above behaviors remind us of the quantum Zeno effects in the process of quantum measurement [21]. The phenomenon shown in Fig. 3 is similar to the decay probability which can be suppressed by the increasing coupling between system and measuring apparatus in quantum Zeno effects.

B. The case of mixed state

Now, we study the dynamics of disentanglement of mixed entangled state and assume that the two qubits are initially in a Werner state [13] which is given by

$$\rho_s = P|\Phi\rangle\langle\Phi| + \frac{1-P}{4}I_{4\times4}, \tag{22}$$

where $|\Phi\rangle$ is the maximally entangled state given by Eq. (8), the parameter $P\in[0,1]$, and $I_{4\times4}$ denotes a 4×4 identity matrix. This state is a mixed state except for the extreme case of $P=1$. Only when $P>1/3$, is the Werner state ρ_s entangled.

We assume the initial state of the whole system ρ_{tot} in a direct product form

$$\rho_{\text{tot}} = \rho_s \otimes |\psi_E\rangle\langle\psi_E|, \tag{23}$$

where $|\psi_E\rangle$ is the initial state of the environment. After the time evolution, we can obtain the reduce density matrix of the two-qubit system in the basis spanned by $\{|00\rangle, |11\rangle, |01\rangle, |10\rangle\}$ as follows:

$$\rho_{1,2} = \frac{1}{2}\begin{pmatrix} \frac{1+P}{2} & PF(t) \\ PF^*(t) & \frac{1+P}{2} \end{pmatrix} \oplus \left(\frac{1-P}{4}\right)I_{2\times2}, \tag{24}$$

where the decoherence factor $F(t)$ is the same as Eq. (14).

From Eq. (24), the concurrence is derived as

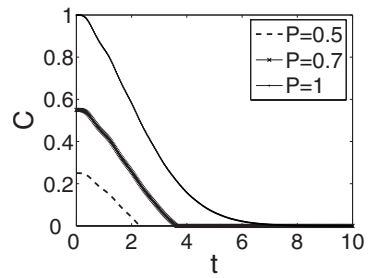

FIG. 4. Concurrence versus time at the critical point $\lambda=2$ and coupling strength $g=0.1$ for parameters $P=0.5$, 0.7, and 1.

$$C = \max\left\{0, P\left(|F| + \frac{1}{2}\right) - \frac{1}{2}\right\}. \tag{25}$$

When $P=1$, it reduces to Eq. (12) for the pure maximally entangled state. While in the region $1/3<P<1$, the concurrence vanishes when the decoherence factor

$$|F| \leqslant (P^{-1}-1)/2. \tag{26}$$

Thus there exists a finite disentanglement time t_d, after which the entanglement is zero. According to the results of heuristic analysis in Eq. (20), $|F(t)|_c \approx e^{-\gamma t^4}$, in the condition of weak coupling and $\lambda\to2$, we can approximately give the disentanglement time

$$t_d = \left(\frac{1}{\gamma}\ln\frac{2P}{1-P}\right)^{1/4}. \tag{27}$$

Then, the disentanglement time increases as the probability P increases from $1/3$ to 1.

In Fig. 4, we also numerically calculate the concurrence versus time for different probabilities. For the mixed states corresponding to $P=0.5$ and 0.7, the disentanglement process takes only a finite time, while for the pure state case $(P=1)$, disentanglement is only completed asymptotically, and will take an infinite time. Numerical results are consistent with the above analytical results that the disentanglement time increases with the increase of P.

IV. DYNAMICAL ENTANGLEMENT EVOLUTION OF TWO QUTRITS

Now, we consider the case of two qutrits and use the negativity [15] to quantify entanglement. For the systems with spin larger than $1/2$, a nonentangled state necessarily has a positive partial transpose (PPT) according to the Peres-Horodecki criterion [15]. In the case of two spin halves, and the case of $(1/2,1)$ mixed spins, a PPT is also sufficient. Vidal and Werner [16] developed the Peres-Horodecki criterion and presented a measure of entanglement called negativity that can be computed efficiently, and the negativity does not increase under local manipulations of the system. The negativity of a state ρ is defined as

$$\mathcal{N}(\rho) = \sum_i |\mu_i|, \tag{28}$$

where μ_i is the negative eigenvalue of ρ^{T_2} and T_2 denotes the partial transpose with respect to the second subsystem. If

$\mathcal{N}>0$, then the two-spin state is entangled. The negativity has been used to characterize the entanglement in a large spin system very well [22–24]. By means of negativity, Derkacz *et al.* have also studied the process of disentanglement in a pair of three-level atoms interacting with the vacuum [8].

A. The case with initial pure state

Similar to the study of the two-qubit case, we write a general initial state of the many-body system as

$$|\Psi(0)\rangle = (a|00\rangle + b|11\rangle + c|22\rangle) \otimes |\psi_E\rangle, \tag{29}$$

where $|0\rangle$, $|1\rangle$, $|2\rangle$ denote the spin-one state with magnetic quantum number 1, 0, −1, respectively. From the evolution operator (7), the state vector at time t is given by

$$|\Psi(t)\rangle = a|00\rangle \otimes U_0|\psi_E\rangle + b|11\rangle \otimes U_1|\psi_E\rangle + c|22\rangle \otimes U_2|\psi_E\rangle, \tag{30}$$

where the unitary operator U_0, U_1, and U_2 are obtained from the unitary operator $U(t)$ by replacing operator $\hat{\Lambda}$ with numbers $\lambda+g$, λ, and $\lambda-g$, respectively.

In the basis spanned by $\{|00\rangle, |11\rangle, |22\rangle, |01\rangle, |10\rangle, |02\rangle, |20\rangle, |12\rangle, |21\rangle\}$, the reduced density matrix of the two-qutrit system is

$$\rho_{1,2} = \begin{pmatrix} |a|^2 & ab^*F_1(t) & ac^*F_2(t) \\ a^*bF_1^*(t) & |b|^2 & bc^*F_3(t) \\ a^*cF_2^*(t) & b^*cF_3^*(t) & |c|^2 \end{pmatrix} \oplus Z_{2\times 2} \oplus Z_{2\times 2} \oplus Z_{2\times 2}, \tag{31}$$

where

$$F_1(t) = \langle\psi_E|U_1^\dagger U_0|\psi_E\rangle,$$

$$F_2(t) = \langle\psi_E|U_2^\dagger U_0|\psi_E\rangle,$$

$$F_3(t) = \langle\psi_E|U_2^\dagger U_1|\psi_E\rangle \tag{32}$$

are the decoherence factors.

The partial transpose with respect to the second system gives

$$\rho_{1,2}^{T_2} = \mathrm{diag}(|a|^2, |b|^2, |c|^2) \oplus B_1 \oplus B_2 \oplus B_3, \tag{33}$$

where the three 2×2 matrices

$$B_1 = \begin{pmatrix} 0 & ab^*F_1(t) \\ a^*bF_1^*(t) & 0 \end{pmatrix},$$

$$B_2 = \begin{pmatrix} 0 & ac^*F_2(t) \\ a^*cF_2^*(t) & 0 \end{pmatrix},$$

$$B_3 = \begin{pmatrix} 0 & bc^*F_3(t) \\ b^*cF_3^*(t) & 0 \end{pmatrix}. \tag{34}$$

Then, from the above matrix $\rho_{1,2}^{T_2}$, one can obtain the negativity as

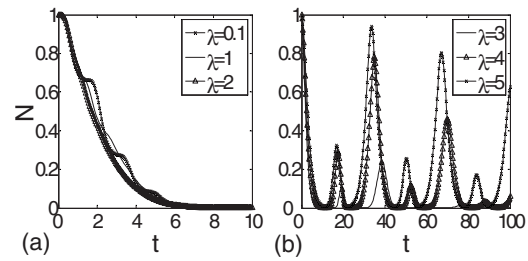

FIG. 5. (a) Negativity versus time with different cases of $\lambda=0.1$, 1, and 2. The coupling $g=0.1$ and the size of environment $L=300$. (b) shows the cases of $\lambda=3$, 4, and 5. The highest one (solid line with up triangles) corresponds to the case $\lambda=5$ and the lowest one (dashed line with points) corresponds to $\lambda=3$.

$$\mathcal{N} = |ab^*F_1(t)| + |ac^*F_2(t)| + |bc^*F_3(t)|. \tag{35}$$

For the maximally entangled state, $a=b=c=1/\sqrt{3}$, and the negativity simplifies to

$$\mathcal{N} = \frac{1}{3}(|F_1(t)| + |F_2(t)| + |F_3(t)|). \tag{36}$$

From the above equation, we can find the negativity is a linear combination of three decoherence factors. Also with the vacuum state of environment, the decoherence factors $|F_\nu(t)| = \langle\psi_E|U_j^\dagger U_i|\psi_E\rangle$ are given by Eq. (14) by the replacements $\Omega_k^{(0)} \to \Omega_k^{(i)}$, $\Omega_k^{(1)} \to \Omega_k^{(j)}$, $\theta_k^{(0)} \to \theta_k^{(i)}$, $\theta_k^{(1)} \to \theta_k^{(j)}$. Here, $F_\nu(t)$ denotes the three factors $F_1(t)$, $F_2(t)$, and $F_3(t)$. $U_j^\dagger U_i$ correspond to $U_1^\dagger U_0$, $U_2^\dagger U_0$, and $U_2^\dagger U_1$ in the three factors Eq. (32). The parameters $\Omega_k^{(n)}$ and $\theta_k^{(n)}$ $(n=0,1,2)$ can be obtained by substituting $\Lambda_0=\lambda+g$, $\Lambda_1=\lambda$, and $\Lambda_2=\lambda-g$ into Eqs. (5) and (6).

During the similar analysis in the case of two qubits, we can also introduce the cutoff number K_c and define the partial product for the three decoherence factors. Through the small k approximation, we can obtain the three partial sums corresponding to the three factors. Therefore, under the condition of weak coupling g and $\lambda \to 2$, in a finite time the three factors $F_1(t)$, $F_2(t)$, and $F_3(t)$ will decay exponentially with time in a similar form as Eq. (20).

We numerically calculate the dynamics of negativity. In Fig. 5(a), it shows phenomena similar to that in Fig. 1(a). When the coupling g is weak and $\lambda \to 2$, the dynamical behaviors of the three decoherence factors in negativity (36) are nearly identical. Each of the factors decay with time just as in Eq. (20), thus it can be understood that negativity also decays monotonously with time in the vicinity of $\lambda=2$. In Fig. 5(b), we consider the cases of larger couplings. Comparing it with Fig. 1(b), the behavior of negativity is different than concurrence. More revivals are found in the behavior of the negativity, and they result from the linear superposition of the three decoherence factors.

In Fig. 6, we numerically study the effects of different couplings g on the dynamics of negativity. Similar to the dynamic behaviors of the concurrence. With a properly large coupling such as $g=1$, the decay of negativity will be much sharper. However, very strong coupling $(g=15)$ will make negativity oscillate rapidly. To the strong coupling limit case

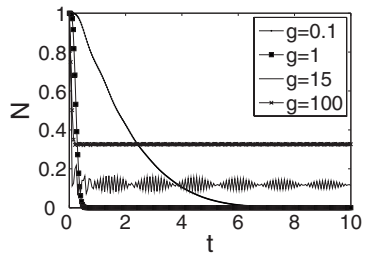

FIG. 6. Negativity versus time with different coupling strengths $g=0.1$, 1, 15, and 100 at the critical point $\lambda_c=2$.

of $g=100$, negativity decays from the initial value $\mathcal{N}=1$ to a steady value $1/3$, which is different than the concurrence of the two qubits. Let us carry out the approximate analysis just as in the case of two qubits. We can obtain three partial sums S_1, S_2, and S_3, corresponding to the three decoherence factors in Eq. (32), which are similar to Eq. (18). When $g\to\infty$ and $\lambda\to 2$, we have $S_2\to 0$ and $S_1=S_3\approx -2E(K_c)t^2$, where $E(K_c)$ is in Eq. (19), thus negativity will decay sharply to a steady value of $1/3$. We can see that different dynamic properties of the factors cause the behaviors of negativity shown in Fig. 6 is different than the concurrence in Fig. 3.

B. The case of mixed state

We then consider the mixed state, namely, the two-qutrit Werner state

$$\rho_s = P|\Phi\rangle\langle\Phi| + \frac{1-P}{9}I_{9\times 9}, \tag{37}$$

where $|\Phi\rangle$ is the maximally entangled state of two qutrits and $|\Phi\rangle=(|00\rangle+|11\rangle+|22\rangle)/\sqrt{3}$. Assume that the whole system is initially in $\rho_{\text{tot}}=\rho_s\otimes|\psi_E\rangle\langle\psi_E|$. After time evolution operator in Eq. (7), we can obtain the reduced density matrix of the two qutrits at arbitrary time t. Then, we make the partial transpose with respect to the second system on the reduced density matrix, and obtain

$$\rho_{1,2}^{T_2} = \frac{1}{9}\,\text{diag}(1+2P,1+2P,1+2P)\oplus B_1\oplus B_2\oplus B_3, \tag{38}$$

where the three 2×2 matrices

$$B_k = \frac{1}{3}\begin{pmatrix} \frac{1-P}{3} & PF_k(t) \\ PF_k^*(t) & \frac{1-P}{3}\end{pmatrix}, \quad k=\{1,2,3\}. \tag{39}$$

From partially transposed reduced density matrix, the negativity is given by

$$\mathcal{N} = \frac{1}{3}\sum_{k=1}^{3}\max\left\{0, P\left(|F_k(t)|+\frac{1}{3}\right)-\frac{1}{3}\right\}. \tag{40}$$

Since $|F_k(t)|\leqslant 1$, the existence of nonzero negativity needs the parameter P satisfying the condition $1/4<P\leqslant 1$. From the above equation, we can also see that the disentanglement occurs only when all three factors satisfy $|F_k(t)|\leqslant(P^{-1}-1)/3$.

Furthermore, we study the case where a d-dimensional Werner state is the initial state. Thus we give the initial state of the system as

$$\rho_s = \frac{P}{d}\sum_{i,j=0}^{d-1}|ii\rangle\langle jj| + \frac{1-P}{d^2}I_{d^2\times d^2}, \tag{41}$$

where the basis vector $|ii\rangle$ is the eigenvector of $s_z=s_{1z}+s_{2z}$ with the eigenvalue $2i+1-d$. Then the initial state of the whole system is also performed by a direct product form as $\rho_{\text{tot}}=\rho_s\otimes|\psi_E\rangle\langle\psi_E|$. After a process similar to that described above, we have the matrix $\rho_{1,2}^{T_2}$ denoting the reduce density matrix after the partial transpose over the second subsystem at time t, which is shown as

$$\begin{aligned}\rho_{1,2}^{T_2} &= \frac{P}{d}\sum_{i,j=0}^{d-1}|ij\rangle\langle ji|F_{i,j}(t) + \frac{1-P}{d^2}I_{d^2\times d^2} \\ &= \frac{1}{d^2}\,\text{diag}[1+(d-1)P,\ldots,1+(d-1)P]_{d\times d} \\ &\quad \oplus_{i<j}\frac{1}{d}\begin{pmatrix}\frac{1-P}{d} & PF_{i,j}(t) \\ PF_{i,j}^*(t) & \frac{1-P}{d}\end{pmatrix},\end{aligned} \tag{42}$$

where the decoherence factors $F_{i,j}(t)=\langle\psi_E|U_j^\dagger U_i|\psi_E\rangle$, and the corresponding time evolution operator U_i can be obtained from Eq. (7) by replacing operator $\hat{\Lambda}$ with value $\lambda+g/2(2i+1-d)$, respectively. It is apparent that we should only focus on the 2×2 matrices and obtain the negativity as

$$\mathcal{N} = \frac{1}{d}\sum_{i<j}\max\left\{0, P\left(|F_{i,j}(t)|+\frac{1}{d}\right)-\frac{1}{d}\right\}, \tag{43}$$

from which we can see that negativity will completely vanish when all the norms satisfy $|F_{i,j}(t)|\leqslant(P^{-1}-1)/d$ simultaneously.

V. CONCLUSION

In summary, we have studied the dynamics of entanglement in a pure dephasing system. By making use of the concept of concurrence, we studied two qubits coupled to an Ising spin chain in a transverse field. When the two qubits initially start from a pure entangled state, we obtain analytical results of concurrence which are just simple products of the initial concurrence $C(0)$ and the decoherence factor $F(t)$. Thus the dynamic properties of concurrence are absolutely determined by the decoherence factor. Especially in the case of weak coupling, the concurrence decays exponentially with time when $\lambda\to\lambda_c$. Moreover, we found that the decay of the decoherence factor is of the form $\exp(-\Gamma t^4)$, which is not a Gaussian form similar to Refs. [11,12]. This is certainly due to the initial state of the environment we have chosen.

Furthermore, when the two qubits are initially in the Werner state, we have found that the complete disentangle-

ment takes place in a finite time just as the "sudden death" of entanglement discovered in Ref. [5]. And due to the process of spontaneous emission, the sudden death of entanglement in Ref. [5] can occur in an arbitrary entangled state (pure or mixed). However, in our system with dephasing effects, when the two entangled qubits are in a pure state, no such phenomena exist.

We also considered two qutrits coupled to the Ising spin chain. When the qutrits initially started from a pure state, we obtained an expression of negativity which is a linear combination of three decoherence factors. With weak coupling, negativity also decays monotonously in the condition of $\lambda \to 2$. When the qutrits were initially in a Werner state, the complete disentanglement could occur in a finite time, and then the properties of negativity were the three decoherence factors. Indeed, the correlated environment, especially when QPT happens, greatly affects the decoherence and disentanglement process. The entanglement decay in other environments which displays a QPT [25], or quantum chaos [26] deserves further investigation.

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[1] A. Einstein, B. Podolsky, and N. Rosen, Phys. Rev. **47**, 777 (1935).
[2] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, Cambridge, England, 2000).
[3] C. H. Bennett, G. Brassard, C. Crépeau, R. Jozsa, A. Peres, and W. K. Wootters, Phys. Rev. Lett. **70**, 1895 (1993).
[4] A. K. Ekert, Phys. Rev. Lett. **67**, 661 (1991).
[5] T. Yu and J. H. Eberly, Phys. Rev. Lett. **93**, 140404 (2004).
[6] M. S. Zubairy, G. S. Agarwal, and M. O. Scully, Phys. Rev. A **70**, 012316 (2004).
[7] P. J. Dodd, Phys. Rev. A **69**, 052106 (2004).
[8] L. Derkacz and L. Jakóbczyk, Phys. Rev. A **74**, 032313 (2006).
[9] K. Roszak and P. Machnikowski, Phys. Rev. A **73**, 022313 (2006).
[10] F. M. Cucchietti, J. P. Paz, and W. H. Zurek, Phys. Rev. A **72**, 052113 (2005).
[11] H. T. Quan, Z. Song, X. F. Liu, P. Zanardi, and C. P. Sun, Phys. Rev. Lett. **96**, 140604 (2006).
[12] F. M. Cucchietti, S. Fernandez-Vidal, and J. P. Paz, Phys. Rev. A **75**, 032337 (2007).
[13] R. F. Werner, Phys. Rev. A **40**, 4277 (1989).
[14] W. K. Wootters, Phys. Rev. Lett. **80**, 2245 (1998).
[15] A. Peres, Phys. Rev. Lett. **77**, 1413 (1996); M. Horodecki, P. Horodecki, and R. Horodecki, Phys. Lett. A **223**, 1 (1996).
[16] G. Vidal and R. F. Werner, Phys. Rev. A **65**, 032314 (2002).
[17] S. Sachdev, *Quantum Phase Transition* (Cambridge University Press, Cambridge, England, 1999).
[18] Y. D. Wang, F. Xue, and C. P. Sun, eprint arXiv:quant-ph/0603014 (unpublished).
[19] C. Emary and T. Brandes, Phys. Rev. Lett. **90**, 044101 (2003).
[20] J. Emerson, Y. S. Weinstein, S. Lloyd, and D. G. Cory, Phys. Rev. Lett. **89**, 284102 (2002).
[21] K. Koshino and A. Shimizu, Phys. Rep. **412**, 191C275 (2005).
[22] J. Schliemann, Phys. Rev. A **68**, 012309 (2003).
[23] X. Wang, H. B. Li, Z. Sun, and Y. Q. Li, J. Phys. A **38**, 8703 (2005).
[24] Z. Sun, X. Wang, and Y. Q. Li, New J. Phys. **7**, 83 (2005).
[25] N. Lambert, C. Emary, and T. Brandes, Phys. Rev. Lett. **92**, 073602 (2004).
[26] H. Fujisaki, T. Miyadera, and A. Tanaka, Phys. Rev. E **67**, 066201 (2003).