Theory of four-wave mixing with matter waves without the undepleted pump approximation

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Starting from the Gross-Pitaevskii nonlinear Schrödinger equation, we derive a set of nonlinear coupled equations describing four-wave mixing with matter waves and solve them analytically without utilizing the undepleted pump approximation.

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I. INTRODUCTION

The realization of Bose-Einstein condensation in small atomic samples [1] has made it possible to provide us with an atom laser [2], i.e., a source of atoms with the required long-range quantum coherence, and to extend nonlinear optics to the ''matter-wave'' domain [3], i.e., to the field of nonlinear atom optics [4,5]. In analogy to the optical case of phase conjugation by four-wave mixing, Meystre and his coworkers have developed a theory describing atomic phase conjugation [5] and four-wave mixing [6] from a Bose-Einstein condensate. Their theory is not based on the Gross-Pitaevskii nonlinear Schrödinger equation familiar in the description of Bose-Einstein condensates, but is based on the nonlinear Schrödinger equation of atom optics [5]. In the former equation, the effective nonlinearity results from short-range interactions between ground state atoms, while in the latter case it results from the near-resonant dipole-dipole interaction between ground and excited atoms. It was thought that the former nonlinearity might not be strong enough to be utilized to observe four-wave mixing with matter waves for existing Bose-Einstein condensates and hence we had to resort to the latter one for such an observation. However, a recent theoretical calculation showed that the nonlinearity associated with the interactions between ground state atoms is large enough to observe four-wave mixing with wave packets created from existing Bose-Einstein condensates. As a matter of fact, in a recent spectacular experiment on four-wave mixing with matter waves, Deng *et al.* nicely demonstrated this conclusion [3]. This important experiment opens up, to the best of our knowledge, a new area in the study of interacting quantum fluids, and implicitly shows how the physics of Bose-Einstein condensates has a multidisciplinary influence. In particular, it is sure to stimulate further ample theoretical study on four-wave mixing with matter waves and atomic phase conjugation based on the Gross-Pitaevskii nonlinear Schrödinger equation or its extension by including dissipations. In this paper, we shall, from the Gross-Pitaevskii nonlinear Schrödinger equation, develop a theory of the four-wave mixing with matter waves without using the undepleted pump approximation. The paper is organized as follows. Section II is divided into two subsections. The first one is devoted to the description of the four-wave mixing system and the derivation of corresponding governing equations, and the second one deals with solving the nonlinear equations. Some details used in obtaining the analytical solutions are included in the Appendix. In Sec. III, we provide a discussion on the results obtained without using the undepleted pump approximation for four-wave mixing with matter waves, and make comparisons between our results and those under the undepleted approximation. Section IV concludes our paper with a summary.

II. THEORY OF FOUR-WAVE MIXING WITH MATTER WAVES

A. Derivation of governing equations

In this subsection, we shall derive a set of equations governing four-wave mixing with matter waves within the framework of the Gross-Pitaevskii nonlinear Schrödinger equation. For a Bose-Einstein condensate in a trapping potential U, the macroscopic wave function Ψ satisfies the Gross-Pitaevskii nonlinear Schrödinger equation [3,7,8]

$$i\hbar\frac{\partial\Psi}{\partial t}=\left(-\frac{\hbar^2}{2M}\nabla^2+U+U_0|\Psi|^2\right)\Psi, \tag{1}$$

where M is the atomic mass, U_0 describes the strength of the atom-atom interaction ($U_0>0$ for sodium atoms), and $|\Psi|^2$ is the atomic number density.

In order to investigate four-wave mixing with matter waves, we consider the situation where there exist three overlapping wave packets with macroscopic wave functions Ψ_j and momenta $\mathbf{P}_j$ ($j=1,2,3$) and their interactions lead to the creation of the four-wave mixing wave packet with its macroscopic wave function Ψ_4 and its momentum $\mathbf{P}_4$ that satisfies the momentum conservation [3]

$$\mathbf{P}_4=\mathbf{P}_1-\mathbf{P}_2+\mathbf{P}_3. \tag{2}$$

Therefore the total macroscopic wave function in Eq. (1) has the form $\Psi=\Psi_1+\Psi_2+\Psi_3+\Psi_4$. We further assume that each component of the total macroscopic wave function satisfies its own Gross-Pitaevskii nonlinear Schrödinger equation as follows:

$$i\hbar\frac{\partial\Psi_m}{\partial t}=\left(-\frac{\hbar^2}{2M}\nabla^2+U+U_0|\Psi_m|^2\right)\Psi_m, \tag{3}$$

where $m=1,2,3,4$. In dealing with the four-wave mixing with matter waves in this paper, what we have in our mind is the same physical picture and the same situation as the ones described in Ref. [3]. Therefore we omit the schematic plot of the four-wave mixing which is clearly shown in Fig. 1 of Ref. [3], and derive the governing equations for the four-wave mixing in the form of differential equations with respect to time t, different from the usual form in the field of optical four-wave mixing where the derivative is normal with respect to a spatial variable.

Substituting the total macroscopic wave function $\Psi=\Psi_1+\Psi_2+\Psi_3+\Psi_4$ into Eq. (1), ultilizing Eqs. (2) and (3), and noting $\Psi_m \propto \exp(i\mathbf{P}_m\cdot\mathbf{r}/\hbar)$, we easily obtain, by keeping only the phase-matching terms, the following equations:

$$i\frac{\partial\Psi_1}{\partial t}=R\Psi_2+\Omega_{34}\Psi_1; \quad i\frac{\partial\Psi_2}{\partial t}=R^*\Psi_1+\Omega_{34}\Psi_2, \tag{4a}$$

$$i\frac{\partial\Psi_3}{\partial t}=R^*\Psi_4+\Omega_{12}\Psi_3; \quad i\frac{\partial\Psi_4}{\partial t}=R\Psi_3+\Omega_{12}\Psi_4, \tag{4b}$$

where

$$\Omega_{ij}=\frac{2U_0}{\hbar}(|\Psi_i|^2+|\Psi_j|^2), \tag{5a}$$

$$R=\frac{2U_0}{\hbar}(\Psi_1\Psi_2^*+\Psi_3^*\Psi_4). \tag{5b}$$

In obtaining Eq. (4), we have restricted to plane-wave approximation for simplicity but it is also appropriate for the experiments of Ref. [3] in which the trap is switched off before four-wave mixing processes. Equation (4) can be further simplified to the form

$$i\frac{\partial\Phi_1}{\partial t}=R\Phi_2; \quad i\frac{\partial\Phi_2}{\partial t}=R^*\Phi_1, \tag{6a}$$

$$i\frac{\partial\Phi_3}{\partial t}=R^*\Phi_4; \quad i\frac{\partial\Phi_4}{\partial t}=R\Phi_3, \tag{6b}$$

by letting $\Phi_j=\Psi_j\exp(-i\int_0^t\Omega_{34}dt)$, $j=1,2$, $\Phi_k=\Psi_k\exp(-i\int_0^t\Omega_{12}dt)$, $k=3,4$. Note that $|\Psi_l|=|\Phi_l|$, $l=1,2,3,4$, and also that R and Ω_{ij} in Eq. (5) can be expressed in terms of Φ_l as follows:

$$\Omega_{ij}=\frac{2U_0}{\hbar}(|\Phi_i|^2+|\Phi_j|^2), \tag{7a}$$

$$R=\frac{2U_0}{\hbar}(\Phi_1\Phi_2^*+\Phi_3^*\Phi_4). \tag{7b}$$

Equation (6) is the set of nonlinear coupled equations describing the four-wave mixing with matter waves within the framework of the Gross-Pitaevskii nonlinear Schrödinger equation.

The variable R represents explicitly the grating structure formed by the mixing of waves 1 and 2 as well as the mixing of the waves 3 and 4. This structure causes the coupling between waves 1 and 2 as shown in Eq. (6a), and the coupling between waves 3 and 4 as shown in Eq. (6b). In the situation of the four-wave mixing experiment by Deng *et al.* [3], waves 1 and 2 serve as the pump waves, wave 3 is the probe wave, and wave 4 denotes the created matter wave. Equation (6) is a set of four equations coupled nonlinearly together and hence is very difficult to be solved analytically. When two pump waves 1 and 2 are supposed to be much stronger than both the probe wave 3 and the created wave 4, one can neglect the time variation of the pump waves and neglect the grating formed by waves 3 and 4 [amount to take the R in Eq. (5) approximately as a time-independent quantity $R\approx(2U_0/\hbar)\Psi_1\Psi_2^*$]. This is the so-called undepleted pump approximation. Great simplification results from such an approximation since the above-mentioned approximated R is considered as a time-independent quantity and one needs only to solve Eq. (6b) which represents now two linear coupled equations for waves 3 and 4. In this paper, we do not resort to the undepleted pump approximation. This is done by considering simultaneously the set of four nonlinear coupling equations (6) governing four-wave mixing (FWM) with matter waves.

B. Solving FWM equations without the undepleted pump approximation

Some of the authors [9] have already developed a method to deal with a number of nonlinear coupled equations. Such a method is easily modified to solve Eq. (6) analytically. First, it can be shown from Eq. (6) that there exist the following conserved quantities (see the Appendix for their detailed derivations):

$$|\Psi_1(t)|^2+|\Psi_2(t)|^2=n_{10}+n_{20}, \tag{8a}$$

$$|\Psi_3(t)|^2+|\Psi_4(t)|^2=n_{30}+n_{40}, \tag{8b}$$

$$|\Psi_1(t)|^2+|\Psi_4(t)|^2=n_{10}+n_{40}, \tag{8c}$$

$$|\Psi_2(t)|^2+|\Psi_3(t)|^2=n_{20}+n_{30}, \tag{8d}$$

$$|\Psi_2(t)|^2-|\Psi_4(t)|^2=n_{20}-n_{40}, \tag{8e}$$

$$|\Psi_1(t)|^2-|\Psi_3(t)|^2=n_{10}-n_{30}, \tag{8f}$$

where $n_{m0}=|\Psi_m(t=0)|^2$ denotes the initial number density of the mth wave packet ($m=1,2,3,4$). The last two equations in Eq. (8) are not independent of the other four equations in Eq. (8) but are a direct consequence of them. Note that Eq. (8) clearly manifests the particle-number conservation required by the four-wave mixing with matter waves [3], and how the redistribution of four matter waves among themselves is restricted by these conservation laws during the four-wave mixing process. Subtraction of Eqs. (8e) and (8f) immediately leads to the relation

$$\{|\Psi_1(t)|^2+|\Psi_4(t)|^2-|\Psi_2(t)|^2-|\Psi_3(t)|^2\}$$
$$=(n_{10}+n_{40}-n_{20}-n_{30})\equiv\frac{\hbar}{2U_0}G. \tag{9}$$

The next step is to explicitly find out the functional dependence of the grating structure characterized by the variable R on the time t. It can be proved from Eqs. (6), (7), and (9) that $\partial \ln R/\partial t=iG=$const (see the Appendix for its derivation), and hence

$$R=R_0\exp(iGt), \tag{10}$$

with the time-independent quantity R_0 as

$$R_0=\frac{2U_0}{\hbar}\{\Psi_1(0)\Psi_2^*(0)+\Psi_3^*(0)\Psi_4(0)\}. \tag{11}$$

Now we are ready to reduce a set of highly nonlinear equations in Eq. (6) into two subsets of linear coupled equations as follows:

$$i\frac{\partial\Phi_1}{\partial t}=R_0\exp(iGt)\Phi_2, \tag{12a}$$

$$i\frac{\partial\Phi_2}{\partial t}=R_0^*\exp(-iGt)\Phi_1, \tag{12b}$$

$$i\frac{\partial\Phi_3}{\partial t}=R_0^*\exp(-iGt)\Phi_4; \tag{13a}$$

$$i\frac{\partial\Phi_4}{\partial t}=R_0\exp(iGt)\Phi_3, \tag{13b}$$

where $\Phi_j=\Psi_j\exp(-i\Omega_{34}t)$, $j=1,2$, $\Phi_k=\Psi_k\exp(-i\Omega_{12}t)$, $k=3,4$, and use has been made of the fact that Ω_{12} and Ω_{34} are time-independent quantities by means of Eq. (8) and their definition in Eq. (5a). Note that in Eqs. (12) and (13), one subset of equations decoupled from another subset of equations, and the two subsets of equations are linear and have identical mathematical structure. These linear coupled equations are solved and we get the analytical expressions of the macroscopic wave functions,

$$\Psi_1(t)\exp[-i(\Omega_{34}+G/2)t]$$
$$=\Psi_1(0)\cos\xi t-i\frac{2R_0\Psi_2(0)+G\Psi_1(0)}{2\xi}\sin\xi t, \tag{14a}$$

$$\Psi_2(t)\exp[-i(\Omega_{34}-G/2)t]$$
$$=\Psi_2(0)\cos\xi t+i\frac{G\Psi_2(0)-2R_0^*\Psi_1(0)}{2\xi}\sin\xi t, \tag{14b}$$

$$\Psi_3(t)\exp[-i(\Omega_{12}-G/2)t]$$
$$=\Psi_3(0)\cos\xi t+i\frac{G\Psi_3(0)-2R_0^*\Psi_4(0)}{2\xi}\sin\xi t, \tag{14c}$$

$$\Psi_4(t)\exp[-i(\Omega_{12}+G/2)t]$$
$$=\Psi_4(0)\cos\xi t-i\frac{2R_0\Psi_3(0)+G\Psi_4(0)}{2\xi}\sin\xi t, \tag{14d}$$

where $\Omega_{ij}\equiv\Omega_{ij}^{(0)}=2U_0(n_{i0}+n_{j0})/\hbar$, $\xi=\sqrt{|R_0|^2+(G/2)^2}$, the parameters G and R_0 are given by Eqs. (7) and (9), respectively, and all of them are time-independent quantities and are determined by the initial conditions. The above analytical expressions of the macroscopic wave functions have already satisfied the corresponding initial conditions, for Eq. (4) and its analytical solutions (14) explicitly describe the four-wave mixing with matter waves without the undepleted pump approximation within the framework of the Gross-Pitaevskii nonlinear Schrödinger equation. They are the central results of the present paper.

III. DISCUSSION OF THE FWM RESULTS

Let us first discuss further some points related to the undepleted pump approximation. This approximation is based on the assumption that throughout the FWM process, two pump waves 1 and 2 are much stronger than both the probe wave 3 and the created wave 4. Hence if this assumption is satisfied, both the macroscopic wave functions of the two pumps can be taken as constant, and the grating structure formed by mixing the probe wave 3 and the created wave 4 is compared with the much stronger grating formed by mixing the two pumps. These two factors combined amounts mathematically to take the R in Eq. (5) approximately as a time-independent quantity, i.e., $R\approx(2U_0/\hbar)\Psi_1\Psi_2^*\approx$const as we have mentioned before.

Obviously, the correctness of the undepleted pump approximation depends crucially on the assumption mentioned before that the two pumps are much stronger than the other two waves throughout the FWM process. Note that some data of the FWM experiment by Deng *et al.* [3] violate clearly such an assumption. For instance, one set of data given in Ref. [3], p. 220, paragraph three, is $N_1\sim4.8\times10^5, N_2\sim5.3\times10^5, N_3\sim5.1\times10^5, N_4\sim1.8\times10^5$.

We need to clarify two points concerning our results without the undepleted pump approximation. First, the sum of the number densities of the two pumps is conserved as shown in Eq. (8a) but any one of the two pump densities does vary with respect to time as shown in Eq. (14). This is significantly different from the conclusion under the undepleted approximation that any *one* of the wave functions (and hence densities) of the two pumps is assumed to be time independent. Second, the fact that the amplitude of the R expression in Eq. (10) characterizing the grating structure turns out to be time independent does not imply that the pumps are unde-

pleted. It only means that the grating structure formed by both the mixing of the pumps and the mixing of probe wave 3 and signal wave 4, not by only one of them, manifests only the phase variation with respect to time. As a matter of fact, the part of the grating structure produced only by the mixing of the pumps does generally vary with respect to time, which is obvious by looking at the derivation of Eq. (10) in the Appendix.

To illustrate our theory on the four-wave mixing with matter waves, let us now discuss a simple but important special situation where there exists no matter-wave 4 at the beginning, and thus $\Psi_4(t=0)=0$. In this specific case, we obtain from Eqs. (9) and (11)

$$|R_0|^2=\left(\frac{2U_0}{\hbar}\right)^2 n_{10}n_{20},$$

$$G=\frac{2U_0}{\hbar}(n_{10}-n_{20}-n_{30}).$$

The parameter ξ in Eq. (14) becomes

$$\xi=\frac{U_0}{\hbar}\sqrt{4n_{10}n_{20}+(n_{10}-n_{20}-n_{30})^2}, \tag{15}$$

and hence $[n_4(\tau)\equiv|\Psi_4(\tau)|^2]$

$$n_4(\tau)=\left(\frac{2U_0}{\hbar}\right)^2 n_{10}n_{20}n_{30}\tau^2F, \tag{16}$$

where τ is the characteristic interaction time of the four-wave mixing and it is proportional to the diameter of the condensate [3], and

$$F\equiv\frac{\sin^2(\xi\tau)}{(\xi\tau)^2}. \tag{17}$$

Equation (16) explicitly expresses number density of the four-wave mixing signal and its nonlinear dependence on the numbers of atoms in the initial wave packets. It is easily shown that Eq. (16) in taking $F\approx 1$, i.e.,

$$n_4(\tau)=\left(\frac{2U_0}{\hbar}\right)^2 n_{10}n_{20}n_{30}\tau^2, \tag{18}$$

is the corresponding result under the undepleted pump approximation. The number density of each momentum component of the condensate in Eqs. (16) and (18) can be taken as $n_{j0}\approx N_{j0}/V$, $j=1,2,3$, and $n_4(\tau)\approx N_4(\tau)/V$. Here V is the volume of the condensate, N_{j0} is the initial total particle number of the jth momentum component of the condensate, and $N_4(\tau)$ is the total particle number of the fourth momentum component of the condensate at time τ. In the Thomas-Fermi limit [3,8], the volume of the condensate $V\propto N^{3/5}$, and $\tau\propto N^{1/5}$ where N is the total particle number of the condensate independent of time t, and can be put into form $N=\Sigma_{j=1}^{3}N_{j0}$ in this specific case. The scaling $V\propto N^{3/5}$ in the Thomas-Fermi limit applies of course to the Thomas-Fermi model of atoms in a harmonic potential considered here. Hence one, from Eq. (18), obtains $N_4/N\propto(2U_0/\hbar)^2(N_{10}N_{20}N_{30})N^{-9/5}$, a result derived under the undepleted approximation. This kind of nonlinear behavior is clearly manifested in the initial linear growth seen in Fig. 3 of Ref. [3] and has already been theoretically predicted in the same Ref. [3].

Now, we are ready to point out two pronounced differences between the result without the undepleted approximation in Eq. (16) and the one with the approximation in Eq. (18). The first difference is as follows. The result under the undepleted approximation manifests the dependence of the relative density N_4/N on a unique variable $\eta\equiv(N_{10}N_{20}N_{30})N^{-9/5}$, and hence plotting the relative density N_4/N versus the variable η is well defined. However, choosing the variable η as the abscissa in plotting the relative density N_4/N may not lead to a well-behaved curve for the situation without using the undepleted approximation because N_4/N in this case, as shown in Eq. (16), is not the single-valued function of the variable η due to the existence of the factor F depending on the variable

$$\begin{aligned}\xi\tau&\propto\sqrt{4N_{10}N_{20}+(N_{10}-N_{20}-N_{30})^2}N^{-2/5}\\&\propto\frac{\sqrt{4N_{10}N_{20}+(N_{10}-N_{20}-N_{30})^2}}{(N_{10}N_{20}N_{30})^{2/9}}\eta^{2/9}\\&\equiv Q(N_{10},N_{20},N_{30})\,\eta^{2/9}.\end{aligned} \tag{19}$$

Consequently, for every fixed η, N_4/N may display scattered values depending on the values of the function $Q(N_{10},N_{20},N_{30})$ or on the total number N of atoms in the condensate and its initial distribution among each wave packet. In other words, the figure describing the relative density N_4/N versus the variable η will probably display (sometimes, possibly wide) spread around a well-behaved curve. The wide spread tends to occur at the place where the F factor varies fast or the F's derivative is significant (note that this can occur even if the variable η is small but with a large Q value) and it is easily confused with the spread coming from experimental errors. In addition, this non-single-valued property of N_4/N on η might also explain the puzzle in Ref. [3] that the observed conversion efficiency N_4/N takes different values (say, sometimes 10% and sometimes 6%) under similar conditions. Reference [3] attributes this difference to the influence of some uncontrolled experimental conditions. It is pointed out that even in the linear regime in Fig. 3 of Ref. [3], there are some relatively large spreads, and we do not know whether the spreads originate from the above reason or from the experimental errors. However, what we are sure of at present is that the variable η may not be an appropriate abscissa in plotting the relative density N_4/N if the undepleted pump assumption fails.

The second difference comes from the saturation effect characterized by the F factor in Eq. (16). The relative density N_4/N evaluated by the results under the undepleted approximation fails to give a correct estimation if the F factor deviates significantly from unity which occurs, roughly speaking, at large N. Let us illustrate this point by using the expressions of the F factor in Eq. (17) and its argument $\xi\tau$ in Eq.

(19). For the typical data in Ref. [3], $N_{10}\sim N_{20}\sim N_{30}\propto N$ and hence $\xi\tau$ in Eq. (19) has approximately the relation $\xi\tau\sim(U_0/\hbar)N\tau/V\propto N^{3/5}$. This relation clearly demonstrates that a large N has a stronger tendency to make the F factor deviate from unity.

Let us give some further discussion on the saturation effect of the relative density N_4/N. The growth of the created wave 4 is at the expense of decreasing both the probe wave 3 [see, for example, Eq. (8a)] and the pump wave 1 [see Eq. (8c)] and, at the same time, of increasing another pump 2 [see, for example, Eq. (8e)]. It is seen from Eq. (5b) that the mixing of two waves (waves 1 and 2, or waves 3 and 4) is roughly speaking more effective in contributing the grating structure when these two waves have comparable magnitudes. Consequently, the part of the grating structure formed by the mixing of the probe wave 3 and created wave 4 becomes progressively more important as N_4/N is growing from zero magnitude towards approaching the order of N_3/N, while the one formed by the mixing of the two pumps having roughly equal magnitude initially becomes less and less effective in creating wave 4 during such a process, which leads to a slower growth of wave 4, i.e., the saturation effect. Finally, such a growing period could cease and a reverse process sets in as the mixing of probe wave 3 and created wave 4 becomes strong enough to make them as two new pumps while the old pumps serve as the new probe and new created waves in the reverse process. Such a reverse process may cause the already produced wave 4 to become zero again (similiar in some sense to revivals). Of course, this is possible only when the characteristic interaction time $\tau\propto N^{1/5}$ is sufficiently long so that $\xi\tau\geqslant\pi$.

Before ending this section, we would like to estimate the order of the total atoms N by the condition $\xi\tau=\pi$. For this purpose, we consider a condensate in an isotropic harmonic potential with a trap frequency $\nu_T=\omega_T/2\pi$. The Thomas-Fermi limit results in the relation between the volume $V=4\pi R^3/3$ and the total atom number N as follows:

$$V=\frac{4\pi}{3}\left(\frac{15U_0}{4\pi M\omega_T^2}\right)^{3/5}N^{3/5}. \tag{20}$$

The characteristic interaction time is taken as $\tau\approx V^{1/3}/v$ with the lowercase letter $v=P/M$ denoting the characteristic speed of the wave packets. We take $N_{10}\sim N_{20}\sim N_{30}\sim N/3$ to estimate $\xi\tau=\pi$ which, by using Eq. (15) and the above discussion, becomes

$$\pi=\xi\tau\approx\left(\frac{3}{4\pi}\right)^{2/3}\left(\frac{\sqrt{5}U_0}{3\hbar v}\right)\left(\frac{15U_0}{4\pi M\omega_T^2}\right)^{-2/5}N^{3/5}. \tag{21}$$

Setting $N\equiv N_\pi$ in the above equation, we arrive at

$$N_\pi\approx\left(\frac{4\pi}{3}\right)^{10/9}\left(\frac{3\pi\hbar v}{\sqrt{5}U_0}\right)^{5/3}\left(\frac{15U_0}{4\pi M\omega_T^2}\right)^{2/3}, \tag{22}$$

where the interaction strength $U_0=4\pi\hbar^2\bar{a}/M$ with $\bar{a}\approx 53a_{bohr}$ [10], $\omega_T\approx 2\pi\times 60$ Hz, $v=P/M$, and $P\approx 2\hbar k$, $k=2\pi/(589\text{ nm})$ [3]. Therefore, $\xi\tau=\pi$ gives a rough estimation $N=N_\pi\approx 3.6\times 10^6$. For the same parameters as those taken in the above estimation, we obtain $\xi\tau\approx\pi(N/N_\pi)^{3/5}$. This relation demonstrates that $N\sim 1.1\times 10^6$, a typical value for the experiment of Ref. [3], leads to $\xi\tau\sim\pi/2$, which also illustrates that the F factor in Eq. (16) should not generally be taken to be unity for that experiment. However, it should be kept in mind that the above estimation is based on the condition $N_{10}\sim N_{20}\sim N_{30}\sim N/3$. It varies if this condition is not satisfied.

IV. CONCLUSION

In summary, the theory of four-wave mixing with matter waves has been developed in this paper within the framework of the Gross-Pitaevskii nonlinear Schrödinger equation. We have derived a set of four highly nonlinear-coupled equations describing four-wave mixing with matter waves. We have obtained the analytical expressions of the macroscopic wave functions for the two pump wave packets, the probe wave packet, and the four-wave mixing signal. Our results are those without making the undepleted pump approximation, and are compared with the results under the undepleted pump approximation. Our results also display that the variable $\eta\equiv(N_{10}N_{20}N_{30})N^{-9/5}$ may not be an appropriate abscissa in plotting the relative density N_4/N if the undepleted pump assumption fails. The reason is that in this case the figure of N_4/N versus η may not be a well-defined curve but may show some kind of (possibly wide) spread.

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APPENDIX

In this Appendix, we prove Eqs. (8)–(10). It is seen from Eq. (6) that

$$-iR=\frac{\partial\Phi_1}{\partial t}\Big/\Phi_2=-\frac{\partial\Phi_2^*}{\partial t}\Big/\Phi_1^*, \tag{A1a}$$

$$-iR=\frac{\partial\Phi_4}{\partial t}\Big/\Phi_3=-\frac{\partial\Phi_3^*}{\partial t}\Big/\Phi_4^*. \tag{A1b}$$

These two equations immediately result in Eqs. (8a) and (8b), respectively, by noting that $|\Psi_l|=|\Phi_l|$, $l=1,2,3,4$.

By using Eq. (6), we obtain

$$i\frac{\partial|\Phi_1|^2}{\partial t}=-R^*\Phi_1\Phi_2^*+R\Phi_1^*\Phi_2, \tag{A2a}$$

$$i\frac{\partial|\Phi_4|^2}{\partial t}=-R^*\Phi_3^*\Phi_4+R\Phi_3\Phi_4^*. \tag{A2b}$$

Summing these two equations and utilizing the R expression in Eq. (7b), we arrive at

$$\frac{\partial}{\partial t}(|\Phi_1|^2+|\Phi_4|^2)=R(\Phi_1\Phi_2^*+\Phi_3^*\Phi_4)^* - R^*(\Phi_1\Phi_2^*+\Phi_3^*\Phi_4) = 0. \tag{A3}$$

Following the same routine as above, we get

$$\frac{\partial}{\partial t}(|\Phi_2|^2+|\Phi_3|^2)=0. \tag{A4}$$

Equations (8c) and (8d) are a direct consequence of these two equations, respectively. In addition, Eq. (9) is obviously proved by combining these two equations and noting again the fact that $|\Psi_l|=|\Phi_l|$, $l=1,2,3,4$.

The proof of Eq. (10) is straightforward. Utilizing the R expression in Eq. (7b), and the governing equation (6), one yields

$$\frac{\hbar}{2U_0}\frac{\partial R}{\partial t}=\left(\Phi_1\frac{\partial\Phi_2^*}{\partial t}+\Phi_2^*\frac{\partial\Phi_1}{\partial t}\right)+\left(\Phi_4\frac{\partial\Phi_3^*}{\partial t}+\Phi_3^*\frac{\partial\Phi_4}{\partial t}\right) = iR(|\Phi_1|^2+|\Phi_4|^2-|\Phi_2|^2-|\Phi_3|^2), \tag{A5}$$

which leads to Eq. (10) immediately. Equation (11) follows from Eq. (5b), one of the R expression equivalent to Eq. (7b).

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