Quantum-state transfer characterized by mode entanglement

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We study the quantum-state transfer (QST) of a class of tight-bonding Bloch electron systems with mirror symmetry by considering the mode entanglement. Some rigorous results are obtained to reveal the intrinsic relationship between the fidelity of QST and the mirror mode concurrence (MMC), which is defined to measure the mode entanglement with a certain spatial symmetry and is just the overlap of a proper wave function with its mirror image. A complementarity is discovered as the maximum fidelity is accompanied by a minimum of the MMC. At the instant that is just half of the characteristic time required to accomplish a perfect QST, the MMC can reach its maximum value of 1. A large class of perfect QST models with a certain spectral structure is discovered to support our analytical results.

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I. INTRODUCTION

Quantum entanglement is a fascinating feature of quantum theory of many-body systems [1]. The concurrence [2], as a widely used measure of pairwise entanglement defined for spin–1/2 systems, has been intensively investigated. Through various concurrences defined by different authors, people have explored the relations between entanglement and some physical observables such as energy and momentum, etc. [3,4], as well as the relations between entanglement and some physical phenomena, such as quantum correlation [5] and quantum phase transitions, etc. [4,6–8].

On the other hand, people have proposed many protocols for quantum-state transfer (QST) recently [9–13]. In these schemes based on quantum spin systems, almost without any spatial or dynamical control over the interactions among qubits, the quantum state can be transferred with high fidelity through a quantum channel, or quantum data bus, which is necessary for scalable quantum computations based on realistic silicon devices. The physical process of QST through a quantum spin system can be understood as a dynamical permutation or translation preserving the initial shape of a quantum state, which can be realized as a specific evolution of the total quantum spin system from an initial wave function localized around a single site of the lattice to a distant one. The basic feature of QST is characterized by fidelity, which is usually the overlap of the identical image of an initial state with its transferred counterpart.

This paper will be devoted to understanding the intrinsic relation between quantum entanglement and QST for the engineered quantum spin chains, or quantitatively, between concurrence and fidelity. Some rigorous results are obtained to reveal the essential relationship between these two fascinating issues for the tight-bonding Bloch electrons. Actually, the QST from one location to another can be considered as perfect if the fidelity can reach its maximum value 1 at some instants. Literally, the perfect QST is a dynamic process starting from an initially factorized state (product state) to a finally factorized state through a middle process with the superposition of factorized states. Since a superposition of single-particle states of Bloch electrons can be understood as a mode entanglement [14], the studies of QST can be naturally referred to the various phenomena of quantum entanglement.

Motivated by arguments about the entanglement concurrence and the quantum correlations [2,14], we first define the mirror mode concurrence (MMC) $C(t)$ to characterize the mode entanglement of a wave packet in Bloch electron systems with mirror symmetry. It will be proved that the MMC is no less than the overlap of the wave packet at time t with its mirror image. By quantitatively comparing the MMC with the time-dependent fidelity $F(t)$ of QST, a complementary relation is discovered as the increase of $F(t)$ is accompanied by a decrease of $C(t)$ (and vice versa). In particular, at the instant $\tau/2$, where τ is the characteristic time to accomplish a perfect QST with $F(\tau)=1$, the MMC can reach its maximum

$$C(\tau/2) = \max[C(t)] = 1. \quad (1)$$

An engineered Bloch electron model with a certain spectrum structure, which admits perfect QST, is discovered and used to demonstrate this complementary relation through numerical simulations.

II. ONE-DIMENSIONAL BLOCH ELECTRON SYSTEM WITH MIRROR SYMMETRY

We consider a one-dimensional Bloch electron system in an engineered crystal lattice of N sites with mirror symmetry with respect to the center of the lattice. The model Hamiltonian with tight-bonding approximation is written as

$$H = \sum_{j=1}^{N-1} J_j a_j^\dagger a_{j+1} + \text{H.c.} \quad (2)$$

in terms of the fermion creation (annihilation) operator $a_j^\dagger(a_j)$, where the site-dependent coupling constants J_j are

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real. The single-particle space is spanned by N basis vectors

$$\begin{aligned}|1\rangle &= |1,0,0,\ldots,0,0\rangle,\\ |2\rangle &= |0,1,0,\ldots,0,0\rangle,\\ &\vdots\\ |N\rangle &= |0,0,0,\ldots,0,1\rangle\end{aligned} \tag{3}$$

where $|n_1,n_2,\ldots,n_N\rangle$ ($n_j=0,1$) denotes the Fock state of fermion systems. Then the reflection operator R is defined as $R|j\rangle=|N+1-j\rangle$. Obviously, the mirror symmetry of $[R,H]=0$ means that $J_j=J_{N-j}$.

We describe the localized electron state around the lth site as a superposition $|\psi_l\rangle=\Sigma_j c_j|j\rangle$ with the summation over the small domain containing the site l. This assumption means that $|\psi_l\rangle$ is a wave packet around the site l. If l' denotes another site far from the site l, we can approximately assume the vanishing overlap $\langle\psi_l|\psi_{l'}\rangle\simeq 0$ for two wave packets $|\psi_l\rangle$ and $|\psi_{l'}\rangle$. With this assumption the perfect QST is described as the dynamic process that the initial state $|\psi_l\rangle$ can evolve exactly into its mirror image. Mathematically, the time evolution operator $U(t)=\exp(-iHt)$ becomes the reflection operator R at the instant τ, i.e., $U(\tau)=R$. We define the fidelity as

$$F_j(t) = |\langle R\psi_j|U(t)|\psi_j\rangle| = |\langle\psi_j|R^\dagger U(t)|\psi_j\rangle|. \tag{4}$$

A perfect QST can be depicted by the maximized fidelity $F_j(\tau)=1$.

Now we can intuitively recognize that QST phenomenon is associated with the mode entanglement. In the terminology of mode entanglement, the single-electron state

$$|E\rangle = \alpha|1\rangle + \beta|N\rangle \equiv \alpha|1,0,\ldots,0\rangle + \beta|0,0,\ldots,1\rangle \tag{5}$$

can be regarded as an entangled state if the single fermion at the first site and Nth site can be probed in principle [14]. In this sense $|\psi_l\rangle$ can be viewed as an N-component entanglement. The perfect QST from $|1\rangle$ to $|N\rangle$ through the middle state $|\psi(t)\rangle=U(t)|1\rangle$ can be understood as a dynamic process starts from a localized (unentangled) state $|1\rangle$ to another localized state $|N\rangle$ through the entangled state $|\psi(t)\rangle$.

III. MIRROR MODE CONCURRENCE AS THE FINGERPRINT OF PERFECT QST

Actually a QST is a process, during which mode entanglement is generated first and then destroyed. To quantitatively characterize this dynamic feature, we define the mirror mode concurrence

$$C(t) = \sum_{j=1}^{N/2} C_{j,N+1-j} \tag{6}$$

with respect to a pure state, evolved from a localized initial wave packet $|\psi(t)\rangle=U(t)|\psi(0)\rangle$. Here each term $C_{j,N+1-j}$ in the summation concerns two separated sites, the site j and its mirror image $l=N+1-j$, and is defined by the pairwise mode concurrence [14]

$$C_{jl} = 2\max\left(0,|Z_{jl}| - \sqrt{X_{jl}^{+}X_{jl}^{-}}\right), \tag{7}$$

constructed in terms of the correlation functions

$$Z_{jl} = \langle a_j^\dagger a_l\rangle, \quad X_{jl}^{+} = \langle\hat{n}_j\hat{n}_l\rangle, \tag{8}$$

and

$$X_{jl}^{-} = \langle(1-\hat{n}_j)(1-\hat{n}_l)\rangle, \tag{9}$$

where the average $\langle\ \rangle$ is defined with respect to the pure state $|\psi(t)\rangle$.

The physical significance is twofold and explicitly reveals the close relationship between the mode entanglement and the dispersion of the wave packet in time evolution. First we notice that Z_{jl} and X_{jl}^{-} are the nonzero elements of the two-mode reduced density matrix [14]

$$\rho_{jl} = \mathrm{tr}_{N-2}[|\psi(t)\rangle\langle\psi(t)|] = \begin{pmatrix} X_{jl}^{+} & & & \\ & Y_{jl}^{+} & Z_{jl}^{*} & \\ & Z_{jl} & Y_{jl}^{-} & \\ & & & X_{jl}^{-}\end{pmatrix} \tag{10}$$

for a system conserving total particle number, where tr_{N-2} means tracing over the variables except for the two on the sites j and $l=N+1-j$. In the single-particle subspace we have $X_{jl}^{+}=0$ and thus

$$C(t) = \sum_{j=1}^{N} |Z_{j,N+1-j}|. \tag{11}$$

It is obvious that MMC is a generalization of the usual entanglement measure—concurrence—and thus characterizes the quantum entanglement in some sense.

Second, the MMC defined above has a geometric interpretation for the dynamic dispersion of the wave packet. We rewrite the MMC as

$$C(t) = \sum_{j=1}^{N} |\psi(j,t)||\psi(N+1-j,t)| \tag{12}$$

where $\psi(j,t)=\langle j|\psi(t)\rangle$. It is easy to show that

$$C(t) \geqslant \left|\sum_{j=1}^{N}\langle\psi(t)|j\rangle\langle j|R|\psi(t)\rangle\right| = |\langle\psi(t)|R|\psi(t)\rangle| \tag{13}$$

where we have used $RR^\dagger=R^\dagger R=1$. The above equation clearly implies that $C(t)$ is no less than the overlap integral of the state $|\psi(t)\rangle$ with its mirror image. In particular, for a large class of states $|\psi(t)\rangle=\Sigma_{j=1}^{N}c_j|j\rangle$ listed in two situations as follows, $C(t)$ is exactly equal to the overlap integral. (i) The electronic wave function is completely localized in a finite domain $D=[1,N/2]$ with no overlap with its mirror image $[N/2,N]$. In this case, the MMC vanishes exactly. (ii) The coefficients of each pair of mirror-symmetric nonzero components in $|\psi(t)\rangle$ have the same or opposite signs.

For a perfect QST accomplished at the instant $t=\tau$, the evolution operator $U(\tau)$ becomes the reflection operator R and $|\psi(0)\rangle$ evolves exactly into its mirror image $R|\psi(0)\rangle$. Since the initial wave packet $|\psi(0)\rangle$ is usually a very localized wave function, the wave function $|\psi(\tau)\rangle=R|\psi(0)\rangle$ and its mirror image $|\psi(0)\rangle$ almost do not overlap with each other (see the illustration in Fig. 1). Thus we have

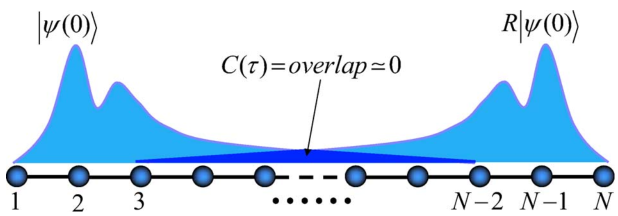

FIG. 1. (Color online) Illustration of the mirror mode concurrence $C(t)$ at $t=\tau$. $C(\tau)$ is just the integral of the wave function $R|\psi(0)\rangle$ with its mirror image when $|\psi(0)\rangle$ satisfies either of the two situations (i) and (ii).

$$C(\tau) = C(0) \geqslant |\langle \psi(0)|R|\psi(0)\rangle| \simeq 0. \tag{14}$$

Therefore, at $t=\tau$, the MMC $C(\tau)$ almost vanishes when the fidelity $F(t)$ reaches its maximum $(F(\tau)=1)$.

From the above argument we see that there exists a quite interesting relationship between entanglement and fidelity. We provide a model more universal than the QST model in Ref. [10]. Their mode is a mapping to the collective spin system with SU (2) dynamic symmetry [by $J_j=J_0\sqrt{j(N-j)}$ more concretely], but our model only requires a much smaller mirror symmetry ($J_j=J_{N-j}$ more generally) and thus has much wider applications. In fact we have shown many examples in Ref. [12] as well as in the following discussions. Further arguments about the complementarity relationship between entanglement and fidelity will also be presented in such a general framework.

IV. MAXIMAL MODE ENTANGLEMENT

The above analysis has confirmed our intuition about the complementary relation between the fidelity of QST and the MMC of mode entanglement. As for the other feature of this complementary relation, we need to consider when the MMC can reach its maximum.

Obviously there exists the inequality

$$C(t) \leqslant \frac{1}{2}\sum_{j=1}^{N} \left[|\psi(j,t)|^2 + |\psi(N+1-j,t)|^2\right] = 1, \tag{15}$$

which takes the equals sign only when the wave function evolves into its mirror image, i.e.,

$$|\psi(j,t)| = |\psi(N+1-j,t)| \tag{16}$$

at some instants t. This means that $C(t)$ will reach its maximum $\max(C(t))=1$ at the instants when Eq. (16) holds.

In order to determine the time when $C(t)$ reaches its maximum, we need to solve Eq. (16) for time t. To this end we use a time-independent real symmetric matrix W to diagonalize the Hamiltonian H or the evolution operator $U(t)$ as $WU^{\dagger}(t)W^{T}=A(t)$, where $A(t)$ is a diagonal matrix. With these notations, the above equation (16) can be transformed into

$$|\langle \psi_W|A(t)|W_j\rangle| = |\langle \psi_W|Q(t)|W_j\rangle|, \tag{17}$$

where

$$|\psi_W\rangle = W|\psi(0)\rangle,$$

$$|W_j\rangle = W|j\rangle,$$

$$Q(t) = WU^{\dagger}(t)U(\tau)W^{T}. \tag{18}$$

We notice that, in general, $|W_j\rangle = W|j\rangle$. $|\psi_W\rangle$ and $|W_j\rangle$ are real for a real initial state $|\psi(0)\rangle$. Then the solutions to Eq. (17) are sufficiently given by $Q(t)=A^{\dagger}(t)$ or $Q(t)=A(t)$, of which the nontrivial one is just $t=\tau/2$. Indeed, since $\langle\alpha|A|\beta\rangle=\langle\beta|A|\alpha\rangle$ for any two real vectors $|\alpha\rangle$ and $|\beta\rangle$, we have

$$\begin{aligned}|\langle \psi_W|Q(t)|W_j\rangle| &= |\langle \psi_W|A^{\dagger}(t)|W_j\rangle| \\ &= |\langle W_j|A^{\dagger}(t)|\psi_W\rangle| \\ &= |\langle \psi_W|A(t)|W_j\rangle|.\end{aligned} \tag{19}$$

Therefore, the solution $t=\tau/2$ is obviously given by $Q(t)=A^{\dagger}(t)$ or

$$U^{\dagger}(t)U(\tau) = U(t). \tag{20}$$

We summarize the above argument as a proposition: If $F(t)$ reach its maximum 1 at the instant $t=\tau$, then at time $t=\tau/2, C(\tau/2)=1$. In the Appendix, we will prove its inverse proposition: if $C(t)$ reach its maximum 1 at the instant $t=\tau/2$, then at time $t=\tau$, $F(\tau)=1$. Furthermore, we can generalize these conclusions for the more general situation even with a higher-dimensional Hamiltonian (also see the Appendix).

The solution $t=\tau/2$ to Eq. (17) indicates that the time required to form the maximal mode entanglement is just half of the time needed to implement the perfect QST. Furthermore we can prove that, for a real vector $|\psi(0)\rangle$, the MMC $C(t)$ is symmetric with respect to both $t=\tau/2$ and $t=\tau$, namely,

$$C\left(\frac{\tau}{2}-t\right) = C\left(\frac{\tau}{2}+t\right),$$

$$C(\tau-t) = C(\tau+t). \tag{21}$$

Actually, for the second equation in Eqs. (21) we have

$$C(\tau \pm t) = \sum_{j=1}^{N} |\langle \psi(0)|U_{\pm}(t)R^{\dagger}|j\rangle||\langle \psi(0)|U_{\pm}(t)|j\rangle|, \tag{22}$$

where $U(t)_{+}=U^{\dagger}(t)$ and $U(t)_{-}=U(t)$. Obviously the second equation in Eqs. (21) holds since we have

$$|\langle \psi(0)|U(t)V|j\rangle| = |\langle \psi(0)|U^{\dagger}(t)V|j\rangle| \tag{23}$$

for $V=1, R^{\dagger}$. Also, the first equation in Eqs. (21) will give a similar proof.

Numerical methods are now employed to give a demonstration of the above analytical results. We concern a class of schemes that admit perfect QST, which are presented in Ref. [11]. The couplings of the Hamiltonian H are given such that

$$J_j = J_0\sqrt{(j+\theta_j k)(N-j+\theta_j k)} \tag{24}$$

where $\theta_j=1-(-1)^j, k=0, 1, 2, \ldots,$ and J_0 is a constant. This model possesses a commensurate structure of energy spec-

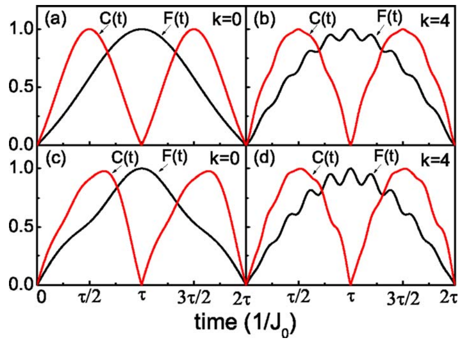

FIG. 2. (Color online) Plots of $C(t)$ and $F(t)$ of the states $|\psi(t)\rangle$ evolve from real [(a) and (b)] and complex [(c) and (d)] initial wave packets $|\psi_{1,2}(0)\rangle$. The evolutions of $|\psi(t)\rangle$ in (a) and (c) [(b) and (d)] are driven by the four-site Hamiltonian with $k=0(k=4)$. For (a) and (b), $F(\tau)=1$ and $C(\tau/2)=1$, while for (c) and (d), $F(\tau)=1$ and $\max[C(t)], C(\tau/2)\neq 1$.

trum that is matched with the corresponding parity. We demonstrate the exact numerical results of the models with $N=4$ and $k=0,4$ in Figs. 2(a) and 2(b). Actually, when $k=0$ [Fig. 2(a)], the model is just the one proposed in Ref. [10]. We have used the localized initial wave packet as $|\psi_{1,2}(0)\rangle = c_1|1\rangle + c_2|2\rangle$, where $c_1=5/6, c_2=\sqrt{11/36}$. From Figs. 2(a) and 2(b) we can observe that $C(0)=C(\tau)=0, C(\tau/2)=1$, and $C(t)$ is symmetric with respect to $t=\tau,\tau/2$. These results are in agreement with our analytical results. It also implies the complementary relation between MMC and fidelity, for inside the range from $t=\tau/2$ to $3\tau/2$, the increase of $F(t)$ is accompanied by a decrease of $C(t)$ (and vice versa).

It is pointed out that our results about MMC $C(t)$ at $t=\tau/2$ are based on the condition that the initial wave packet $|\psi(0)\rangle$ is real except for a global phase. One may be interested in the situation when c_1 and c_2 are not real for $|\psi_{1,2}(0)\rangle$. For this situation, e.g., $c_1=(1+i)/2, c_2=1/5+i\sqrt{23/50}$, the numerical calculation shows that $C(t)$ is not just symmetric with respect to $t=\tau/2, C(\tau/2)\neq 1, \max[C(t)]$ is very close, yet not equal to 1 and $C(\tau/2)\neq \max[C(t)]$ [see Figs. 2(c) and 2(d)].

V. PERFECT QST OF BLOCH ELECTRONS IN AN ENGINEERED LATTICE

Based on the above recognitions about the relation between a perfect QST and mode entanglement, we can construct various lattice models with mirror symmetry to achieve perfect QST. Furthermore we can characterize these QSTs with the MMC. Actually, a large class of models for QST have been discovered by us most recently [11] by generalizing the spin model in Ref. [10].

Now we further generalize the perfect QST model to a much larger class. The Hamiltonian is given in Eq. (2) with the engineered coupling constants

$$J_j = J_0\sqrt{(j+\xi_j)(N-j+\xi_j)}, \tag{25}$$

where

$$\xi_j = [1-(-1)^j]l/(2m+1) \tag{26}$$

for the given $m,l \in 0, 1, 2, 3, \ldots$ We notice that it will return to the previous models in Refs. [10,11] when $m=0$.

Numerical analysis shows that the above Hamiltonian possesses a commensurate structure of energy spectrum by an experiential formula

$$\varepsilon_n = N_n E_0 - (N+1)J_0, \tag{27}$$

where the energy unit is

$$E_0 = \frac{2J_0}{2m+1}, \tag{28}$$

$N_n=n(2m+1)-l$ for $n=1,2,\ldots,N/2$, and $N_n=n(2m+1)+l$ for $n=N/2+1,\ldots,N$. Numerical results show that the above experiential formula (27) still holds when $N=3000$. It can be checked that the energy spectrum is matched with the corresponding parity (the eigenvalue of R) as

$$p_n = (-1)^{N_n}\exp\left\{i\left[\left(m+\frac{1}{2}\right)N+1\right]\pi\right\}. \tag{29}$$

The corresponding eigenstates $|\varphi_n\rangle = \Sigma_{j=1}^N c_j(n)|j\rangle$ can be determined by the matrix equation $H|\varphi_n\rangle = \varepsilon_n|\varphi_n\rangle$.

According to Refs. [11,12], the characteristic time to perform a perfect QST is $\tau=\pi/E_0$, provided that $l/(2m+1)$ is an irreducible fraction. Now we can show that, at $t=\tau$, the time evolution operator

$$U(t) = \sum_n \exp(-i\varepsilon_n t)|\varphi_n\rangle\langle\varphi_n| \tag{30}$$

is just the mirror reflection operator R by neglecting a global phase, namely,

$$U(\tau) = \sum_n (-1)^{N_n}|\varphi_n\rangle\langle\varphi_n| = (-1)^l R. \tag{31}$$

Thus, the present model admits perfect QSTs when

$$\xi_j = [1-(-1)^j]l/(2m+1). \tag{32}$$

In order to verify the prediction about the relation between the MMC and fidelity, a numerical analysis is carried out for the present QST model. We investigate the four-site case with $m=1$ and $l=2$. The real initial wave packet is also $|\psi_{1,2}(0)\rangle$. Detail behaviors of the MMC and fidelity between the instants $t=0$ and $t=2\tau$ are shown in Fig. 3. We notice, in Fig. 3, that $C(t)=0$, 1, 0, for $t=0,\tau/2,\tau$ respectively and $C(t)$ is symmetric with respect to $t=\tau,\tau/2$. Obviously, it is in agreement with our prediction.

VI. SUMMARY

In summary we have defined the mirror mode concurrence to describe how a perfect quantum-state transfer can be achieved for a large class of lattice models of fermion systems with mirror symmetry. By investigating the property of MMC of these perfect QST models, a complementary relation between the MMC and fidelity is revealed. Actually our definition of MMC is just a part of total concurrence [4,15].

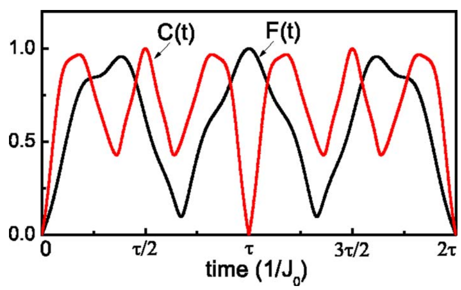

FIG. 3. (Color online) Plots of $C(t)$ and $F(t)$ of the state $|\psi(t)\rangle = U(t)|\psi_{1,2}(0)\rangle$ for the Hamiltonian with $N=4, m=1$, and $l=2$. It shows that $C(0)=C(\tau)=0, C(\tau/2)=1$, and $C(t)$ is symmetric with respect to $t=\tau, \tau/2$.

However, when the symmetry of our systems is taken into consideration, MMC is a better measurement in characterizing the process of a perfect QST. A class of QST models is discovered to support our observations. Therefore, a perfect QST can now be understood as a process of establishing an entanglement and then destroying it at the correlated instants. Finally we remark that our main results are valid in other perfect QST models with general symmetries such as translation, rotation, etc. It is very interesting to further investigate the QST vs entanglement relation based on solid state systems with the symmetries described by point groups or the crystallographic space groups.

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APPENDIX: A GENERAL PROOF FOR COMPLEMENTARITY $F(\tau)=1 \Leftrightarrow C(\tau/2)=1$

We have proved that the MMC $C(t)$ will reach its maximum 1 at the instant $t=\tau/2$ where τ is the instant, at which the fidelity $F(t)$ reaches its maximum $F(\tau)=1$. We now prove the inverse proposition: If $C(t)$ reaches its maximum 1 at the instant $t=\tau/2$, then at time $t=\tau, F(\tau)=1$, namely, we have a theorem in a sufficient and necessary statement

$$F(\tau) = 1 \Leftrightarrow C(\tau/2). \tag{A1}$$

In this appendix, we will prove the above theorem for a general model even for higher-dimensional fermion systems with a Hamiltonian,

$$H = \sum_{i\neq j}^{N} J_{ij} a_i^\dagger a_j, \tag{A2}$$

on the one-particle Fock space spanned by N basis vectors $\{|j\rangle\}, j=1,2,3,\ldots,N$. Suppose the Hamiltonian has a symmetry S and $[S,H]=0$, and the basis vectors can be decomposed into two subspaces $\{|n_j\rangle | j=1,2,3,\ldots,N/2\}$ and $\{|m_j\rangle | j=1,2,3,\ldots,N/2\}$ such that

$$S|n_j\rangle = |m_j\rangle, \quad S|m_j\rangle = |n_j\rangle, \tag{A3}$$

then perfect QST requires that at a certain instant $t=\tau, U(\tau)=\exp(-iH\tau)=S$. This case of Eq. (A2) is just a generalization of the situation of mirror symmetry Hamiltonian.

Through the definition of total concurrence

$$C(t) = \sum_{j}^{N/2} C_{n_j,m_j} = \sum_{j}^{N/2} 2|\langle\psi(t)|a_{n_j}^\dagger a_{m_j}|\psi(t)\rangle|, \tag{A4}$$

we can first prove the proposition from $F(\tau)=1$ to $C(\tau/2)=1$.

As for an initial state $|\psi(0)\rangle$, the fidelity of a state $|\psi(t)\rangle = U(t)|\psi(0)\rangle$ reads as

$$F(t) = |\langle S\psi(0)|U(t)|\psi(0)\rangle| = |\langle\psi(0)|S^{+}U(t)|\psi(0)\rangle|, \tag{A5}$$

and a perfect QST at $t=\tau$ can be depicted by the maximized fidelity $F(\tau)=1$ when

$$U(\tau/2)U(\tau/2) = U(\tau) = S$$

satisfies Eq. (A2). We calculate the total concurrence of $|\psi(t)\rangle$ as

$$\begin{aligned} C(t) &= \sum_{j}^{N/2} 2|\langle\psi(0)|U^{+}(t)a_{n_j}^\dagger a_{m_j}U(t)|\psi(0)\rangle| \\ &= \sum_{j=1}^{N/2} 2|\langle\psi(0)|U^{+}(t)|n_j\rangle\langle m_j|SS^{+}U(t)|\psi(0)\rangle| \\ &= \sum_{j=1}^{N/2} 2|\langle\psi(0)|U^{+}(t)|n_j\rangle||\langle n_j|U^{+}(t)|\psi(0)\rangle|. \end{aligned} \tag{A6}$$

Then at the instant $t=\tau/2$

$$C(\tau/2) = \sum_{j=1}^{N/2} 2|\langle\psi(0)|U^{+}(\tau/2)|n_j\rangle||\langle n_j|U^{+}(\tau/2)|\psi(0)\rangle|. \tag{A7}$$

For real $|\psi(0)\rangle$ we have $|\langle\psi(0)|U^{+}(\tau/2)|n_j\rangle| = |\langle n_j|U^{+}(\tau/2)|\psi(0)\rangle|$ and then

$$C(\tau/2) = \sum_{j=1}^{N/2} 2|\langle\psi(0)|U^{+}(\tau/2)|n_j\rangle|^2 = 1. \tag{A8}$$

Thus we have

$$\max[C(t)] = C(\tau/2) = 1. \tag{A9}$$

Now we prove the proposition from $C(\tau/2)=1$ to $F(\tau)=1$. According to Eq. (A5), if we require $C(\tau/2)=\max[C(t)]=1$ at some instant $t=\tau/2$, then

$$|\langle\psi(0)|U^{+}(\tau/2)|n_j\rangle| = |\langle m_j|U(\tau/2)|\psi(0)\rangle|. \tag{A10}$$

Therefore we have

$$|\langle\psi(0)|U^{+}(\tau/2)|n_j\rangle| = |\langle m_j|U(\tau/2)|\psi(0)\rangle|,$$

or

$$|\langle\psi(0)|U^{+}(\tau/2)|n_j\rangle| = |\langle\psi(0)|U^{+}(\tau/2)|m_j\rangle|,$$

or

$$|\langle\psi(0)|U^{+}(\tau/2)|n_j\rangle| = |\langle\psi(0)|U^{+}(\tau/2)S|n_j\rangle|$$

This means $U^{+}(\tau/2)S=U^{+}(\tau/2)$ or $U^{+}(\tau/2)S=U(\tau/2)$. It has a trivial solution $S=1$ and an approved nontrivial solution $S=U(\tau)$. With the nontrivial solution $S=U(\tau)$, there will be a perfect QST, i.e.,

$$F(\tau) = |\langle S\psi(0)|U(\tau)|\psi(0)\rangle| = |\langle\psi(0)|S^{+}U(\tau)|\psi(0)\rangle| = 1. \tag{A11}$$

As is stands, we have verified the theorem $F(\tau) = 1 \Leftrightarrow C(\tau/2)$ in a general situation.

[1] Michael A. Nielsen and Isaac L. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, Cambridge, U.K., 2001).
[2] W. K. Wootters, Phys. Rev. Lett. **80**, 2245 (1998).
[3] K. M. O'Connor and W. K. Wootters, Phys. Rev. A **63**, 052302 (2001).
[4] X. F. Qian, T. Shi, Y. Li, Z. Song, and C. P. Sun, Phys. Rev. A **72**, 012333 (2005).
[5] X. Wang and P. Zanardi, Phys. Lett. A **301**, 1 (2002); X. Wang, Phys. Rev. A **66**, 034302 (2002); R. Xin, Z. Song, and C. P. Sun, Phys. Lett. A **342**, 30 (2005).
[6] A. Osterloh, L. Amico, G. Falci, and R. Fazio, Nature (London) **416**, 608 (2002).
[7] S. J. Gu, S. S. Deng, Y. Q. Li, and H. Q. Lin, Phys. Rev. Lett. **93**, 086402 (2004).
[8] Y. Chen, P. Zanardi, Z. D. Wang, and F. C. Zhang, e-print quant-ph/0407228.
[9] S. Bose, Phys. Rev. Lett. **91**, 207901 (2003).
[10] M. Christandl, N. Datta, A. Ekert, and A. J. Landahl, Phys. Rev. Lett. **92**, 187902 (2004).
[11] T. Shi, Y. Li, Z. Song, and C. P. Sun, Phys. Rev. A **71**, 032309 (2005).
[12] Y. Li, Z. Song, and C. P. Sun, e-print quant-ph/0504175.
[13] P. Karbach and J. Stolze, Phys. Rev. A **72**, 030301(R) (2005).
[14] P. Zanardi and X. Wang, J. Phys. A **35**, 7947 (2002).
[15] M.-F. Yang, Phys. Rev. A **71**, 030302(R) (2005).