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Factoring the unitary evolution operator and quantifying entanglement

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Abstract

The unitary evolution can be represented by a finite product of exponential operators. It leads to a perturbative expression of the density operator of a close system. Based on the perturbative expression scheme, we present an entanglement measure. This measure has the advantage that it is easy to compute for a general dynamical process. © 1999 Published by Elsevier Science B.V. All rights reserved.

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1. Introduction

Over the past decade information theory has been generalized to include quantum mechanical systems, for example, a two-level quantum system has come to be known as a qubit in this context. The additional freedom introduced with the quantum mechanical superposition principle has opened up a variety of capabilities that go well beyond those of conventional information techniques. There are two distinct directions in which progress is currently being made: quantum computation and error correction or prevention on the one hand [1], and nonlocality and distillation, on the other hand [2]. In each of those progresses, quantum entanglement that provides a good measure of quantum correlations plays an important role.

There are a number of good measures of the amount of entanglement for two quantum systems in a pure state, a good measure of entanglement for mixed states is also found though it is hard to compute for a general state (see, for example [3]).

In this Letter, considering the entanglement from the other aspect, we prefer to discuss the change of entanglement due to the state changes rather than to compute straightforwardly the entanglement of an arbitrary state.

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In the framework of quantum information theory, the state change allowed by quantum mechanics may be treated in terms of quantum operations [3], a simple example is the unitary evolution experienced by a close quantum system. The final state of the system is related to the initial state by a unitary transformation U,

$$\rho \to \varepsilon(\rho) = U\rho U^{+} .$$

Unitary evolution is widely in use of quantum gates and circuits [4] as a quantum operation. In addition to the unitary evolution, environment coupling to a quantum system or a measurement performed on the quantum system changes the state too [6,7]. The connection of quantum operations to quantum measurements is easy to explain. Standard text book treatments describe quantum measurement in terms of a complete set of orthogonal projection operators for the system being measured. This formalism, however, does not describe many of the measurements that can be performed on a quantum system. The most general type of measurement that can be performed on a quantum system is known as generalized measurement [6]. Generalized measurement can be understood within the framework of unitary evolution, because most generalized measurements can be realized through many dynamical processes (see [8] and references therein), and the state change due to environment may be also treated in terms of the unitary evolution, since an arbitrary open system may be enlarged by including the environment to be a close system. In this sense, unitary evolution is one of the most general types of state change possible in quantum mechanics.

The rest of present paper is organized as follows: In Section 2, we present a general method to factorize the unitary evolution operator $U(t)$ for a close system. The results may be generalized in the treatment of many autonomous dynamical systems. Section 3 contains our results on the entanglement change occurring in a dynamical process. Finally, in Section 4, we present two typical examples and some conclusions.

2. Factorizing the unitary evolution operator U(t)

As noted above, the unitary evolution operator is one of the most general types of state change possible in quantum mechanics, the point of this section is to factorize the evolution operator into a set of independent one. For this end, we discuss the following cases:

- *Case A.* [9]. The Hamiltonian can be written as a finite sum,

$$H(t) = \sum_{i}^{m} a_i(t) H_i , \tag{2.1}$$

where $a_i(t)$ are a set of linearly independent complex valued functions of time, and H_i are constant operators. In addition, the set of operators $H_i (i = 1, \ldots, m)$ may be enlarged by repeated commutation to a Lie algebra L with finite dimension $n (n \geq m)$. With this presupposition, the unitary evolution operator can be uncoupled into a set of independent operators.

$$U(t) = U_1(t) U_2(t) \ldots U_n(t) , \tag{2.2}$$

where each component $U_i(t)$ is an operator satisfying

$$\frac{d}{dt} U_i(t) = \dot{g}_i(t) H_i U_i(t) , \quad U_i(0) = 1 . \tag{2.3}$$

With the scalar function $g_i(t)$ being the solution to a set of nonlinear differential equations

$$\frac{d}{dt} g_i(t) = \sum_{i=1}^{n} \eta_{ik} a_k(t) , \quad g_i(0) = 0 , \tag{2.4}$$

where η_{ik} are nonlinear function of g's. Thus we have factorized the unitary evolution operator into the form:

$$U(t) = \prod_{i=1}^{n} \mathrm{e}^{g_i(t)H_i}. \tag{2.5}$$

Especially, for a general case of a dynamically closed quantum system which consists of two interacting subsytems A and B, the total Hamiltonian may be written as a sum of three terms

$$H = H_A + H_B + H_{\mathrm{int}}, \tag{2.6}$$

the first two terms represent the free Hamiltonian of subsystem A and B, respectively, and the last term describes the interaction between the A and B. Following the procedure stated above, we arrive at

$$U(t) = \prod_{i=AB,\mathrm{int},\ldots,M} \mathrm{e}^{g_i(t)H_i}. \tag{2.7}$$

Here, $H_i(i = A, B, \mathrm{int}, \ldots M)$ are elements of Lie algebra with finite dimension enlarged by $H_A, H_B, H_{\mathrm{int}}$

- *Case B.* In the case of the dimension of the Lie algebra enlarged by $\{H_i\}$ is infinite, we can factorize the unitary evolution operator using the general Baker–Campbell–Hausdorff [10] formula. To start with, we give the evolution operator of the system under consideration

$$U(t) = \mathrm{e}^{-itH} = \mathrm{e}^{-it(H_A+H_B+H_{\mathrm{int}})}, \tag{2.8}$$

where $H_A, H_B, H_{\mathrm{int}}$ are the same as in Eq. (2.6), the Eq. (2.8) can approximately be written as

$$U(t) = \mathrm{e}^{-1/2[H_A+H_B,H_{\mathrm{int}}]t^2}\mathrm{e}^{-iH_{\mathrm{int}}t}\mathrm{e}^{-i(H_A+H_B)t} + O(t^3). \tag{2.9}$$

This splitting formula is hold in the case that t has to be safely smaller than a typical energy of the system. Thus, even in the simplest case, a better method is needed. Let n be a positive integer. The exponential function satisfies the scaling identity

$$\exp(-iHt) = \left[\exp(-iHt/2^n)\right]^{2n}. \tag{2.10}$$

When n is sufficiently large, the argument $t/2^n$ is in some sense small. Eqs. (2.9) and (2.10) together give

$$\begin{aligned} U(t) = {} & \mathrm{e}^{-\frac{1}{2}[H_A+H_B,H_{\mathrm{int}}]\tau^2}\mathrm{e}^{-iH_{\mathrm{int}}\tau}\mathrm{e}^{-i(H_A+H_B)\tau}\mathrm{e}^{-\frac{1}{2}[H_A+H_B,H_{\mathrm{int}}]\tau^2}\mathrm{e}^{-iH_{\mathrm{int}}\tau}\mathrm{e}^{-i(H_A+H_B)\tau}\ldots\mathrm{e}^{-\frac{1}{2}[H_A+H_B,H_{\mathrm{int}}]\tau^2} \\ & \times\mathrm{e}^{-iH_{\mathrm{int}}\tau}\mathrm{e}^{-i(H_A+H_B)\tau} + O(\tau^3), \end{aligned} \tag{2.11}$$

where $\tau = t/2^n$. Still higher-order formulae are known. We would like to point out that, in quantum computation [11], the $n = 1$ is widely taken in use and it is large enough to avoid the decoherence during quantum computing.

3. Quantification of entanglement

In the previous section we have factorized the time evolution operator $U(t)$. The question remains open about how does the entanglement change in a dynamical process. Of course, this question is not entirely well defined unless we state what physical circumstances characterized the amount of entanglement. This suggests that there is no unique measure of entanglement. Before we define the measure of entanglement we expand the density operator for a close system. Suppose that the two interacting subsystems are initially separable [12], i.e., their initial density operator (state) can be written in a form

$$\rho(0) = \rho_A(0) \otimes \rho_B(0), \tag{3.1}$$

to use the entanglement for quantum information processing, however, we need a inseparable state, more precisely, a state in pure entanglement form. The procedure of converting a separable state to inseparable one can be performed, as stated in Section 1, through a unitary evolution operator $U(t)$ (in addition, a partial trace is also needed sometimes.)

$$\rho(t) = U(t)\rho(0)U^{+}(t). \tag{3.2}$$

If the interaction between the two subsystems is small, it is natural to attempt some sort of Taylor series expansion of the exponential in Eqs. (3.2), (2.8) and (2.11), which give for *Case A*

$$\begin{aligned}\rho(\lambda,t) = {} & \rho_A^0(t)\otimes\rho_B^0(t) + \lambda\sum_{i\neq A,B}\left(\frac{\partial f_i}{\partial\lambda}H_i\,\rho_A^0(t)\otimes\rho_B^0(t) + \rho_A^0(t)\otimes\rho_B^0(t)\frac{\partial f_i^{*}}{\partial\lambda}H_i\right)\\ & + \frac{\lambda^2}{2}\sum_{i,j\neq A,B}\left(\frac{\partial f_i}{\partial\lambda}\frac{\partial f_j}{\partial\lambda}H_iH_j\,\rho_A^0(t)\otimes\rho_B^0(t) + \rho_A^0(t)\otimes\rho_B^0(t)\frac{\partial f_i}{\partial\lambda}\frac{\partial f_j}{\partial\lambda}H_iH_j\right.\\ & \left. + \frac{\partial f_i}{\partial\lambda}H_i\,\rho_A^0(t)\otimes\rho_B^0(t)\frac{\partial f_j^{*}}{\partial\lambda}H_j\right) + O(\lambda^2)\end{aligned} \tag{3.3}$$

and for *Case B*

$$\begin{aligned}\rho(\lambda,t) = {} & \rho_A^0(t)\otimes\rho_B^0(t) - \frac{\lambda}{2}\left(\frac{t}{2^n}\right)^2\sum_{i=0}^{2^{2n}-1}\left\{\left[H_A+H_B,H_{\mathrm{int}}(t_i)\right]_-,\rho_A^0(t)\otimes\rho_B^0(t)\right\}_+\\ & - i\lambda\left(\frac{t}{2^n}\right)\sum_{i=0}^{2^n-1}\left[H_{\mathrm{int}}(t_i),\rho_A^0(t)\otimes\rho_B^0(t)\right]_- + O(\lambda^2) + O\left(\left(\frac{t}{2^n}\right)^3\right),\end{aligned} \tag{3.4}$$

where λ denotes the coupling constant, and $\rho_i^0(t)$ represents the state of subsystem i at time t with $\lambda = 0$. The results presented above suggest that we may take the form

$$\delta D(\rho) = \|\rho(t) - \rho_A^0(t)\otimes\rho_B^0(t)\|^2 = \mathrm{Tr}\left(\rho(t) - \rho_A^0(t)\otimes\rho_B^0(t)\right)^2 \tag{3.5}$$

as a measure of entanglement change in the time evolution process. Noticing the initial state is separable, the measure of entanglement change (3.5) is a measure of entanglement in reality. Although the definition of the measure for entanglement is not unique, it has to satisfy the three conditions stated below [3]:

1. $D(\rho) = 0$ if and only if ρ is separable.
2. Local unitary operators leave $D(\rho)$ invariant, i.e.

$$D(\rho) = D\left(U_A\otimes U_B\,\rho\,U_A^{+}\otimes U_B^{+}\right).$$

3. The expected entanglement cannot increase under Local general measurements + Classical communication + Postselection (LGM + CC + PS) given by $\sum_i V_i^{+}V_i = 1$, i.e.,

$$\sum \mathrm{Tr}(\rho_i)\,D(\rho_i/\mathrm{Tr}\rho_i) \le D(\rho),$$

where $\rho_i = V_i\,\rho\,V_i^{+}$. For the measure of entanglement change proposed above, (i) follows from the fact that $D(\rho)$ is a true metric, and (ii) is obvious. Property (iii) is satisfied too [3]. We believe that there are numerous other nontrivial choices for measure of entanglement, one of the choices could not be said to be more important than any other, the present choice has the advantage that it is easy to compute for any dynamical process.

Our discussion so far has centered on the entanglement change in a dynamical process. To complete it we still need to show that this definition can be generalized for any process that quantum mechanics allowed. For a general process, the quantum operator that change a state of the system should be factorized by the subsystem's

operators. For instance, a control not operation in quantum computation given by (in fact, control not is a unitary evolution operator)

$$O = |0\rangle_1\langle 0| \otimes 1_2 + |1\rangle_1\langle 1| \otimes \sigma_2^x \,, \tag{3.6}$$

where 1_2 is the unit operator for the second qubit, σ_2^x stands for the x pauli matrix of the second qubit. $|1\rangle_1$ and $|0\rangle_1$ represent two state of the first qubit. This control not operator is factorisable, i.e. O can be written in the form

$$O = \sum_i O_1^i \otimes O_2^i \,,$$

where O_1^i and O_2^i denote operators for the first and second qubit, respectively. Hence, according to the definition Eq. (3.5), the control not operator in the form (3.6) does not change the entanglement of the system.

We would like to point out that the discussions presented here are for the unitary evolutions, for non-unitary evolution such as a trace over some of the degree of freedom, we should find an auxiliary unitary process instead of the non-unitary one.

4. Example and conclusion

In order to understand how our program for calculating the amount of entanglement change works, we present in this section two examples, one of them consists of two interacting qubits (two identical two-level system) in a laser beam (see Cirac [5]), and the another two independent qubits coupling simultaneously to a bath.

Example 1. The Hamiltonian describing the system in this example has the following form (set $\hbar = 1$):

$$\begin{aligned} &H = H_A + H_B + H_{\rm int} + H_{\rm f} \,, \\ &H_i = \tfrac{1}{2}\omega\sigma_z^{\,i} \,, \quad (i = A,B) \\ &H_{\rm int} = g\left(\sigma_A^+\sigma_B^- + \sigma_A^-\sigma_B^+\right) + \lambda\left(\sum_{i=A,B}\sigma_i^+ a + \sigma_i^- a^+\right), \\ &H_{\rm f} = \omega_{\rm f} a^+ a \,, \end{aligned} \tag{4.1}$$

where $\sigma_i^z, \sigma_i^-, \sigma_i^+$ describe the pauli operator of the i qubit, g denotes the coupling constant, and $H_{\rm f}$ stands for the free Hamiltonian of the laser beam. Suppose the state is initially in the form

$$\begin{aligned} &\rho(0) = \rho_A(0) \otimes \rho_B(0) \otimes \rho_{\rm f}(0) \,, \\ &\rho_A(0) \otimes \rho_B(0) = |e_A, e_B\rangle\langle e_A, e_B|, \\ &\rho_{\rm f}(0) = \sum_n p(n)|n\rangle\langle n|, \end{aligned} \tag{4.2}$$

where $|e_i\rangle$ denotes the excited state of the qubit i and $|n\rangle$ stands for a Fock state of the laser beam field. In the Schrödinger picture, the density operator that obeys von Neumann equation is given by

$\rho_{AB}(t) = {\rm Tr}_{\rm f}\,\rho(t) =$

$$\begin{pmatrix} \sum_n p^2(n) f_{gg}^2(n,t) & \sum_n p(n+1)p(n) f_{gg}(n+1,t) f_{EG}(n,t) & \sum_n p(n+2)p(n) f_{ee}(n+2) f_{gg}(n) \\ \sum_n p(n+1)p(n) f_{gg}(n+1,t) f_{EG}(n,t) & \sum_n p^2(n) f_{EG}^2(n,t) & \sum_n p(n) p(n+1) f_{EG}(n+1) f_{ee}(n) \\ \sum_n p(n+2)p(n) f_{ee}(n+2) f_{gg}(n) & \sum_n p(n) p(n+1) f_{EG}(n+1) f_{ee}(n) & \sum_n p^2(n) f_{ee}^2(n,t) \end{pmatrix}, \tag{4.3}$$

where we take $|g_A,g_B\rangle$,$|E,G\rangle$, and $|e_A,e_B\rangle$ as a set of basis, and

$$|g_A,g_B\rangle = |g_A\rangle \otimes |g_B\rangle, |e_A,e_B\rangle = |e_A\rangle \otimes |e_B\rangle, |E,G\rangle = \frac{1}{\sqrt{2}}\left(|g_A\rangle \otimes |e_B\rangle + |e_A\rangle \otimes |g_B\rangle\right),$$

$$\rho_A^0(t) \otimes \rho_B^0(t) = \rho_A^0(0) \otimes \rho_B^0(0),$$

$$f_{gg}(n,t) = \tfrac{1}{4}\sin 2\phi \sin\theta e^{-iE_+t} + \tfrac{1}{2}\sin 2\phi \cos^2\frac{\theta}{2} e^{-iE_-t} - \tfrac{1}{2}\sin 2\phi e^{-iE_0t},$$

$$f_{ee}(n,t) = \sin^2\phi \sin^2\frac{\theta}{2} e^{-iE_+t} + \cos^2\phi e^{-iE_0t} + \sin^2\phi \cos^2\frac{\theta}{2} e^{-iE_-t},$$

$$f_{EG}(n,t) = \sin\theta \sin\phi \sin\frac{\Omega t}{2},$$

$$E_\pm = \Omega\frac{\cos\theta \pm 1}{2} + \omega_f(n+1), \quad E_0 = (n+1)\omega_f,$$

and $\theta = \pi/2, \tan\phi = \sqrt{\frac{n+2}{n+1}}$, $\Omega^2 = (16n+24)g^2$. Eqs. (3.5) and (4.3) together give

$$\delta D(\rho) = \sum_{i,j=1,2,3} \left(\rho_{AB}^{ij}\right)^2 - 2\rho_{AB}^{33} + 1, \tag{4.4}$$

where ρ_{AB}^{ij} denotes the element of matrix ρ_{AB} given by Eq. (4.3), which represents the entanglement change or entanglement of subsystems A and B at time t.

Example 2. The Hamiltonian describing dissipation of the two qubits has the following form (see the last reference of [1]) (setting $\hbar = 1$)

$$H = \omega_0(\sigma_a^z + \sigma_b^z) + \sum_{l=a,b}\int d\omega\left[g_{\omega l}A_l(a_{\omega l}^+ + a_{\omega l})\right] + \int d\omega \cup_{l=a,b}(\omega a_{\omega l}^+ a_{\omega l}), \tag{4.5}$$

where $\boldsymbol{\sigma}_i$ describe the pauli's matrix of the i qubit, $a_{\omega l}$ stands for the bath mode ω coupling to the l qubit, and $\cup_{l=a,b} a_{\omega l}^+ a_{\omega l} = a_{\omega l}^+ a_{\omega l}$ for $a_{\omega a} = a_{\omega b}$, whereas $\cup_{l=a,b} a_{\omega l}^+ a_{\omega l} = a_{\omega a}^+ a_{\omega a} + a_{\omega b}^+ a_{\omega b}$ for $a_{\omega a} \neq a_{\omega b}$. The coupling coefficients are denoted by $g_{\omega l}$, and the qubit operator A_l in general is expressed as a linear superposition of three pauli's operators, i.e. $A_l = \lambda^{(1)}\sigma_l^x + \lambda^{(2)}\sigma_l^y + \lambda^{(3)}\sigma_l^z$. The ratio $\lambda^{(1)}:\lambda^{(2)}:\lambda^{(3)}$ is determined by the type of the dissipation. For instance, $\lambda^{(1)} = \lambda^{(2)} = 0$ for phase damping and $\lambda^{(3)} = 0$ for amplitude damping [7]. Phase damping induces pure dephasing, whereas amplitude damping induces loss and dephasing simultaneously. Many source of decoherence in quantum computers are described by amplitude damping [13].

Without any loss of generality, we discuss in detail the case with $\lambda^{(1)} = \lambda^{(2)} = 0$ i.e. phase damping. Some words of caution are now in order. As mentioned above, the bath can also cause another unwanted effect in computation process, i.e. amplitude dissipation. It is easy, however, to make system have small loss rate of amplitude dissipation [11], so a considerable number of operations are allowed to perform.

In the case of phase damping, the unitary evolution operator may be factorized in the following form

$$U(t) = U_0(t)U_I(t), \tag{4.6}$$

where $U_0(t) = e^{-iH_0t}$ with $H_0 = \omega_0(\sigma_a^z + \sigma_b^z) + \int d\omega \cup_{l=a,b}(\omega a_{\omega l}^+ a_{\omega l})$ is the free evolution operator, while $U_I(t)$ denotes the evolution operator in the interaction picture. A readily calculation shows that

$$U_I(t) = U_I^a(t)U_I^b(t),$$
$$U_I^i(t) = u_I^i(t)|e_i\rangle\langle e_i| + v_I^i|g_i\rangle\langle g_i|, \tag{4.7}$$

where $|e_i\rangle$ and $|g_i\rangle$ are the eigenstates of σ_i^z with eigenvalues $+1$ and -1, respectively, and $u_{\rm I}^i$ and $v_{\rm I}^i$ satisfy $(i-a,b)$:

$$i\hbar\frac{\partial}{\partial t}u_{\rm I}^i = \sum d\omega g_{\omega i}\left(a_{\omega i}^+ e^{i\omega t} + a_{\omega i} e^{-i\omega t}\right)u_{\rm I}^i,$$

$$i\hbar\frac{\partial}{\partial t}v_{\rm I}^i = -\sum d\omega g_{\omega i}\left(a_{\omega i}^+ e^{i\omega t} + a_{\omega i} e^{-i\omega t}\right)v_{\rm I}^i. \tag{4.8}$$

The Wei–Norman's algebraic method [9] that provides a way to factorize the evolution operator gives

$$u_{\rm I}^i = \prod_\omega e^{f_\omega^i(t)} e^{A_\omega^i(t)a_\omega^+} e^{B_\omega^i(t)a_\omega} \tag{4.9}$$

and

$$v_{\rm I}^i = \prod_\omega e^{h_\omega^i(t)} e^{C_\omega^i(t)a_\omega^+} e^{D_\omega^i(t)a_\omega}. \tag{4.10}$$

Here,

$$A_\omega^i(t) = -\frac{g_{\omega i}}{\omega}(e^{i\omega t}-1),$$

$$B_\omega^i(t) = -\left(A_\omega^i(t)\right)^*,$$

$$f_\omega^i(t) = -i\frac{g_{\omega i}^2}{\omega}t + \frac{g_{\omega i}^2}{\omega^2}(1-e^{-i\omega t}),$$

$$C_\omega^i(t) = -A_\omega^i(t), \quad D_\omega^i(t) = -B_\omega^i(t), \quad h_\omega^i(t) = f_\omega^i(t).$$

Now, we turn our attention to compute the reduced density operator of the two-qubit system, first of all, we calculate the total density operator, which follows straightforwardly from Eq. (4.5)

$$\rho(t) = U_0 U_{\rm I}\,\rho(0) U_{\rm I}^+ U_0^+, \tag{4.11}$$

where $\rho(0)$ denotes the initial density operator (state), which may be written in a separable form

$$\rho(0) = \rho_a(0)\otimes\rho_b(0)\otimes\rho_{\rm B}(0).$$

Here, $\rho_i(0)(i=a,b)$ represents the initial state of qubit i, and $\rho_{\rm B}(0)$ stands for the initial state of the bath. In following, we use the notation, $|e_a,g_b\rangle$ to indicate the eigenstates of σ_a^z and σ_b^z with eigenvalues 1 and -1, $\mathrm{Tr}_{\rm B}$ indicate a trace over the bath, and $\rho_i^0(t)$ to represent the free two-qubit state i.e.

$$\rho^0(t) = \mathrm{Tr}_{\rm B} U_0(t)\,\rho(0) U_0^+(t).$$

With this notation, in a subspace spanned by $\{|11\rangle = |e_a,e_b\rangle, |12\rangle = |e_a,g_b\rangle, |21\rangle = |g_a,b_2\rangle, |22\rangle = |g_a,g_b\rangle\}$, the state of the two-qubit system at time t takes the form

$$\rho_{ab}(t) = \mathrm{Tr}_{\rm B}\,\rho(t) = \begin{pmatrix} \rho_{1111} & \rho_{1112} & \rho_{1121} & \rho_{1122} \\ \rho_{1211} & \rho_{1212} & \rho_{1221} & \rho_{1222} \\ \rho_{2111} & \rho_{2112} & \rho_{2121} & \rho_{2122} \\ \rho_{2211} & \rho_{2212} & \rho_{2221} & \rho_{2222} \end{pmatrix}, \tag{4.12}$$

where $\rho_{ijkl} = \rho_{ijkl}^0 F_{ijkl}(i,j,k,l=1,2)$, $\rho_{ijkl}^0 = \mathrm{Tr}_{\rm B}\langle ij|\,\rho^0(t)|kl\rangle$, and

$$F_{ijkl} = \mathrm{Tr}_{\rm B}\langle ij| \sum_{c,d,e,f=1}^{2} \left(U_{\rm I}^a\right)_i \left(U_{\rm I}^b\right)_j |cd\rangle\langle ef| \left(\left(U_{\rm I}^b\right)^+\right)_k \left(\left(U_{\rm I}^a\right)^+\right)_l |kl\rangle.$$

Here, $(U_{\rm I}^a)_i = {}_a\langle i|U_{\rm I}^a|i\rangle_a$ and $|2\rangle_a = |e_a\rangle, |1\rangle_a = |g_a\rangle$. In order to get more information about the reduced density operator, we make some discussion on the quantity F_{ijkl}. It can be easily verified that $F_{ijkl} = 1$ for $i = k, j = l$, while $F_{ijkl} = F_{klij}^*$ for $i \neq k$ and $j \neq l$. Moreover, the quantity results from the interaction between the two-qubit system and the bath, hence it depends on the states of the bath. Although different bathes result in different results F_{ijkl}, the physical results discussed here do not rely on the bath. In this sense, we may consider a simple case with zero temperature. In this case, F_{ijkl} is given that

$$F_{ijkl} = F_{ijkl}(t) = \exp\left(-\int_0^\infty \left[\Delta_{ik}(\omega,t) + \Delta_{jl}^*(\omega,t) \right] \rho(\omega)\, d\omega \right), \tag{4.13}$$

where

$$\Delta_{ij}(\omega,t) = 2\frac{(g_{\omega i} - g_{\omega j})^2 \sin^2 0.5\omega t}{\omega^2},$$

$\rho(\omega)$ stands for the bath spectrum distribution. Eq. (4.12) suggests that F_{ijkl} approaches zero with the passage of time except some moments at which

$$\int_0^\infty \left[\Delta_{ij}(\omega,t)\, \Delta_{kl}^*(\omega,t) \right] \rho(\omega)\, d\omega = 0.$$

This attractive results might be used in preventing information loss stored in quantum states. Now we come back to the entanglement change, Eqs. (3.5) and (4.11) together give

$$\delta D\left(\rho_{ab}(t) \,\|\, \rho_a^0(t) \otimes \rho_b^0(t) \right) = \sum_{i,j,k,l=1}^{2} \left(\rho_{ijkl} - \rho_{ijkl}^0 \right)\left(\rho_{klij} - \rho_{klij}^0 \right). \tag{4.14}$$

In summary, we propose a new method to compute the entanglement change in a dynamical process. We see the above treatment in Sections 2 and 3 does not refer to specific entangled systems. This is a desired property as it makes our measure of entanglement universal. Especially, the results yielded by present paper can be easily generalized to more than two subsystems, this is just the case of many qubits interacting simultaneously with environment. In addition to the measure stated above, the quantum relative entropy defined as

$$D\left(\rho(t) \,\|\, \rho_A^0(t) \otimes \rho_B^0(t) \right) = \mathrm{Tr}\left[\rho(t)\left(\ln\rho(t) - \ln\rho_A^0(t)\,\rho_{\rm B}^0(t) \right) \right]$$

and the Bures metric given by

$$D\left(\rho(t) \,\|\, \rho_A^0(t) \otimes \rho_B^0(t) \right) = 2 - 2\sqrt{F\left(\rho, \rho_A^0 \otimes \rho_B^0 \right)},$$

with $F(\rho, \rho_A^0 \otimes \rho_B^0) = [\mathrm{Tr}[\sqrt{\rho_B^0 \otimes \rho_A^0}\,\rho\sqrt{\rho_B^0 \otimes \rho_A^0}\,]^{1/2}]^2$ are other good measures of entanglement. With this modified definitions, the measures of entanglement can be given in easy way.

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