Phase separation of a trapped Bose-Fermi gas mixture: Beyond the Thomas-Fermi approximation

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Phase separation of a Bose-Fermi gas mixture in a trapping potential is strongly influenced by the interaction between the fermion and the boson. The stability condition for the mixture at zero temperature is deduced. It is found that mixture stability depends on the fermion-boson interaction, the total number of the bosons and the fermions, and the trapping frequencies. The stability conditions for the mixture at finite temperature are also derived and discussed.

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I. INTRODUCTION

Since the realization of dilute alkali-metal atomic vapor condensates (BEC) in 1995 [1], large efforts have been made to study many-body effects and macroscopic properties of the gases, which may be more transparently demonstrated in BEC than in other many-body systems. For fermionic atomic vapor, however, it is difficult to achieve a degenerate gas since the evaporative cooling of a pure fermionic gas is ineffective at very low temperature. Theoretical and experimental advances discover a number of interesting phenomena not accessible in the previous BEC systems. One of the most stunning progresses is the recent experimental demonstration of a condensate mixture, in which two or more internal states of condensates coexist [2]. The realization of bicondensate mixture is ascribed to the sympathetic cooling mechanism, i.e., the cooling is through the energy exchange between cold and thermal atoms. Most recently, DeMarco and Jin [3] report their observation of a degenerate Fermi gas by using an evaporative cooling strategy. Although the techniques use a two-component Fermi gas, the new experiment stimulates the experimental and theoretical study of the Bose-Fermi gas mixture [4–29].

The two-component BEC mixture of dilute gases are interesting from both theoretical and experimental viewpoints. Experiments have been conducted to study the creation of topological excitations in two-component BEC gas [10], quantum tunneling effects [11], metastable effects [12], Rabi oscillations [13], the dynamics of component separations [14], and relative phase coherence [15] in a binary mixture of Bose gases. Theoretical work on trapped two-component BEC gases has included the static and stability properties [16–19], the dynamics of the relative phase [20,21], and the collective modes [22], and the phase diagram and collective modes for spinor BEC [23,24]. As to the Bose-Fermi mixtures, the density profiles of the mixtures trapped in a harmonic potential at nonzero temperature under the Thomas-Fermi approximation [25–28] have been investigated, and the effects of phonon exchange on the fermion-fermion interacting strength [29] are discussed at temperatures below the BEC transition. In most of the studies on the stability of the Bose-Fermi mixture, the effects of finite temperature have not been discussed.

In this paper, we shall analyze the stability of a Bose-Fermi gases mixture at zero and finite temperature. We first study the stability of the Bose-Fermi gases mixture at zero temperature by using a variation method [30,31], which provides us with an clear understanding of the phase separation at zero temperature. We then study the properties of the mixtures at finite temperature. The stability conditions for both the homogeneous and inhomogeneous mixtures are deduced and discussed.

The paper is organized as follows. In Sec. II, we study phase separation of a trapped Bose-Fermi gas mixture at zero temperature. The phase separation of the trapped mixture at finite temperature is discussed in Sec. III. Section IV is a brief summary.

II. PHASE SEPARATION OF THE BOSE-FERMI GAS MIXTURE AT ZERO TEMPERATURE

We study the Bose-Fermi gas mixture by using a variation method. This method was first introduced in [30] to study the BEC ground state of a Bose system in a harmonic trap, and later generalized to study BEC with attractive interactions [31]. Throughout this section we assume that the number of boson is much larger than the fermions, in this situation, the distribution of the bosons remains unchanged.

To begin with, we consider a second-quantized grand canonical Hamiltonian of interacting Bose and Fermi gases

$$
\begin{aligned}
H &= H_b + H_f + V_{bf},\\
H_b &= \int d^3r\,\phi^\dagger(r)\left(\frac{-\hbar^2\nabla^2}{2m_b} - \mu_b + \frac{1}{2}m_b\omega_b r^2\right)\phi(r)\\
&\quad + \frac{g_{bb}}{2}\int\int d^3r\,d^3r'\,\phi^\dagger(r)\phi^\dagger(r')\delta(r-r')\phi(r')\phi(r),\\
H_f &= \int d^3r\,\psi^\dagger(r)\left(\frac{-\hbar^2\nabla^2}{2m_f} - \mu_f + \frac{1}{2}m_f\omega_f r^2\right)\psi(r),\\
V_{bf} &= g_{bf}\int d^3r\,d^3r'\,\phi^\dagger(r)\psi^\dagger(r')\delta(r-r')\psi(r')\phi(r),
\end{aligned}
\tag{2.1}
$$

where $\phi(r)$ and $\psi(r)$ denote boson and fermion field operators with masses m_b and m_f, respectively. For weakly interacting dilute gases, the interactions between the bosonic atoms are modeled by δ potentials and the interactions among the fermionic atoms may be neglected, since the interactions between atoms at very low temperature is suppressed for polarized systems. g_{bb} and g_{ba} stand for boson-boson and boson-fermion coupling constant, respectively:

$$g_{bb}=\frac{4\pi\hbar^2}{m_b}a_{bb}, \quad g_{bf}=\frac{2\pi\hbar^2}{m_{bf}}a_{bf},$$

a_{bb} (a_{bf}) are the s-wave scattering length between boson and boson (boson and fermion), and m_{bf} is a reduced mass of the boson and the fermion. The chemical potentials μ_b and μ_f are determined through the conditions

$$N_b=\left\langle \int d^3r\,\phi^\dagger(r)\phi(r)\right\rangle, \quad N_f=\left\langle \int d^3r\,\psi^\dagger(r)\psi(r)\right\rangle. \tag{2.2}$$

At $T=0$, the self-consistent mean-field theory, assuming that all N bosonic particles in a gas populated the same state denoted by single-particle wave-function $\Phi(r)$, leads to a nonlinear Schrödinger equation (or the Gross-Pitaevskii equation) for $\Phi(r)=\langle\phi(r)\rangle$

$$\left[-\frac{\hbar^2}{2m_b}\nabla^2+\frac{1}{2}m_b\omega_b^2r^2+g_{bb}n_b(r)\right]\Phi(r)=E_b\Phi(r). \tag{2.3}$$

Here we omit quantities $g_{bf}n_f(r)$, because they are smaller than $g_{bb}n_b(r)$ in the case of $N_b\gg N_f$. This is relevant to a experiment on degenerate fermionic gas. In order to get a degenerate fermionic gas, the boson particles appear in the system only as a coolant. so the number of bosons is always much larger than the number of fermions. In the same approximation as in the bosons, the fermionic wave function is given by a Slater determinant

$$\Psi(r_1,r_2,\ldots,r_{N_f}) = \frac{1}{\sqrt{N_f!}}\begin{bmatrix} \Psi_1(r_1) & \Psi_1(r_2) & \cdots & \Psi_1(r_{N_f}) \\ \Psi_2(r_1) & \Psi_2(r_2) & \cdots & \Psi_2(r_{N_f}) \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \Psi_{N_f}(r_1) & \Psi_{N_f}(r_2) & \cdots & \Psi_{N_f}(r_{N_f}) \end{bmatrix}, \tag{2.4}$$

where $\Psi_i(r)$ is the single-particle states determined by the Hartree-Fock self-consistent equation

$$\left[-\frac{\hbar^2}{2m_f}\nabla^2+\frac{1}{2}m_f\omega_f^2r^2+g_{bf}n_b(r)\right]\Psi_i(r)=E_i\Psi_i(r). \tag{2.5}$$

The density of the fermions is given by

$$n_f(r)=\sum_\alpha |\Psi_\alpha(r)|^2, \tag{2.6}$$

where α denotes the index of occupied states. In the semiclassical (Thomas-Fermi) approximation, the particles are assigned classical position and momenta, but the effects of quantum statistics are taken into account. Under this approximation, Eqs. (2.3) and (2.5) for the boson and fermion wave function are equivalent to [25,32]

$$\frac{1}{2}m_b\omega_b^2r^2+g_{bb}n_b(r)=\mu_b,$$

$$\frac{\hbar^2}{2m_f}[6\pi^2n_f(r)]^{2/3}+\frac{1}{2}m_f\omega_f^2r^2+g_{bf}n_b(r)=e_F. \tag{2.7}$$

We obtain $n_b(r)=1/g_{bb}[\mu_b-(1/2)m_b\omega_b^2r^2]$ from the first line of Eqs (2.7). Substituting $n_b(r)$ into the second line of Eqs (2.7), we yield

$$\frac{\hbar^2}{2m_f}[6\pi^2n_f(r)]^{2/3}+\frac{1}{2}m_f\omega_f^2r^2+\frac{g_{bf}}{g_{bb}}\left(\mu_b-\frac{1}{2}m_b\omega_b^2r^2\right)=e_F. \tag{2.8}$$

This equation shows that the fermions experience a potential minimum in the center of the trap if $g_{bf}/g_{bb}<m_f\omega_f^2/m_b\omega_b^2$. In this case, the entire distribution behaves like a fermionic core within the Bose condensate. The fermion density is a constant throughout the Bose condensate if $g_{bf}/g_{bb}=m_f\omega_f^2/m_b\omega_b^2$. Whereas the fermions are repelled from the center of the trap and localized near the edge of the Bose condensate, if $g_{bf}/g_{bb}>m_f\omega_f^2/m_b\omega_b^2$, i.e., a phase separation occurs in this system. We would like to note that the distribution of BEC remains unchanged in the above discussions, since we assume that the Bose system is much larger than the Fermi one. To drive Eq. (2.8), we assume that the Thomas-Fermi approximation(TFA) is valid. The coupling constant g_{bb} and g_{bf} may take any value as long as the TFA is available, and the phase separation depends mainly on ratio g_{bf}/g_{bb}. In what follows we discuss the separation of the mixture from the other aspect. We note the solution of Eq. (2.5) requires prior knowledge of the boson density profile $n_b=|\Phi(r)|^2$. To obtain the density profile, we have to solve the Gross-Pitaevskii Eq. (2.3). There are a large number of literatures devoted to solve the Gross-Pitaevskii Eq. [33], we here use a variation method [30,31] to solve the problem. For a isotropic trapping potential, we may assume the trial wave function for $\Phi(r)$ in Eq.(2.3) to be

$$\Phi(r)=\sqrt{N_b}\,\omega^{3/4}\left(\frac{m_b}{\pi\hbar}\right)^{3/4}e^{-m_b\omega r^2/2\hbar}, \tag{2.9}$$

where ω is the effective frequency and is taken as a variational parameter. Substituting Eq. (2.9) into Eq. (2.3), we obtained the ground-state energy

$$E_b[\Phi]=E_b(\omega)=\frac{3}{4}N_b\hbar\omega+\frac{3}{4}N_b\hbar\frac{\omega_b^2}{\omega}+g_{bb}N_b^2\left(\frac{\omega m_b}{2\pi\hbar}\right)^{3/2}. \tag{2.10}$$

If $E_b(\omega)$ is plotted as a function of ω, one sees that a stable local minimum exists only up to a certain maximum number of atoms for $g_{bb}<0$ [31]. The critical point occurs where

$$\left.\frac{\partial E_b(\omega)}{\partial\omega}\right|_{(\omega=\omega_c,N_b=N_{bc})}=0,$$

and

$$\frac{\partial^2 E_b(\omega)}{\partial\omega^2}_{\omega=\omega_c,N_b=N_{bc}}>0. \tag{2.11}$$

Here, ω_c stands for the variational parameter that minimizes the ground-state energy. By using Eq. (2.10) we can get the equation

$$\hbar\omega_c^2-\hbar\omega_b^2+2g_{bb}N_b\left(\frac{m_b}{2\pi\hbar}\right)^{3/2}\omega_c^{5/2}=0. \tag{2.12}$$

If there are no interactions between the bosons, i.e., $g_{bb}=0$, the solution of Eq. (2.12) is $\omega_c=\omega_b$. For weak boson-boson interaction $N_bg_{bb}\ll 1$, we may give the solution of Eq. (2.12) by perturbative expansions. To first order of g_{bb}, this solution is

$$\omega_c=\omega_b-\frac{g_{bb}N_b}{\hbar}\left(\frac{m_b\omega_b}{2\pi\hbar}\right)^{3/2}. \tag{2.13}$$

Substituting ω_c into Eq. (2.9), we obtain the distribution of the boson system $|\Phi(r)|^2=n_b(r)$. Subsequently, we can get the distribution of the fermions by solving the second equation of Eqs. (2.7). From the distribution of the fermions, we may acquire the knowledge of phase separation.

There is a shortcut to study the phase separation of the system; i.e., for the fixed distribution of bosons if $\partial^2 n_f(r)/\partial r^2|_{r=0}>0$, there is no phase separation between bosons and fermions, else a phase separation occurs. In what follows, we consider the case under the Thomas-Fermi approximation and beyond the Thomas-Fermi approximation.

For the case under the Thomas-Fermi approximation, the phase separation condition can be derived by using the second equation of Eqs (2.7), it is given that

$$g_{bf}>\frac{m_f\omega_f^2\pi^{3/2}\hbar^{5/2}}{2N_b(\omega_c m_b)^{5/2}}. \tag{2.14}$$

This equation shows that the mixture stability depends on the total number of the bosons, the interaction strength between fermions and bosons and the interaction strength among the bosons. Physically, the fermions experience two potentials, one of them is the trapping potential and another comes from the interaction between the bosons and the fermions. If the boson-fermion interaction does not change the minimum of the total potential (trapping potential plus effective potential from boson-fermion interaction), there is no phase separation, else the phase separation occurs. For a possible experiment conducted in $^{39}K-^{40}K$, we may estimate the critical value for g_{bf}, which was given in Eq. (2.14). Assuming $\omega_f\sim\omega_c=\omega$, defining $a_0=\sqrt{\hbar/m_b\omega}$, and taking $N_b=10^4$, we get the critical value for g_{bf} is $g_{bf}^c=2.712\times10^{-4}\hbar\omega a_0^3$. In other words, if the boson-fermion coupling constant is greater than g_{bf}^c, the phase separation occurs in the mixture.

In the above discussions, we calculate the distribution of the boson system by means of variation method, while we consider the fermion system under the Thomas-Fermi approximation. In the following, instead of the Thomas-Fermi approximation, we give a full quantum-mechanical description for the fermions. We start with the expression of kinetic energy density $t(r)=-[(\hbar^2)/(2m_f)]\Sigma_i\Psi_i^*(r)\nabla^2\Psi_i(r)$ given in [34]

$$\frac{\partial t(r)}{\partial r}=\frac{(M+2)\hbar\omega_f}{n_f^{2/3}(0)}n_f^{2/3}(r)n_f'(r)+\frac{\hbar^2}{12m_f}n_f^{2/3}(r)n_f'(r)\int_0^r\frac{1}{n_f^{5/3}(s)}\frac{\partial}{\partial s}\nabla^2 n_f(s), \tag{2.15}$$

where $n_f'(r)=\partial n_f(r)/\partial r$ and M is a shell number, the shell below it is fully filled. By using Eq. (2.15), the distribution of the fermions may be derived to satisfy

$$\frac{\hbar^2}{12}\frac{n_f'''(r)}{n_f(r)}+m_f\omega_f^2 r+g_{bf}\frac{\partial n_b(r)}{\partial r}=0, \tag{2.16}$$

this is the equation that corresponds to the second line of Eqs. (2.7) for the case beyond the Thomas-Fermi approximation. Integrating Eq. (2.16), we may get a equation of $n''(r)$ that characters the stability of the mixture, i.e.,

$$\int g_{bf}\frac{\partial n_b(r)}{\partial r}n_f(r)dr<-\int m_f\omega_f^2 r n_f(r)dr. \tag{2.17}$$

Assuming

$$n_b(r)\sim e^{-[(m_b\omega_c)/\hbar]r^2},\quad n_f(r)\sim e^{-[(m_f\Omega)/\hbar]r^2},$$

the condition for phase separation is

$$g_{bf}>\frac{\pi^{3/2}\hbar^{5/2}(m_b\omega_c+m_f\Omega)\omega_f^2}{2m_bm_f^{3/2}\omega_c\Omega^{5/2}N_f}; \tag{2.18}$$

namely, if this condition holds, there is a phase separation occurring in the mixture. The key difference between the conditions (2.14) and (2.18) are that the later one is proportional to $1/N_f$. This fact results from how we treat the kinetic energy of the fermions in full quantum-mechanical frame. We would like to point out that the result Eq. (2.18) depends on the trial distribution given below Eq. (2.17). As Ref. [34] shows, the Gaussian trial distribution is a good approximation when there are hundred fermions in a trap. If we choose

$\omega_c \sim \omega_f = \omega$, $N_b = 10^4$, $N_f = 10^2$, we obtain $g_{bf}^c = 5.56 \times 10^{-5} \hbar \omega a_0^2$, where a_0 is the same as in last paragraph.

III. PHASE SEPARATION OF THE BOSE-FERMI GAS MIXTURE AT FINITE TEMPERATURE

In this section, we pay attention to discussing the above problem at finite temperature. For the boson and fermion system, thermodynamical properties are trivial if there is not interaction between them. But in this case, the sympathetic cooling scheme does not take any effects and the degenerate fermions have not been achieved. The thermodynamical properties may be changed when the interaction between the fermions and bosons is turn on, then a new phenomenon, the phase separation, may occur in this system. For a homogeneous fermion and boson mixture system, the Helmholtz free energy can be written as [35,36]

$$\beta F = -\frac{V}{\lambda_2^3} f_{52}(z_f) + \frac{1}{2} a_{ff} \rho_f N_f \lambda_f^2 + \ln(1 - z_b) - \frac{V}{\lambda_b^3} g_{5/2}(z_b) + 2 a_{bb} \rho_b N_b \lambda_b^2 + a_{bf}(\lambda_b^2 + \lambda_f^2) N_f N_b / V, \tag{3.1}$$

where the index f refers to the fermionic component, whereas the index b stands for the bosonic one, N_i is the number of particles in component i, $\lambda_i = h/\sqrt{2\pi m_i k_B T}$ denotes the thermal wave length of component i, $f_n(z)$, and $g_n(z)$ represent the Fermi and Bose integral, respectively. a_{ij} is the s-wave scattering length between component i and j. Equation (3.1) is based on the pseudopotential form of the atom-atom interaction, and may be assumed accurate when the system is dilute, i.e., $\rho_i a_{ii}^3 \ll 1$ and $a_{ii}/\lambda_i \ll 1$, where ρ_i is the density of the component i. This condition is well satisfied for the samples of alkali-metal atoms in experiments to date [1,33,37,38].

From Eq. (3.1), we obtain the chemical potential for each component straightforwardly,

$$\beta \mu_b = \beta \mu_b^0 + 4 a_{bb} \rho_b \lambda_b^2 + a_{bf}(\lambda_b^2 + \lambda_f^2) N_f / V,$$

$$\beta \mu_f = \beta \mu_f^0 + a_{ff} \rho_f \lambda_f^2 + a_{bf}(\lambda_b^2 + \lambda_f^2) N_b / V, \tag{3.2}$$

where μ_i^0 are the chemical potentials of ideal gas. There are three terms in the equation for the chemical potential (3.2). While the second term comes from the interaction within the component, the third term is from the interaction between the fermion and boson component. As known, a homogenous binary mixture is stable only when the symmetric matrix $\hat{\mu}$ given by

$$\hat{\mu} = \begin{bmatrix} \frac{\partial \mu_b}{\partial \rho_b} & \frac{\partial \mu_b}{\partial \rho_f} \\ \frac{\partial \mu_f}{\partial \rho_b} & \frac{\partial \mu_f}{\partial \rho_f} \end{bmatrix} \tag{3.3}$$

is non-negatively definite. In other words, all eigenvalues of matrix $\hat{\mu}$ given in Eq. (3.3) are nonnegative. Mathematically, for homogeneous fermion and boson mixture the stability conditions are

$$\frac{\partial \mu_b}{\partial \rho_b} \geqslant 0, \frac{\partial \mu_f}{\partial \rho_f} \geqslant 0, \tag{3.4}$$

and

$$det \begin{bmatrix} \frac{\partial \mu_b}{\partial \rho_b} & \frac{\partial \mu_b}{\partial \rho_f} \\ \frac{\partial \mu_f}{\partial \rho_b} & \frac{\partial \mu_f}{\partial \rho_f} \end{bmatrix} \geqslant 0. \tag{3.5}$$

For ideal gas, we have $\rho_b = [1/(\lambda_b^3)] g_{3/2}(z_b)$, $\rho_f = [1/(\lambda_f^3)] f_{3/2}(z_f)$, this leads to

$$\beta \frac{\partial \mu_f^0}{\partial \rho_f} = \frac{\lambda_f^3}{f_{1/2}(z_f)}, \quad \beta \frac{\partial \mu_b^0}{\partial \rho_b} = \frac{\lambda_b^3}{g_{1/2}(z_b)}. \tag{3.6}$$

It follows from Eqs. (3.4) and (3.5) that

$$4 a_{bb} \lambda_b^2 + \frac{\lambda_b^3}{g_{1/2}(z_b)} \geqslant 0, \tag{3.7}$$

$$a_{ff} \lambda_f^2 + \frac{\lambda_f^3}{f_{1/2}(z_f)} \geqslant 0, \tag{3.8}$$

and

$$Z(T, a_{bf}, a_{ff}, a_{bb}) = Z = \left(4 a_{bb} \lambda_b^2 + \frac{\lambda_b^3}{g_{1/2}(z_b)}\right)\left(a_{ff} \lambda_f^2 + \frac{\lambda_f^3}{f_{1/2(z_f)}}\right) - a_{bf}^2(\lambda_b^2 + \lambda_f^2)^2 \geqslant 0. \tag{3.9}$$

It is well known that a homogeneous imperfect gas with attractive interaction is not stable. The fermions with attractive interaction could form a BCS state, which consists of two fermionic particles interacting with each other but not with the other fermions from the Fermi gas, whereas bosons with attractive interaction could collapse into liquid. We would like to note that we here discuss the system with repulsive interactions, so the phenomena mentioned above cannot occur. It is obvious that the stability conditions (3.7) and (3.8) hold always for $a_{bb} > 0$, $a_{ff} > 0$. As Eqs. (3.7)–(3.9) show, the stability conditions do not involve the densities of both components. At first sight, this seems to be confusion, in fact, there is no contradiction. One can demonstrate that at low density, the Helmholtz free energy of the Bogoliubov gas reduce to a quadratic form in N_b and N_f. To have a minimum, this form should be positive definite, i.e., $\det\|(\partial^2 F)/(\partial N_b \partial N_f)\| \geqslant 0$. Therefore, the corresponding stability criterion involves only density-independent constants. This condition is similar to the stability conditions for the two-component Bose-Einstein condensate in a trapped untracold gas [17,39–44]. When

$T\to\infty,\lambda_i\to 0$, hence $Z\sim 1/\rho_b\rho_f$. Thus, at high temperature, the homogeneous binary gas mixture is always stable and no phase separation occurs. In the case considered here, Fermi temperature $T_F=[h^2/(2mk_B)][(3N_f/8\pi V)]^{2/3}$ is much lower than BEC transition temperature $T_c=[(h^2)/(2\pi mK_B)][(N_b)/(2.612V)]^{2/3}$. Therefore, with the temperature decreases, it first passes the BEC transition point T_c. When $T\to T_c$, $g_{1/2}(1)\to\infty$, we have

$$Z(T,a_{bf},a_{bb},a_{ff})\sim 4a_{bb}\lambda_b^2\left(a_{ff}\lambda_f^2+\frac{\lambda_f^3}{f_{1/2}(z_f)}\right)-a_{bf}^2(\lambda_b^2+\lambda_f^2)^2.$$

In particular, when $T\ll T_F$, the stability condition becomes (setting $m_f=m_b$)

$$a_{bb}a_{ff}-a_{bf}^2\geq 0. \tag{3.10}$$

It does not depend on temperature and agrees with the stability conditions of two-component BEC at zero temperature [39,44]. Although it is difficult to reach this regime of very low temperature, it attracts much more attention, because both superfluidity and shell effects are expected to occur at temperatures much smaller than the Fermi temperature [6,45]. Z given by Eq. (3.9) as a function of the temperature is illustrated in Fig. 1. We see that the system is always stable when $T\to 0$ and $T\to\infty$, and the system is unstable for $T_{c1}<T<T_{c2}$, where T_{c1} and T_{c2} are roots of $Z(T,a_{bb},a_{bf},a_{ff})=0$. In particular, T_{c1} and T_{c2} depend on a_{bf} a_{ff}, and a_{bb}. As a_{bf} decreases (for fixed a_{bb} and a_{ff}), T_{c1} tends to T_{c2} (in Fig. 1 going from dotted line to solid line). The critical temperature T_{c1} and T_{c2} characterize the onset of the phase separation, which is quite different from the Bose-Einstein condensation and the degenerate fermions. The critical temperature of the BEC transition and of the onset of degenerate fermionic gas depends mainly on the density of the system $N_i/V(i=b,f)$. Especially, the BEC and the degenerate fermionic gas may happen even if $a_{bf}=0$. For the phase separation, however, nothing will happen if $a_{bf}=0$. For a fixed temperature and the coupling constant a_{bf}, Z vs a_{bb} and a_{ff} is shown in Fig. 2, which represents the dependence of the stability on the interaction strength inside each component.

Until now, we considered only a homogeneous Fermi-Bose gas mixture at finite temperature. In reality, however, experiments with ultracold atoms are performed by trapping and cooling atoms in an external potential that can be generally modeled by an isotropic harmonic oscillator $V(r)=(m/2)\omega_t^2r^2$, where ω_t is the trapping frequency. An exact criterion for the stability of an inhomogeneous Bose-Fermi mixture should involve calculating the Helmholts free energy as a function at all eigenstates of the trapping potential. Fortunately, in the system considered here, it is a good approximation to take use of the local-density approximation (LDA), which treats the system as being locally homogeneous. The LDA requires that the level spacing $\hbar\omega_t$ of the trapping potential is much smaller than the Fermi energy. Of course, the local-density approximation always breaks down at the edge of the gas cloud where the density vanishes and the effective Fermi energy becomes zero. Under the LDA, the stability conditions can still be calculated by means of the equations derived above with the understanding that the effective chemical potentials are spatially dependent through

$$\mu_b=\mu_b^0-(1/2)m_b\omega_b^2r^2,\quad \mu_f=\mu_f^0-(1/2)m_f\omega_f^2r^2.$$

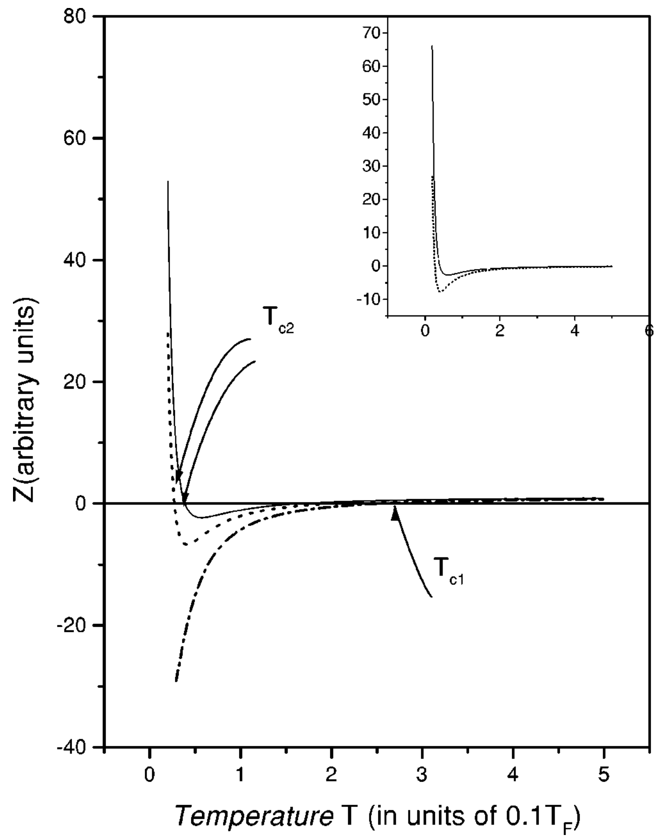

FIG. 1. Plot of Z given by Eq. (3.9) as a function of temperature T. The parameters chosen are $N_b=1000$, $N_f=10\,000$, $a_{bb}=0.05$, $a_{ff}=0.01$. Dashed-dotted line, $a_{bf}=0.3$; dotted line, $a_{bf}=0.02$; solid line, $a_{bf}=0.01$. The dotted line in the inset is the same as the dotted line in the figure, while the solid line in the inset is for the gases in a trap with trapped frequency 166 Hz.

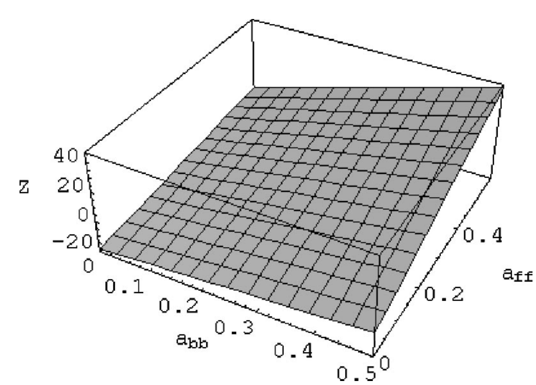

FIG. 2. Plot of Z as a function of a_{ff} and a_{bb}. The parameters chosen are temperature $T=0.1T_F$, $a_{bf}=0.2$.

Thus, a local stability condition is the same as given in Eq. (3.9) but replacing $z_i(i=1,2)$ by

$$\tilde{z}_1=z_1 e^{-(\beta/2)m_b\omega_b^2 r^2},\quad \tilde{z}_2=z_2 e^{-(\beta/2)m_f\omega_f^2 r^2}.$$

As the inset of Fig. 1 shows, the regime of temperature in which the system is unstable, decrease for the case of $r\neq 0$. Physically, the total energy of the system increases when it is in a trap compared with the case without trapped potential. Meanwhile, the chemical potentials decrease in this process within the TFA. The above changes in chemical potentials and in the total energy are equal to a loss of particles in the system. In this sense, the system trapped in potentials is more stable than the case without the trapping potentials.

Alternatively, as given in Ref. [35], the local grand potential for the mixture is

$$\Omega(r)=-k_BT\frac{1}{\lambda_b^3}[g_{5/2}(z_b)-2a_{bb}\rho_b^2\lambda_b^2]+a_{bf}(\lambda_b^2+\lambda_f^2)\rho_b\rho_f - k_BT\left[\frac{1}{\lambda_f^3}f_{5/2}(z_f)-\frac{1}{2}a_{ff}\rho_f^2\lambda_f^2\right], \tag{3.11}$$

where ρ_b and ρ_f stand for the density distributions of the boson and fermion, respectively:

$$\rho_b(r)=\frac{1}{\lambda_b^3}g_{3/2}(\xi_b),$$

$$\rho_f(r)=\frac{1}{\lambda_f^3}f_{3/2}(\xi_f),$$

with

$$\xi_b=z_b\exp[-\beta V_b(r)-4a_{bb}\lambda_b^2\rho_b(r)-2a_{bf}(\lambda_b^2+\lambda_f^2)\rho_f(r)],$$

$$\xi_f=z_f\exp[-\beta V_f(r)-a_{ff}\lambda_f^2\rho_f(r)-2a_{bf}(\lambda_f^2+\lambda_b^2)\rho_b(r)].$$

As usual, the chemical potential is determined by the normalization relations

$$N_b=\int d^3r\rho_b(r),N_f=\int d^3r\rho_f(r). \tag{3.12}$$

For the axially symmetric trapping potential

$$V_i(r)=\frac{1}{2}m_i\omega_i^2(x^2+y^2+\lambda^2z^2)\quad (i=b,f),$$

where m_i,ω_i, and λ are the mass, trapping frequency, and the anistropy of the trapping frequency. The total grand potential of the system may be calculated by intergrating the local grand potential over the whole volume. It is given that

$$\Omega=-\frac{W_b}{\lambda_b^3\beta^{5/2}}[g_{5/2}(z_b)-2a_{bb}B\lambda_b^{-1}] -\frac{W_f}{\lambda_f^3\beta^{5/2}}\left[f_{5/2}(z_f)-\frac{1}{2}a_{ff}F\lambda_f^{-1}\right]+\frac{W_ma_{bf}}{\lambda_b^3\lambda_f^3\beta^{5/2}}M. \tag{3.13}$$

Here,

$$W_i=\frac{4m_i^2\pi^{3/2}}{\lambda_i^2\omega_i^4}\quad (i=b,f),\quad W_m=\frac{4m_bm_f\pi^{3/2}}{\lambda_b\lambda_f\omega_b^2\omega_f^2},$$

$$B=\sum_{i,j=1}^{\infty}\frac{z_b^{i+j}}{[ij(i+j)]^{3/2}},\quad F=\sum_{i,j=1}^{\infty}(-1)^{i+j}\frac{z_f^{i+j}}{[ij(i+j)]^{3/2}},$$

$$M=\sum_{i,j=1}^{\infty}(-1)^j\frac{z_b^iz_f^j}{[ij(i+j)]^{3/2}},$$

where the TFA is used in derivation of Eq. (3.13). From the total grand thermodynamic potential, the thermodynamical quantities such as the entropy, the specific heat can be calculated straightforwardly.

IV. SUMMARY

In this paper, we investigated Bose-Fermi gas mixture at zero and finite temperature. The temperature effects have been taken into account. By the variation method, we obtained conditions for the phase separation at zero temperature, which agree with those found by other authors by a distinct method. For finite temperature, the conditions for phase separation is derived by analyzing thermodynamic stability. These results show different behavior of phase separation at different temperature regimes. Furthermore, we extend the discussion at zero temperature to the case for finite temperature. It is found that phase separation may occur in traps easily relative to the case without trapping potential. This kind of phase separation results from the effects of trapping.

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