Canonical quantum teleportation

Sixia Yu and Chang-Pu Sun*

The Institute of Theoretical Physics, Academia Sinica, Beijing 100080, People's Republic of China

(Received 26 April 1999; revised manuscript received 13 July 1999; published 18 January 2000)

Canonically conjugated observables such as position and momentum and phase and number are found to play a threefold role in quantum teleportation. First, the common eigenstate of two commuting canonical observables like phase difference and number sum provides the quantum channel between two systems. Second, a similar pair of canonical observables from another two systems is measured in the Bell-operator measurements. Finally, two translations generated by the canonically conjugated observables of a single system constitute the local unitary operation to recover the unknown state. In addition, the necessary and sufficient condition is presented for a reliable quantum teleportation of finite-level systems.

PACS number(s): 03.67.Hk, 03.65.Bz, 03.65.Ca, 42.50.Dv

Quantum teleportation [1], a disembodied transmission of quantum state, has been demonstrated in several experiments both for finite-level systems [2] and continuous variables [3–5]. Along with the resulting discussions [6,7] about its experimental realization, many other aspects such as general schemes [8–10] and some applications [11] of quantum teleportation have also been investigated. All these investigations so far emphasize mainly the *states* of the systems. In this paper we shall show the fundamental roles played by the canonically conjugated (c.c.) *observables* in the drama of quantum teleportation in order to reveal the physical contents of its basic ingredients.

Generally speaking, quantum teleportation consists of three basic steps: (i) to prepare two systems in an Einstein-Podolsky-Rosen (EPR) entangled state or a Bell state and send them to two different places to establish a quantum channel; (ii) at one place, to perform the so-called joint Bell-operator measurements with respect to one system involved in the EPR entanglement and a third system at an unknown state to be transferred; (iii) at another place, to perform necessary unitary operations to the other system involved in the EPR entanglement according to the outcome of the Bell-operator measurements. By this means the unknown state is transferred from one place to another.

In the case of continuous variables, three similar systems 1, 2, and 3 are considered, that are described by canonical observables $\hat{x}_a,\hat{p}_a$ $(a=1,2,3)$ satisfying canonical commutation rules

$$[\hat{x}_a,\hat{p}_b]=i\delta_{ab} \quad (a,b=1,2,3). \tag{1}$$

Systems 1 and 2 are prepared in a common eigenstate of the position difference $\hat{x}_1-\hat{x}_2$ and the momentum sum $\hat{p}_1+\hat{p}_2$ corresponding to eigenvalues x_{12} and p_{12} [12]:

$$|x_{12};p_{12}\rangle=e^{-i\hat{p}_1\hat{x}_2}|x_{12}\rangle_1\otimes|p_{12}\rangle_2, \tag{2}$$

where $|x_{12}\rangle_1$ is an eigenstate of $\hat{x}_1$ with eigenvalue x_{12} and $|p_{12}\rangle_2$ is an eigenstate of $\hat{p}_2$ with eigenvalue p_{12}. System 3 is in an unknown state $|\psi\rangle_3$ to be teleported to the first system.

Then a kind of Bell-operator measurement measuring the position difference $\hat{x}_2-\hat{x}_3$ and the momentum sum $\hat{p}_2+\hat{p}_3$ is performed on systems 2 and 3. This measurement projects systems 2 and 3 to one of the common eigenstates $|x_{23};p_{23}\rangle$ of $\hat{p}_2+\hat{p}_3$ and $\hat{x}_2-\hat{x}_3$ with x_{23} and p_{23} taking values on the real line uniformly. Accordingly, system 1 is transformed into the state $\mathcal{O}_c^\dagger|\psi\rangle_1$, where

$$\mathcal{O}_c=e^{-ip_{23}x_{12}}e^{-ip_{13}\hat{x}_1}e^{ix_{13}\hat{p}_1} \tag{3}$$

with $p_{13}=p_{12}-p_{23}, x_{13}=x_{12}+x_{23}$. At this central stage, the measured observables are exactly two commuting canonical observables: momentum sum and position difference. According to the outcomes x_{23},p_{23} of the measurement and values x_{12},p_{12} known from the state preparation, one is able to perform the unitary operation $\mathcal{O}_c$ to system 1. System 1 is then at the unknown state though no one knows what the unknown state is.

Since p_{12} and p_{23} are the momentum sums of corresponding systems, $p_{13}=p_{12}-p_{23}$ is naturally the momentum difference between systems 1 and 3. Similarly $x_{13}=x_{12}+x_{23}$ can be viewed as the position difference between systems 1 and 3. The unitary operation $\mathcal{O}_c$, being made up of two successive translations up to a phase factor, has therefore a natural physical meaning: it compensates the *position difference* and *momentum difference* between systems 1 and 3. This obvious fact was already noticed in Ref. [3], where the teleportation of continuous variables was first proposed.

We see clearly that the c.c. observables, position and momentum in this case, play a threefold role in the drama of quantum teleportation of continuous variables. First, the common eigenstate of two commuting canonical observables, e.g., the position difference and the momentum sum, provides the quantum channel between two systems. Second, the same commuting canonical pair of another two systems is measured in the Bell-operator measurement. Finally the c.c. observables of a single system generate two translations,

*Electronic address: suncp@itp.ac.cn; Internet www site: http://www.itp.ac.cn/~suncp

which make up the unitary operation to recover the unknown state. So quantum teleportation deserves the name *canonical quantum teleportation*.

Given one pair of c.c. observables one may design one possible canonical quantum teleportation with exactly those three steps. Notice that in the procedure of quantum teleportation the real position and momentum cannot be used because localization of the particle is required. In fact in the recent experimental realization of quantum teleportation of continuous variables [4], a pair of c.c. observables of the photon field, phase quadrature, and number quadrature have been used.

At first glance, in the case of the finite-level systems the three steps of quantum teleportation seem to be three unrelated procedures: Bell-state preparation, Bell-operator measurements [13], or nonlocal measurements [3], and special unitary operations, whose physical meanings need clarifying. We shall then demonstrate that there is also a pair of c.c. observables that plays the same threefold role for finite-level systems. As it turns out, one observable is the number operator and the other one is the phase operator of a finite-level system.

For an infinite-level system as simple as a quantum harmonic oscillator, a Hermitian phase operator does not exist [14–16]. After a series of efforts to solve this problem [17–20] it was clear recently that the quantum phase of a harmonic oscillator can only be described by means of the phase difference between two systems with a rational-number-type spectrum and the quantized phase difference obeys a quantum addition rule [20,21]. Among the early approaches to this dilemma, the truncated Hilbert space approach proposed by Pegg and Barnett [18] describes *in de facto* the phase variable of a finite-level system instead of a harmonic oscillator with infinite energy levels. This approach was also investigated in some detail by others [22,23].

For an s-level system A, the number operator $\mathcal{N}_A$ has spectrum $Z_s=\{0,1,\ldots,s-1\}$ and its eigenstates $|n\rangle_A$ with $n\in Z_s$ span the Hilbert space of the system. In this Hilbert space, taking the phase window as $[0,2\pi)$, one can define the exponential phase operator as

$$e^{in\mathcal{P}_A}=\sum_{m\in Z_s}|m+n\rangle_A\langle m|,\quad n\in Z_s. \tag{4}$$

Here the state $|ks+n\rangle_A$ is identified with the state $|n\rangle_A$ whenever k is an integer. This identification seems to be trivial enough for a single system, but it is crucial for the combination of number operators from different systems. The so-defined exponential operator is obviously unitary, which leads to a Hermitian phase operator $\mathcal{P}_A$ with spectrum $\Xi_s=\{2m\pi/s|m=0,1,\ldots,s-1\}$ and eigenstates

$$|\theta\rangle_A=\frac{1}{\sqrt{s}}\sum_{n\in Z_s}e^{-in\theta}|n\rangle_A,\quad \theta\in\Xi_s. \tag{5}$$

The motivation to define a Hermitian phase operator is, analogous to the well-known canonical position and momentum, to find the c.c. partner for the number operator. However, the canonical relationship between the quantum phase and number cannot be explicitly manifested through their commutator. The quantum phase and number have a very complicated commutator [18] due to the fact that the phase variable has a curved configure space because of its periodicity, which is also the origin of the rational-number-type spectrum of quantized phase difference [20]. Only when the unitary operations instead of Hermitian observables are considered does the canonical relationship between the phase and number manifest itself [24]. As shown explicitly in Eqs. (2) and (3) it is also the operations represented by unitary operators instead of the observables represented by Hermitian operators that play the main role in the case of continuous variables.

As is well known, the unitary operations generated by position and momentum that represent the translations in the momentum and configuration spaces, respectively, satisfy the Weyl form of the commutation relation

$$e^{ix\hat{p}}e^{ip\hat{x}}e^{-ix\hat{p}}e^{-ip\hat{x}}=e^{ixp}. \tag{6}$$

This kind of relation also indicates the canonical relationship, even more intrinsically than the commutator. This is because the exponential phase and number operators also satisfy a similar relation

$$e^{i\theta\mathcal{N}_A}e^{in\mathcal{P}_A}e^{-i\theta\mathcal{N}_A}e^{-in\mathcal{P}_A}=e^{in\theta}. \tag{7}$$

In this sense the quantum phase and number operator are c.c. observables. The exponential phase-difference and number-difference operators of two quantum harmonic oscillators also satisfy this kind of relation, which yields another pair of c.c. observables [24].

As relation (6) indicates, the operator $e^{ix\hat{p}}$ represents a translation by x in the configuration space, so relation (7) ensures that the exponential phase operator $e^{-in\mathcal{P}_A}$ also represents a translation by n (modular s) of the number. Similarly, the exponential number operator $e^{i\theta\mathcal{N}_A}$ represents a translation by θ (modular 2π) of the quantum phase. These are exactly the physical contents of these two unitary operations.

The quantum phase and phase differences were found to observe a quantum addition rule $(\dot{+})$ [20], which assures another quantum phase or phase difference with the same kind of spectrum. The quantum addition of phase operators $\mathcal{P}_A$ and $\mathcal{P}_B$ of two s-level systems A and B, since they are commuting, is simply $\mathcal{P}_A\dot{-}\mathcal{P}_B\equiv\mathcal{P}_A-\mathcal{P}_B$ modular 2π. Similarly, to preserve the spectrum of the number operator, the quantum-number sum can be defined as $\mathcal{N}_A\dot{+}\mathcal{N}_B\equiv\mathcal{N}_A+\mathcal{N}_B$ modular s. Because the quantum phase difference and number sum are commuting, they possess common eigenstates

$$|\theta_{AB};n_{AB}\rangle=e^{-i\mathcal{N}_A\mathcal{P}_B}|\theta_{AB}\rangle_A\otimes|n_{AB}\rangle_B, \tag{8}$$

where $|\theta_{AB}\rangle_A$ is the eigenstate of $\mathcal{P}_A$ with eigenvalue $\theta_{AB}\in\Xi_s$ and $|n_{AB}\rangle_B$ is the eigenstate of $\mathcal{N}_B$ with eigenvalue $n_{AB}\in Z_s$. They form a complete and orthonormal basis of systems A and B. These two observables are measurable in the framework of nonlocal measurements [3].

Now that a complete analog between the well-known c.c. observables, position and momentum, and the less obviously c.c. observables, quantum phase and number, has been established, we can formulate the quantum teleportation of finite-level systems in the same canonical manner. As a quantum channel of the quantum teleportation of finite-level systems, systems A and B are prepared in a common eigenstate $|\theta_{AB};n_{AB}\rangle$ of their quantum phase difference and number sum.

Suppose that another s-level system C is in an unknown state $|\phi\rangle_C$ that will be teleported to the system A. To this end we perform a joint measurement of the quantum phase difference $\mathcal{P}_B\dot{-}\mathcal{P}_C$ and the number sum $\mathcal{N}_B\dot{+}\mathcal{N}_C$ of the systems B and C. With probability $1/s^2$, the total state of the whole system $|\Phi\rangle=|\theta_{AB};n_{AB}\rangle\otimes|\phi\rangle_C$ is projected to state $\mathcal{O}_s^\dagger|\phi\rangle_A\propto\langle\theta_{BC};n_{BC}|\Phi\rangle$, where

$$\mathcal{O}_s=e^{-in_{BC}\theta_{AB}}e^{-in_{AC}\mathcal{P}_A}e^{i\theta_{AC}\mathcal{N}_A} \tag{9}$$

with $\theta_{AC}=\theta_{AB}+\theta_{BC}$ and $n_{AC}=n_{AB}-n_{BC}$ after the measurement. The number sum n_{BC} takes a value in Z_s and the phase difference θ_{BC} takes values in Ξ_s with equal probability, which label the s^2 outcomes of the measurements.

After knowing these phase differences θ_{AB},θ_{BC} and number sums n_{AB},n_{BC}, one can perform a unitary transformation $\mathcal{O}_s$ to system A so that the unknown state of system C appears at the other end of the quantum channel. We note that operation $\mathcal{O}_s$ is made up of an exponential phase operator and an exponential number operator up to a phase factor. From the discussions above we know that these two operations represent a phase translation by values θ_{AC} and a number translation by values n_{AC}. Because θ_{AC} can be regarded as the phase difference and n_{AC} as the number difference between systems A and C, the meaning of these two unitary operations becomes clear: before the unknown state can be recovered the phase difference and number difference between systems A and C must be compensated.

Consider the simple case of two-level systems, where we identify state $|0\rangle$ with $|\uparrow\rangle$ and state $|1\rangle$ with $|\downarrow\rangle$. As in the quantum channel we prepare systems A and B in the state as in Eq. (8) with $\theta_{AB}=\pi$, $n_{AB}=1$. Four possible outcomes of the Bell-operator measurements on systems B and C are labeled by phase difference $\theta_{BC}=0,\pi$ and number sum $n_{BC}=0,1$. We can see that four corresponding unitary operations $\mathcal{O}_2$ in Eq. (9) applied to system A are exactly the same as those in Ref. [1].

Canonical transformations, which preserve the canonical commutators among observables as in Eq. (1) or relations such as Eq. (7) of corresponding unitary operations, can be performed to c.c. observables. Some canonical transformations can result in some new forms of quantum teleportations. The simplest case is to make a canonical transformation only to system B, for example, $\mathcal{P}_B\to-\mathcal{P}_B$ and $\mathcal{N}_B\to-\mathcal{N}_B$, which results in quantum teleportation as follows. The quantum channel is a common eigenstate of $\mathcal{P}_A\dot{+}\mathcal{P}_B$ and $\mathcal{N}_A\dot{-}\mathcal{N}_B$, e.g., state

$$|\Psi_{AB}\rangle=\frac{1}{\sqrt{s}}\sum_{m\in Z_s}|m\rangle_A\otimes|m\rangle_B \tag{10}$$

corresponding to zero number difference and zero phase sum. The observables measured in the second step are $\mathcal{P}_B\dot{+}\mathcal{P}_C$ and $\mathcal{N}_B\dot{-}\mathcal{N}_C$. And the final operation Eq. (9) to recover the unknown state remains unchanged. This scheme is exactly the original teleportation of systems with more than two levels discussed in Ref. [1]. One notes that when $s=2$ the quantum phase difference and number sum are identical with quantum phase sum and number difference, respectively; therefore, these two teleportation schemes are identical in the case of $s=2$.

Now we try to take a general pure state of systems A and B as our quantum channel. Any normalized state can be expressed as $T|\Psi_{AB}\rangle$, where operator T acts only on system A with $\mathrm{Tr}(T^\dagger T)=s$. Then we perform a general Bell-operator measurement on systems B and C. This is equivalent to projection to some orthonormal basis of systems B and C,

$$|k;l\rangle=\frac{1}{\sqrt{s}}\sum_{m\in Z_s}|m\rangle_B\otimes\mathcal{O}_{kl}|m\rangle_C, \tag{11}$$

where s^2 operators $\mathcal{O}_{kl}$ act only on a single system and satisfy the following normalization conditions

$$\mathrm{Tr}(\mathcal{O}_{kl}\mathcal{O}^\dagger_{k'l'})=s\,\delta_{kk'}\delta_{ll'},\quad k,k',l,l'\in Z_s. \tag{12}$$

Numbers k,l label all possible outcomes of the measurements. Given outcomes k,l of the measurements, appearing with equal probability, system A is found to be in state

$$s^2\langle k;l|T|\Psi_{AB}\rangle\otimes|\phi\rangle_C=T\mathcal{O}^\dagger_{kl}|\phi\rangle_A, \tag{13}$$

where operator $\mathcal{O}_{kl}$ is now acting on system A. The only requirement for a reliable quantum teleportation is therefore to have $T\mathcal{O}^\dagger_{kl}$ unitary, which infers that T must be reversible. From Eq. (12) one obtains $\mathrm{Tr}[(T^\dagger T)^{-1}]=s$, which is compatible with $\mathrm{Tr}(T^\dagger T)=s$ iff T is unitary. Therefore to have $T\mathcal{O}^\dagger_{kl}$ unitary is equivalent to having all the operators T and $\mathcal{O}_{kl}$ unitary. This is the necessary and sufficient condition for a reliable quantum teleportation. In other words the quantum channel must be a maximum entangled state and the measurements must be projections to maximum entangled states. The recovering operation at the final stage is simply $\mathcal{O}_{kl}T^\dagger$ depending on the outcomes of the measurements.

From orthonormal bases $|k;l\rangle$ one can construct two commuting canonical observables like the phase difference and number sum, whose common eigenstates are exactly these bases. As a result, the measured observables in the second step of the quantum teleportation may be different from the observables determined in the quantum channel. For example, the quantum channel may be provided by the common eigenstate of the quantum phase sum and number difference, and the quantum phase difference and number sum are Bell operators. By this means one can also teleport an unknown state from one place to another. The general scheme discussed in Ref. [10] is included here as a special example.

The continuous variables case can be analyzed similarly. Let us fix our measurements at the second step to the projections to states $|x_{23},p_{23}\rangle$. All the pure states that can be used as quantum channels should have the form $\Sigma_{n=0}^{\infty}D^{\dagger}|n\rangle_1 \otimes|n\rangle_2$ where D is an arbitrary unitary operator acting on system one only. The operation at the final stage is $M^{\dagger}D$, where M is a unitary operator acting on system one with elements $\langle m|M|n\rangle=\langle x_{23},p_{23}|m,n\rangle$, where $|m,n\rangle=|m\rangle_1 \otimes|n\rangle_2$ denotes the number-state bases with m,n going from zero to infinity. When the elements of D are taken as $\langle m|D|n\rangle=\langle x_{12},p_{12}|m,n\rangle$, the teleportation of continuous variables discussed at the beginning is regained. This discrete formulation of the quantum channel up to a normalization constant was noted in Ref. [25].

When we consider three quantum harmonic oscillators, although the quantized phase differences between any two of them are well defined, it is impossible to perform a quantum teleportation using the quantized phase difference and number sum. This is because the exponential phase operator of a single oscillator, which should be employed to compensate a number difference at the final stage of the quantum teleportation, does not exist.

In conclusion, quantum teleportation is characterized by c.c. observables completely: The quantum channel is provided by the common eigenstate of two commuting canonical observables, the Bell-operator measurement measures a similar pair of canonical observables, and the recovering operation consists of two translations generated by the c.c. observables. By applying suitable canonical transformations to the c.c. observables, one can design new schemes of quantum teleportation. The necessary and sufficient condition for a reliable quantum teleportation of finite systems is to have a maximum entangled state as a quantum channel and the Bell-operator measurements be projections to maximum entangled states. The nonexistence of certain c.c. observables makes quantum teleportation using these variables impossible. All these investigations concern the ideal quantum teleportation. In real experiments where nonideal elements must be considered, it becomes ambiguous as to how to characterize quantum teleportation. In this aspect some efforts have been made [26]. The attention to the c.c. observables in quantum teleportation may help to establish such kinds of criteria both for the continuous and discrete variables.

One of the authors (S.Y.) gratefully acknowledges the support of the K.C. Wong Education Foundation, Hong Kong. This work was also partially supported by the NSF of China.

[1] C. Bennett, G. Brassard, C. Crepeau, R. Jozsa, A. Peres, and W. K. Wooters, Phys. Rev. Lett. **70**, 1895 (1993).

[2] D. Boschi, S. Branca, F. De Martini, L. Hardy, and S. Popescu, Phys. Rev. Lett. **80**, 1121 (1998); D. Boumeester, J.-W. Pan, K. Mattle, M. Eibl, H. Weinfurter, and A. Zeilinger, Nature (London) **390**, 575 (1997); M. A. Nielsen, E. Knill, and R. Laflamme, *ibid.* **396**, 52 (1998).

[3] L. Vaidman, Phys. Rev. A **49**, 1473 (1994).

[4] S. L. Braunstein and H. J. Kimble, Phys. Rev. Lett. **80**, 869 (1998).

[5] A. Furasawa, J. L. Sorensen, S. L. Braunstein, C. A. Fuchs, H. J. Kimble, and E. S. Polzik, Science **282**, 706 (1998).

[6] S. L. Braunstein and H. J. Kimble, Nature (London) **394**, 840 (1997); D. Bouwmeester, J.-W. Pan, M. Daniell, H. Weinfurter, M. Zukowski, and A. Zeilinger, *ibid.* **394**, 841 (1997).

[7] L. Vaidman and N. Yoran, Phys. Rev. A **59**, 116 (1999).

[8] C. H. Bennett, G. Brassard, S. Popescu, B. Schumacher, J. A. Smolin, and W. K. Wooters, Phys. Rev. Lett. **76**, 722 (1996).

[9] M. Tor, e-print quant-ph/9608005.

[10] S. Stenholm and P. J. Bardroff, Phys. Rev. A **58**, 4373 (1998).

[11] C. S. Maierle, D. A. Lidar, and R. A. Harris, Phys. Rev. Lett. **81**, 5928 (1998); M. S. Zubairy, Phys. Rev. A **58**, 4368 (1998).

[12] G. J. Milburn and S. L. Braunstein, e-print quant-ph/9812018.

[13] S. L. Braunstein and A. Mann, Phys. Rev. A **51**, R1727 (1995).

[14] P. A. M. Dirac, Proc. R. Soc. London, Ser. A **114**, 193 (1927).

[15] R. Lynch, Phys. Rep. **367**, 256 (1995).

[16] K. Fujikawa, Phys. Rev. A **52**, 3299 (1995).

[17] R. G. Newton, Ann. Phys. (N.Y.) **124**, 327 (1980).

[18] D. T. Pegg and S. M. Barnett, Phys. Rev. A **39**, 1665 (1989).

[19] A. Luis and L. L. Sánchez-Soto, Phys. Rev. A **48**, 4702 (1993).

[20] S. Yu, Phys. Rev. Lett. **79**, 780 (1997).

[21] S. Yu and Y. Zhang, J. Math. Phys. **39**, 5260 (1998).

[22] A. Vourdas, Phys. Rev. A **41**, 1653 (1990).

[23] L. Luis and L. L. Sánchez-Soto, Phys. Rev. A **47**, 1492 (1993).

[24] S. Yu (unpublished).

[25] S. J. van Enk, e-print quant-ph/9905081.

[26] T. C. Ralph and P. K. Lam, Phys. Rev. Lett. **81**, 5668 (1998).