Quantum-state transmission via a spin ladder as a robust data bus

Ying Li,[1] Tao Shi,[1] Bing Chen,[1] Zhi Song,[1,*] and Chang-Pu Sun[1,2,†,‡]
[1]*Department of Physics, Nankai University, Tianjin 300071, China*
[2]*Institute of Theoretical Physics, The Chinese Academy of Science, Beijing 100080, China*
(Received 7 June 2004; revised manuscript received 24 November 2004; published 3 February 2005)

We explore the physical mechanism to coherently transfer the quantum information of spin by connecting two spins to an isotropic antiferromagnetic spin ladder system as data bus. Due to a large spin gap existing in such a perfect medium, the effective Hamiltonian of the two connected spins can be archived as that of Heisenberg type, which possesses a ground state with maximal entanglement. We show that the effective coupling strength is inversely proportional to the distance of the two spins and thus the quantum information can be transferred between the two spins separated by a longer distance, i.e., the characteristic time of quantum-state transferring linearly depends on the distance.

DOI: 10.1103/PhysRevA.71.022301 PACS number(s): 03.67.Hk, 03.67.Pp, 03.65.−w, 05.50.+q

Transferring a quantum state from a quantum bit to another is not only the central task in the quantum communication but also is often required in scalable quantum computing based on the quantum network [1]. In the latter, one should connect different quantum predeceasing units in different locations with a medium called data bus. The typical examples of quantum state transfer is the quantum storage based on various physical systems [2,3], such as the quasispin wave excitations [4]. For the solid-state based quantum computing at the large scale, it is very crucial to have a solid system serving as such quantum data bus, which can provide us with a quantum channel for quantum communication [5]. Most recently the simple spin chain, a typical solid-state system, is considered as a coherent data bus [6–8]. The quantum transmission of state is achieved by placing two spins at the two ends of the chain. These schemes may admit an efficient state transfer of any quantum state in a fixed period of time of the state evolution, but the crucial problem is the dependence of transferring efficiency on communication distance. Usually the efficiency is inversely proportional to square or higher-order power of the distance of the two spins, and thus such quantum-state transmission can only work efficiently in a much shorter distance.

The aim of this paper is to solve this short-distance transfer problem by replacing the simple spin chain with an isotropic antiferromagnetic spin ladder. Because this kind of spin ladder possesses a finite spin gap, an effective Heisenberg interaction can be induced in the stable ground-state channel to achieve the maximally entangled states that implement a faster quantum states transfer of two spin qubits attached to this spin ladder system. Actually, when the spin gap is sufficiently large comparing to the coupling strength between two spin qubits and the spin ladder, the perturbation method can be performed. Analytical and numerical results show that the spin ladder system is a perfect medium through which the interaction between two distant spins can be mapped to an approximate Heisenberg-type coupling with a coupling constant inversely proportional to the distance between the two separated spins.

It is well known that there are two ways to transfer quantum information: one can first use the channel to share entanglement with separated Alice and Bob and then use this entanglement for teleportation [9], or directly transmit a state through a quantum data bus. For the latter it seems that the long-distance entanglement is not necessary to interface different kinds of physical systems but it will be shown in this paper that there hides an effective entanglement intrinsically. In this sense a quantum state transmission can be generally understood through such quantum entanglement.

We sketch our idea with the model illustrated in Fig. 1. The whole quantum system we consider here consists of two qubits (A and B) and a $(2\times N)$-site two-leg spin ladder. In practice, this system can be realized by the engineered array of quantum dots [10]. The total Hamiltonian

$$H = H_M + H_q \quad (1)$$

contains two parts, the medium Hamiltonian

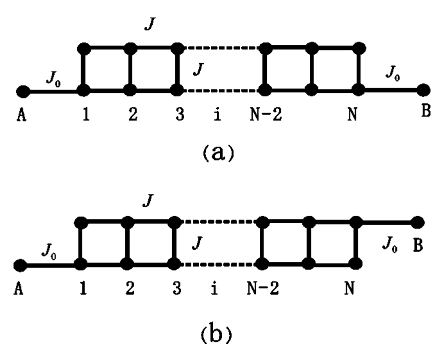

FIG. 1. Two qubits A and B connect to a $(2\times N)$-site spin ladder. The ground state of H with a-type connection [Fig. 1(a)] is singlet (triplet) when N is even (odd), while for b-type connection [Fig. 1(b)], one should have the opposite result.

*Electronic address: songtc@nankai.edu.cn
†Electronic address: suncp@itp.ac.cn
‡Internet www site: http://www.itp.ac.cn/~suncp

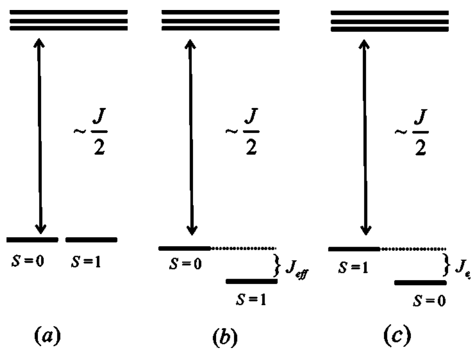

FIG. 2. Schematic illustration of the energy levels of the system. (a) When the connections between two qubits and the medium switch off ($J_0=0$) the ground states are degenerate. (b) and (c) When J_0 switches on, the ground state(s) and the first excited state(s) are either singlet or triplet. This is approximately equivalent to that of two coupled spins.

$$H_M = J\sum_{\langle ij\rangle\perp} \vec{S}_i\cdot\vec{S}_j + J\sum_{\langle ij\rangle\parallel} \vec{S}_i\cdot\vec{S}_j \tag{2}$$

describing the spin-1/2 Heisenberg spin ladder consisting of two coupled chains, and the coupling Hamiltonian

$$H_q = J_0\vec{S}_A\cdot\vec{S}_L + J_0\vec{S}_B\cdot\vec{S}_R \tag{3}$$

describing the connections between qubits A, B, and the ladder. In the term H_M, i denotes a lattice site on which one electron sits, $\langle ij\rangle \perp$ denotes nearest-neighbor sites on the same rung, $\langle ij\rangle \parallel$ denotes nearest neighbors on either leg of the ladder. In term H_q, L and R denote the sites connecting to the qubits A and B at the ends of the ladder. There are two types of the connection between $\vec{S}_A(\vec{S}_B)$ and the ladder, which are illustrated in Fig. 1. According to the Lieb's theorem [13], the spin of the ground state of H with the connection of type a is zero (one) when N is even (odd), while for the connection of type b, one should have an opposite result. For the two-leg spin ladder H_M, analytical analysis and numerical results have shown that the ground state and the first excited state of the spin ladder have spin 0 and 1, respectively [11,12]. It is also shown that there exists a finite spin gap

$$\Delta = E_1^M - E_g^M \sim J/2 \tag{4}$$

between the ground state and the first excited state (see the Fig. 2). This fact has been verified by experiments [11] and is very crucial for our present investigation.

Thus, it can be concluded that the medium can be robustly frozen in its ground state to induce the effective Hamiltonian

$$H_{\rm eff} = J_{\rm eff}\vec{S}_A\cdot\vec{S}_B \tag{5}$$

between the two end qubits. With the effective coupling constant J_{eff} to be calculated in the following, this Hamiltonian depicts the direct exchange coupling between two separated qubits. As the famous Bell states, $H_{\rm eff}$ has singlets and triplets eigenstates $|j,m\rangle_{AB}$: $|0,0\rangle = 1/\sqrt{2}(|\uparrow\rangle_A|\downarrow\rangle_B - |\downarrow\rangle_A|\uparrow\rangle_B)$ and $|1,1\rangle = |\uparrow\rangle_A|\uparrow\rangle_B$, $|1,-1\rangle = |\downarrow\rangle_A|\downarrow\rangle_B$, $|1,0\rangle = 1/\sqrt{2}(|\uparrow\rangle_A|\downarrow\rangle_B + |\downarrow\rangle_A|\uparrow\rangle_B)$, which can be used as the channel to share entanglement for a perfect quantum communication in a longer distance.

The above central conclusion can be proved both with the analytical and numerical methods as follows. To deduce the above effective Hamiltonian we utilize the Fröhlich transformation, whose original approach was used successfully for the superconductivity BCS theory. As a second-order perturbation, the effective Hamiltonian $H_{\rm eff} \cong H_{\rm M} + \frac{1}{2}[H_q, S]$ can be achieved approximately by a unitary operator $U = \exp\{-S\}$, where anti-Hermitian operator S obeys $H_q + [H_M, S] = 0$. Let $|m\rangle$ and E_m are the eigenvectors and eigenvalues of $H_M = H(J_0=0)$, respectively.

From the explicit expressions for the elements $S_{mn} = (H_q)_{mn}/(E_m - E_n)$, $(m \neq n)$, $S_{mm} = 0$, the matrix elements of effective Hamiltonian can be achieved approximately as

$$\langle n|H_{\rm eff}|m\rangle \cong E_m\delta_{mn} + \sum_{k\neq m}\frac{(H_q)_{\rm nk}(H_q)_{\rm km}}{2(E_k - E_m)} - \sum_{k\neq n}\frac{(H_q)_{\rm nk}(H_q)_{\rm km}}{2(E_n - E_k)}. \tag{6}$$

We use $|\psi_g\rangle_M$ ($|\psi_\alpha\rangle_M$) and E_g (E_α) to denote ground (excited) states of H_M and the corresponding eigenvalues. The zero-order eigenstates $|m\rangle$ can then be written as in a joint way

$$|j,m\rangle_g = |j,m\rangle_{AB}\otimes|\psi_g\rangle_M, |\psi_\alpha^{jm}(s^z)\rangle = |j,m\rangle_{AB}\otimes|\psi_\alpha\rangle_M. \tag{7}$$

Here, we have considered that z-component $S^z = S_M^z + S_A^z + S_B^z$ of total spin is conserved with respect to the connection Hamiltonian H_q. Since S_M^z and S_M^2 conserves with respect to H_M we can label $|\psi_g\rangle_M$ as $|\psi_g(s_M, s_M^z,)\rangle_M$, and then $s^z = m + s_M^z$ can characterize the noncoupling spin state $|\psi_\alpha^{jm}(s^z)\rangle$.

When the connections between two qubits and the medium switch off, i.e., $J_0 = 0$, the degenerate ground states of H are just $|j,m\rangle_g$ with the degenerate energy E_g and spin 0, 1, respectively, which is illustrated in Fig. 2(a). When the connections between the two qubits and the medium switch on, the degenerate states with spin 0, 1 should split as illustrated in Figs. 2(b) and 2(c). In the case with $J_0 \ll J$ at lower temperature $kT < J/2$, the medium can be frozen in its ground state and then we have the effective Hamiltonian

$$H_{\rm eff} \cong \sum_{j',m',j,m,s^z}\frac{\left|{}_g\langle j,m|H_q|\psi_\alpha^{j'm'}(s^z)\rangle\right|^2}{E_g - E_\alpha}|j,m\rangle_{gg}\langle j,m|$$
$$= J_{\rm eff}\,.\,{\rm Diag.}\left(\frac{1}{4},\frac{1}{4},\frac{1}{4},-\frac{3}{4}\right) + \varepsilon, \tag{8}$$

where

$$J_{\rm eff} = \sum_\alpha \frac{J_0^2[L(\alpha)R^*(\alpha) + R(\alpha)L^*(\alpha)]}{E_g - E_\alpha}, \tag{9}$$

$$\varepsilon = \sum_\alpha \frac{3J_0^2[|L(\alpha)|^2 + |R(\alpha)|^2]}{4(E_g - E_\alpha)}.$$

This just proves the above effective Heisenberg Hamiltonian (5). Here, the matrix elements of interaction $K(\alpha)$

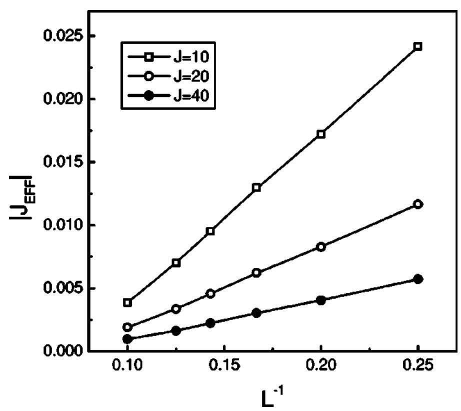

FIG. 3. The spin gaps obtained by numerical method for the systems $L=4$, 5, 6, 7, 8, and 10, with $J=10$, 20, 40, and $J_0=1$ are plotted, which is corresponding to the magnitude of $J_{\rm eff}$. It indicates that $J_{\rm eff}\sim 1/(LJ)$.

$={}_M\langle\psi_g|S_K^z|\psi_\alpha(1,0)\rangle_M$ $(K=S,L)$ can be calculated only for the variables of data bus medium. We also remark that because S^z and S^2 are conserved for H_q, off-diagonal elements in the above effective Hamiltonian vanish.

In temporal summary, we have shown that at lower temperature $kT<J/2$, H can be mapped to the effective Hamiltonian (5), which seemingly depicts the direct exchange coupling between two separated qubits. Notice that the coupling strength has the form $J_{\rm eff}\sim g(L)J_0^2/J$, where $g(L)$ is a function of $L=N+1$, the distance between the two qubits are concerned. Here we take the $N=2$ case as an example. According to Eq. (11) one can get $J_{\rm eff}=-\frac{1}{4}J_0^2/J$ and $\frac{1}{3}J_0^2/J$ when A and B connect the plaquette diagonally and adjacently, respectively. This result is in agreement to the theorem [13,14] about the ground state and the numerical result when $J_0\gg J$. In the general case, the behavior $g(L)$ vs L is very crucial for quantum information since $L/|J_{\rm eff}|$ determines the characteristic time of quantum-state transfer between the two qubits A and B. In order to investigate the profile of $g(L)$, a numerical calculation is performed for the systems $L=4$, 5, 6, 7, 8, and 10, with $J=10$, 20, 40, and $J_0=1$. The spin gap between the ground state(s) and first excited state(s) are calculated, which corresponds to the magnitude of $J_{\rm eff}$. The numerical result is plotted in Fig. 3, which indicates that $J_{\rm eff}\sim 1/(LJ)$. It implies that the characteristic time of quantum-state transfer linearly depends on the distance and then guarantees the possibility to realize the entanglement of two separated qubits in practice.

In order to verify the validity of the effective Hamiltonian $H_{\rm eff}$, we need to compare the eigenstates of $H_{\rm eff}$ with those reduced states from the eigenstates of the total system. In general the eigenstates of H can be written formally as

$$|\psi\rangle=\sum_{jm}c_{jm}|j,m\rangle_{AB}\otimes|\beta_{jm}\rangle_M, \tag{10}$$

where $\{|\beta_{jm}\rangle_M\}$ is a set of vectors of the data bus, which is not necessarily orthogonal. Then we have the condition $\sum_{jm}|c_{jm}|^2_M\langle\beta_{jm}|\beta_{jm}\rangle_M=1$ for normalization of $|\psi\rangle$. In this sense the practical description of the A-B subsystem of two qubits can only be given by the reduced density matrix,

$$\rho_{AB}=Tr_M(|\psi\rangle\langle\psi|)=\sum_{jm}|c_{jm}|^2|j,m\rangle_{AB}\langle j,m| + \sum_{j'm'\neq jm}c^*_{j'm'}c_{jmM}\langle\beta_{j'm'}|\beta_{jm}\rangle_M|j,m\rangle_{AB}\langle j',m'|, \tag{11}$$

where Tr_M means the trace over the variables of the medium. By a straightforward calculation we have

$$|c_{11}|^2=|c_{1-1}|^2=\langle\psi|\left(\tfrac{1}{4}+S_A^z\cdot S_B^z\right)|\psi\rangle,$$

$$|c_{00}|^2=\langle\psi|\left(\tfrac{1}{4}-\vec{S}_A\cdot\vec{S}_B\right)|\psi\rangle, \tag{12}$$

$$|c_{10}|^2=1-2|c_{11}|^2-|c_{00}|^2.$$

Now we need a criterion to judge how close the practical reduced eigenstate by the above reduced density matrix (11) to the pure state for the effective two sites coupling $H_{\rm eff}$. As we noticed, it has the singlet and triplet eigenstates $|j,m\rangle_{AB}$ in the subspace spanned by $|0,0\rangle_{AB}$ with $S^z=S_A^z+S_B^z=0$, we have $|c_{11}|^2=|c_{10}|^2=|c_{1-1}|^2=0$, $|c_{00}|^2=1$; for triplet eigenstate $|1,0\rangle_{AB}$, we have $|c_{11}|^2=|c_{1-1}|^2=|c_{00}|^2=0$, $|c_{10}|^2=1$. With the practical Hamiltonian H, the values of $|c_{jm}|^2$, $i=1,2,3,4$ are numerically calculated for the ground state $|\psi_g\rangle$ and first excited state $|\psi_1\rangle$ of finite systems $L=4$, 5, 6, 7, 8, and 10 with $J=10$, 20, and 40, $(J_0=1)$ in $S^z=0$ subspace, which are listed in Tables I(a), I(b), and I(c). It shows that, at lower temperature, the realistic interaction leads to the results about $|c_{jm}|^2$, which are very close to that described by $H_{\rm eff}$, even if J is not so large in comparison with J_0.

We remark that the above tables reflect all the facts distinguishing the difference between the results about the entanglement of two end qubits generated by $H_{\rm eff}$ and H. Though we have ignored the considerations for the off-diagonal terms in the reduced density matrix, the calculation of the feudality $F(|j,m\rangle)\equiv{}_{\cdot M}\langle j,m|\rho_{AB}|j,m\rangle_M=|c_{jm}|^2$ further confirms our observation that the effective Heisenberg-type interaction of two end qubits can approximate the realistic Hamiltonian very well. Then we can transfer the quantum information between two ends of the $(2\times N)$-site two-leg spin ladder that can be regarded as the channel to share entanglement with separated Alice and Bob. Physically, this is just due to a large spin gap existing in such a perfect medium, whose ground state can induce a maximal entanglement of the two end qubits. We also pointed out that our analysis is applicable for other types of medium systems as data buses, which possess a finite spin gap. Since $L/|J_{\rm eff}|$ determines the characteristic time of quantum state transfer between the two qubits, the dependence of $J_{\rm eff}$ upon L becomes important and relies on the appropriate choice of the medium.

In conclusion, we have presented and studied in detail a protocol to achieve the entangled states and fast quantum state transfer of two spin qubits by connecting two spins to a medium which possesses a spin gap. A perturbation method, the Fröhlich transformation, shows that the interaction between the two spins can be mapped to the Heisenberg-type

TABLE I. The diagonal elements of reduced density matrix, which provide a criteria for the validity of H_{eff}, are calculated numerically for the ground state and first excited state of finite system systems $L=4$, 5, 6, 7, 8, and 10. The results for $J=10$, 20, and 40, $(J_0=1)$ are listed in (a), (b), and (c) respectively. It shows that, at lower temperature, the result based the realistic interaction is very close to that by H_{eff}.

States	j	m	L	4	5	6	7	8	10
(a)			$\vert c_{00}\vert^2$	4.2×10^{-4}	5.9×10^{-4}	7.4×10^{-4}	8.7×10^{-4}	9.7×10^{-4}	1.2×10^{-3}
$\vert\psi_g\rangle$	1	0	$\vert c_{10}\vert^2$	0.9952	0.9954	0.9954	0.9955	0.9956	0.9956
			$\vert c_{11}\vert^2$	2.2×10^{-3}	2.0×10^{-3}	1.9×10^{-3}	1.8×10^{-3}	1.7×10^{-3}	1.6×10^{-3}
			$\vert c_{1-1}\vert^2$	2.2×10^{-3}	2.0×10^{-3}	1.9×10^{-3}	1.8×10^{-3}	1.7×10^{-3}	1.6×10^{-3}
			$\vert c_{00}\vert^2$	0.9989	0.9984	0.9979	0.9975	0.9971	0.9966
$\vert\psi_1\rangle$	0	0	$\vert c_{10}\vert^2$	3.7×10^{-4}	5.2×10^{-4}	7.0×10^{-4}	8.4×10^{-4}	1.0×10^{-3}	1.2×10^{-3}
			$\vert c_{11}\vert^2$	3.7×10^{-4}	5.4×10^{-4}	7.0×10^{-4}	8.3×10^{-4}	9.3×10^{-4}	1.1×10^{-3}
			$\vert c_{1-1}\vert^2$	3.7×10^{-4}	5.4×10^{-4}	7.0×10^{-4}	8.3×10^{-4}	9.3×10^{-4}	1.1×10^{-3}
(b)			$\vert c_{00}\vert^2$	9.7×10^{-5}	1.4×10^{-4}	1.8×10^{-4}	2.1×10^{-4}	2.3×10^{-4}	3.7×10^{-4}
$\vert\psi_g\rangle$	1	0	$\vert c_{10}\vert^2$	0.9989	0.9989	0.9989	0.9989	0.9990	0.9989
			$\vert c_{11}\vert^2$	5.3×10^{-4}	4.8×10^{-4}	4.7×10^{-4}	4.4×10^{-4}	4.0×10^{-4}	3.8×10^{-4}
			$\vert c_{1-1}\vert^2$	5.3×10^{-4}	4.8×10^{-4}	4.7×10^{-4}	4.4×10^{-4}	4.0×10^{-4}	3.8×10^{-4}
			$\vert c_{00}\vert^2$	0.9997	0.9996	0.9995	0.9994	0.9993	0.9991
$\vert\psi_1\rangle$	0	0	$\vert c_{10}\vert^2$	9.1×10^{-5}	1.4×10^{-4}	1.7×10^{-4}	2.0×10^{-4}	2.7×10^{-4}	3.7×10^{-4}
			$\vert c_{11}\vert^2$	9.1×10^{-5}	1.3×10^{-4}	1.7×10^{-4}	2.0×10^{-4}	2.1×10^{-4}	2.7×10^{-4}
			$\vert c_{1-1}\vert^2$	9.1×10^{-5}	1.3×10^{-4}	1.7×10^{-4}	2.0×10^{-4}	2.1×10^{-4}	2.7×10^{-4}
(c)			$\vert c_{00}\vert^2$	2.3×10^{-5}	3.3×10^{-5}	4.2×10^{-5}	5.0×10^{-5}	5.7×10^{-5}	1.8×10^{-4}
$\vert\psi_g\rangle$	1	0	$\vert c_{10}\vert^2$	0.9997	0.9997	0.9997	0.9997	0.9998	0.9996
			$\vert c_{11}\vert^2$	1.3×10^{-4}	1.2×10^{-4}	1.1×10^{-4}	1.1×10^{-4}	8.8×10^{-5}	9.3×10^{-5}
			$\vert c_{1-1}\vert^2$	1.3×10^{-4}	1.2×10^{-4}	1.1×10^{-4}	1.1×10^{-4}	8.8×10^{-5}	9.3×10^{-5}
			$\vert c_{00}\vert^2$	0.9999	0.9999	0.9999	0.9998	0.9998	0.9997
$\vert\psi_1\rangle$	0	0	$\vert c_{10}\vert^2$	2.5×10^{-5}	3.5×10^{-5}	4.6×10^{-5}	1.0×10^{-4}	1.2×10^{-4}	1.7×10^{-4}
			$\vert c_{11}\vert^2$	2.3×10^{-5}	3.3×10^{-5}	4.2×10^{-5}	5.0×10^{-5}	4.2×10^{-5}	6.5×10^{-5}
			$\vert c_{1-1}\vert^2$	2.3×10^{-5}	3.3×10^{-5}	4.2×10^{-5}	5.0×10^{-5}	4.2×10^{-5}	6.5×10^{-5}

coupling. Numerical results show that the isotropic antiferromagnetic spin ladder system is a perfect medium through which the interaction between two separated spins is very close to the Heisenberg-type coupling with a coupling constant inversely proportional to the distance even if the spin gap is not so large comparing to the couplings between the input and output spins with the medium.

This work of Z.S. is supported by the Cooperation Foundation of Nankai and Tianjin University for research of nanoscience. C.P.S. also acknowledges the support of the CNSF (Grant Nos. 90203018 and 10474104), the Knowledge Innovation Program (KIP) of the Chinese Academy of Sciences, the National Fundamental Research Program of China (Grant No. 001GB309310).

[1] *The Physics of Quantum Information*, edited by D. Bouwmeeste, A. Ekert, and A. Zeilinger (Springer, Berlin, 2000).
[2] C. Liu, Z. Dutton, C. H. Behroozi, and L. V. Hau, Nature (London) **409**, 490 (2001).
[3] M. D. Lukin, Rev. Mod. Phys. **75**, 457 (2003).
[4] C. P. Sun, Y. Li, and X. F. Liu, Phys. Rev. Lett. **91**, 147903 (2003).
[5] D. P. DiVincenzo and C. Bennet, Nature (London) **404**, 247 (2000), and references therein.
[6] S. Bose, Phys. Rev. Lett. **91**, 207901 (2003).
[7] V. Subrahmanyam, Phys. Rev. A **69**, 034304 (2004).
[8] M. Christandl, N. Datta, and J. Landahl, Phys. Rev. Lett. **92**, 187902 (2004).
[9] C. H. Bennett, G. Brassard, C. Crepeau, R. Jozsa, A. Peres, and W. Wootters, Phys. Rev. Lett. **70**, 1895 (1993).
[10] D. Loss and D. P. DiVincenzo, Phys. Rev. A **57**, 120 (1998); B. E. Kane, Nature (London) **393**, 133 (1998).
[11] E. Dagotto and T. M. Rice, Science **271**, 618 (1996).
[12] S. White, R. Noack, and D. Scalapino, Phys. Rev. Lett. **73**, 886 (1994); R. Noack, S. White, and D. Scalapino, *ibid.* **73**, 882 (19940).
[13] E. Lieb, Phys. Rev. Lett. **62**, 1201 (1989); E. Lieb and D. Mattis, J. Math. Phys. **3**, 749 (1962).
[14] Z. Song, Phys. Lett. A **231**, 135 (1997); **233**, 135 (1997).