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# Generalization of Cini's model for quantum measurement and dynamical realization of wavefunction collapse

Xia-Ji Liu, Chang-Pu Sun

*Institute of Theoretical Physics, Northeast Normal University, Changchun 130024, China*

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**Abstract**

A quantum dynamical model for the measurement of an $N$-level system is proposed based on the correlation between the states of the detector and the states of the measured system in the measurement process. It is shown that the postulate of wavefunction collapse for the measured system can be realized in a dynamical evolution of the total system formed by the measured system plus the detector, which is governed by the Schrödinger equation.

## 1. Introduction

The postulate of wavefunction collapse in quantum measurement was introduced by von Neumann [1] from outside the basic laws of quantum mechanics. Mathematically, this postulate is expressed as a reduction from a pure state described by a coherent density matrix,

$$\hat{\rho} = |\Phi\rangle\langle\Phi| = \sum_{k,k'} C_k C_{k'}^* |k\rangle\langle k'| , \tag{1.1}$$

to a mixed state described by a density matrix without off-diagonal elements,

$$\hat{\rho}_r = \sum_k |C_k|^2 |k\rangle\langle k| . \tag{1.2}$$

Because this postulate introduces an irreducible source of sudden change of the state vector resulting from the act of measurement, it is not satisfactory to understand the quantum measurement process, or even the system of laws of quantum measurement. Also an external classical measuring apparatus (detector) must be used to produce such a sudden change. This cannot be described by quantum mechanics in the conventional way. However, quantum mechanics is believed to be a quite universal theory that should be valid for the whole world of physics. Naturally, one expects that it can be used to describe the detector as well as its measured system and it causes wavefunction collapse dynamically. For the above reasons, many attempts were made to dispense with the postulate of wavefunction collapse and to build a realistic quantum theory of the measurement process based on a detailed analysis of the interaction between the measured system and the detector [2–5]. Among these studies, the Hepp–Coleman model mainly describes the dynamical process of the appearance of wavefunction collapse for a macroscopic

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detector with infinite particle number $N$. It has been extensively studied in the last twenty years from various points of view [6–8]. More recently, one (CPS) of the present authors generalized the Hepp–Coleman model to dynamically realize wavefunction collapse in both the macroscopic limit with very large particle numbers of the detector and the classical limit with very large quantum numbers [9]. It was found that the essence of the dynamical models for wavefunction collapse is the factorization of the Schrödinger evolution rather than its exact solvability [10]. However, it has to be pointed out that the correlation between the state of the measured system and that of the detector has not been emphasized well in the original Hepp–Coleman model and its generalization [3,6–10]. This problem was well analysed by Cini using a beautiful dynamical model [5]. The present investigation emphasizes both the wavefunction collapse and the correlation collapse. In fact, the correlation between the states of the measured system and the detector is crucial for a realistic process of measurement, which uses a scheme utilizing the macroscopic counting number of the measuring instrument-detector to show the microscopic state of the measured system.

## 2. Dynamics of the generalized Cini model

The original Cini model for the correlation between the states of the measured system S and the measuring instrument-detector D is built for a two-level system interacting with the detector D, which consists of $M$ indistinguishable particles with two possible states $\omega_0$ and $\omega_1$. With the two states $u_+$ and $u_-$ the detector has different strengths of interaction. Then, the large number $N$ of "ionized" particles in the ionized state $\omega_1$ transiting from the un-ionized state $\omega_0$ shows these correlations. In this section, we wish to generalize Cini's model to an $M$-level system. The measured system S with $M$ levels has the model Hamiltonian

$$\hat{H}_S = \sum_{k=1}^{M} E_k |\Phi_k\rangle\langle\Phi_k| . \tag{2.1}$$

Here $|\Phi_k\rangle$ are the eigenstates corresponding to the eigenvalues $E_k$ ($k=1,2,\ldots,M$). The detector D is a two-boson-state system with the free Hamiltonian

$$\hat{H}_D = \hbar\omega_1 a_1^+ a_1 + \hbar\omega_2 a_2^+ a_2 , \tag{2.2}$$

where $a_i$ and $a_i^+$ are the creation and annihilation operators satisfying

$$[a_i, a_j^+] = \delta_{ij}, \qquad [a_i, a_j] = [a_i^+, a_j^+] = 0 . \tag{2.3}$$

In the Schrödinger representation, the interaction is described by

$$\hat{H}_I(t) = \sum_n g e^{-\eta t}(W_n |\Phi_n\rangle\langle\Phi_n|)\{\exp[i(\omega_2-\omega_1)t]a_1^+ a_2 + \exp[i(\omega_1-\omega_2)t]a_2^+ a_1\} , \tag{2.4}$$

where the non-degenerate weights $W_n$ represent the different strengths for the different states $|\Phi_n\rangle$ of the system. The exponential decay factor $e^{-\eta t}$ for $\eta > 0$ is introduced here to turn off the interaction after a suitable time so that the coherence cannot be restored in the evolution process. This point can be explicitly seen in the following discussion. The introduction of the time-dependent factors $\exp[\pm i(\omega_1-\omega_2)t]$ is quite similar to that in Ref. [8] where these factors are used to describe the energy exchange due to the presence of the free Hamiltonian $H_D$. Notice that there was no free Hamiltonian for the detector in the original Hepp–Coleman model such as $H_D$ in our present model. Let $H = H_S + H_D + H_I$ be the total Hamiltonian in the Schrödinger representation for the composite system formed by S plus D. Transforming the problem to the interaction representation with the evolution operator

$$U_0(t) = \exp[(1/i\hbar)(H_S + H_D)t] , \tag{2.5}$$

one has the interaction potential

$$V_\mathrm{I}(t)=\mathrm{e}^{-\eta t}g\sum_{n=1}^{M}W_n\,|\Phi_n\rangle\langle\Phi_n|\,(a_1^+a_2+a_1a_2^+)\,. \tag{2.6}$$

In order to diagonalize $V_\mathrm{I}(t)$, we invoke the canonical transformation as used in Ref. [5],

$$a_1=\frac{1}{\sqrt{2}}(b_1-b_2),\qquad a_2=\frac{1}{\sqrt{2}}(b_1+b_2)\,, \tag{2.7}$$

where the new boson operators $b_i$ and $b_i^+$ satisfy the same bosonic commutation relation. In terms of these operators, $V_\mathrm{I}(t)$ is rewritten in the diagonal form

$$V_\mathrm{I}(t)=g\mathrm{e}^{-\eta t}\sum_{n=1}^{M}W_n\,|\Phi_n\rangle\langle\Phi_n|\,(b_1^+b_1-b_2^+b_2)\,. \tag{2.8}$$

Then, considering that the interaction part $V_\mathrm{I}(t)$ commutes with that at a different time, i.e.

$$[V_\mathrm{I}(t),V_\mathrm{I}(t')]=0\,,$$

one can write the evolution operator as

$$U_\mathrm{I}(t)=\exp\left(\frac{1}{\mathrm{i}\hbar}\int_0^t V_\mathrm{I}(t')\,\mathrm{d}t'\right)=\sum_{n=1}^{M}\exp[-\mathrm{i}t_\eta gW_n(b_1^+b_1-b_2^+b_2)]\,|\Phi_n\rangle\langle\Phi_n|$$

$$\equiv\sum_{n=1}^{M}U_n(t)\,|\Phi_n\rangle\langle\Phi_n|\,, \tag{2.9}$$

where

$$t_\eta=\frac{1-\mathrm{e}^{-\eta t}}{\eta}$$

approaches the real time as $\eta\to 0$. It can be regarded as the $\eta$-deformation of the time $t$; when $t\to\infty$, $t_\eta\to 1/\eta$.

## 3. Correlation of states from the evolution of state

Now, we consider the evolution of the total system starting with an initial state at $t=0$,

$$|\Psi(0)\rangle=\sum_{k=1}^{M}C_k\,|\Phi_k\rangle\otimes|N,0\rangle\,, \tag{3.1}$$

where

$$|m,n\rangle=\frac{a_1^{+m}a_2^{+n}}{\sqrt{m!n!}}\,|0\rangle \tag{3.2}$$

denotes a Fock state of the two-boson system with $n$ particles in the "ionized" state and $m$ particles in the non-ionized state. We hope to show a correlation between the states $|m,n\rangle$ of the detector and the state $|\Phi_l\rangle$ of the system in a dynamical evolution of the state $|\Psi(t)\rangle$ in some limiting case so that one can find the state $|\Phi_l\rangle$ from the state $|m,n\rangle$ of the detector. Notice that the eigenstates of the operator

$$\hat{O}=b_1^+b_1-b_2^+b_2 \tag{3.3}$$

are

$$|\lambda, N-\lambda\} = \frac{1}{\sqrt{\lambda!(N-\lambda)!}} b_1^{+\lambda} b_2^{+(N-\lambda)} |0\rangle$$
$$= \left(\frac{1}{\sqrt{2}}\right)^N \sum_{m=0}^{\lambda} \sum_{n=0}^{N-\lambda} \frac{\sqrt{\lambda!(N-\lambda)!}}{\sqrt{(\lambda-m)!m!(N-\lambda-n)!n!}} |m, n\rangle , \quad (3.4)$$

with the eigenvalues

$$\epsilon_N(\lambda) = 2\lambda - N , \quad (3.5)$$

where $\lambda = 0, 1, 2, \ldots, N$ for a given integer $N$. The original Fock state can be expanded in terms of $|\lambda, N-\lambda\}$ as

$$|N, 0\rangle = \left(\frac{1}{\sqrt{2}}\right)^N \sum_{\lambda=0}^{N} \frac{\sqrt{N!}(-1)^{N-\lambda}}{\sqrt{\lambda!(N-\lambda)!}} |\lambda, N-\lambda\} . \quad (3.6)$$

Then, one obtains the wavefunction at time $t$,

$$|\Psi_{\rm I}(t)\rangle = U_{\rm I}(t) |\Psi(0)\rangle$$
$$= \sum_{\lambda=0}^{N} \sum_{k=1}^{M} C_k \left(\frac{1}{\sqrt{2}}\right)^N \frac{\sqrt{N!}(-1)^{N-\lambda}}{\sqrt{\lambda!(N-\lambda)!}} \exp[-igt_\eta W_k(2\lambda-N)] |\Phi_k\rangle \otimes |\lambda, N-\lambda\} . \quad (3.7)$$

Using Eq. (3.4) again, one has

$$|\Psi_{\rm I}(t) = \sum_{k=1}^{M} C_k |\Phi_k\rangle \otimes \sum_{n=0}^{N} a_n(t, k) |n, N-n\rangle , \quad (3.8)$$

where

$$a_n(t, k) = \frac{(-i)^{N-n}\sqrt{N!}}{\sqrt{n!(N-n)!}} \cos^n(gW_k t_\eta) \sin^{N-n}(gW_k t_\eta) . \quad (3.9)$$

Obviously, the probability of finding $n$ "ionized" particles in the second bosonic mode is

$$P_n = |a_n|^2 = \frac{N!}{n!(N-n)!} \cos^{2n}(gW_k t_\eta) \sin^{2(N-n)}(gW_k t_\eta) \quad (3.10)$$

or

$$P_n = C_n^N p^n (1-p)^{N-n} , \quad (3.11)$$

where

$$C_n^N = \frac{N!}{(N-n)!n!}, \qquad p_k(t) = \cos^2(gW_k t_\eta) .$$

When $N$ is very large so that the Stirling formula is valid, it can be proved that when $n_k = \bar{n}_k = Np_k(t)$, the probability $P_n$ has its maximum,

$$P_{\bar{n}_k} = C_{\bar{n}_k}^N \left(\frac{\bar{n}_k}{N}\right) \left(\frac{N-\bar{n}_k}{N}\right)^{N-\bar{n}_k} . \quad (3.12)$$

Notice that the derivation is the same as that in Ref. [5], but $\bar{n}_k$ depends on the index $k$. As proved in Ref. [5],

$P_{n_k}$ is very strongly peaked around its maximum $P_{\bar{n}_k}$, which becomes unity when $N\to\infty$. Therefore, if the detector is very macroscopic ($N\to\infty$), then $P_{n_k}(N\to\infty)=\delta_{n_k\bar{n}_k}$ which leads to

$$|\Psi_1(t)\rangle_{N\to\infty}=\sum_{k=1}^{M}C_k|\Phi_k\rangle\otimes|\bar{n}_k(t),N-\bar{n}_k(t)\rangle\,. \tag{3.13}$$

When $\bar{n}_k(t)\neq\bar{n}_{k'}(t)$ for $k\neq k'$, there is a one-to-one correlation between the states $|\Phi_k\rangle$ of S and the states of D. In this sense, if the detector is found in the state $|\bar{n}_k(t),N-\bar{n}_k(t)\rangle$, it can be concluded that the system is in the state $|\Phi_k\rangle$. A realistic detector must have a good fringe visibility, which can show the macroscopic differences between any two states of $|\bar{n}_k,\ N-\bar{n}_k\rangle$ for different $k$. It requires that there is no considerable overlap between $|\bar{n}_k,N-\bar{n}_k\rangle$ and $|\bar{n}_{k'},N-\bar{n}_{k'}\rangle$ for $k\neq k'$. In fact, if $\bar{n}_k=\bar{n}_{k'}$ then

$$gW_kt_\eta=gW_{k'}t_\eta+n\pi,\qquad n=0,\,1,\,2,\,\dots$$

or

$$gt_\eta(W_k-W_{k'})=n\pi\,.$$

Otherwise, if one lets the interaction between S and D decay very fast so that the limiting time

$$\frac{1}{\eta}\leqslant\frac{\pi}{g(W_k-W_{k'})}$$

for any $k\neq k'$, then $\bar{n}_k(t)\neq\bar{n}_{k'}(t)$ for any long time evolution. Notice that the abovementioned problem of overlap of correlation states is a disadvantage in the original Cini model, where there is no decay factor of interaction. So, after a time $t_n=n\pi/gW$ the states $|\bar{n},N-\bar{n}\rangle$ and $|N,0\rangle$ do completely overlap. Here, we consider the case of two levels with $W_2=W=1/\sqrt{N}$, $W_1=0$, thus, at $t=t_n$ the correlations vanish for the original Cini model. Introducing the decay factor $e^{-\eta t}$ in the interaction is the key point to avoid vanishing of correlation in our model. In fact, such a decay of the interaction can appear in realistic physics. For example, an atom is prepared in a microwave cavity loaded with an electromagnetic field which can decay at a suitable rate. In this example, the atom and the cavity are regarded as the system and the detector respectively. Note that this example is quite useful for the study of atomic cooling [11].

## 4. Wavefunction collapse

In this section we will use the above modified Cini model to describe the wavefunction collapse for the $N$-level system quantum mechanically. According to the Hepp–Coleman approach and its generalization, if one deals with both the system S and the detector D as two systems of a closed total system C formed by S plus D, it is possible that the quantum dynamics of the total system can result in wavefunction collapse through a suitable interaction between S and D. In this sense, the reduced density matrix for the measured system S has gradually vanishing off-diagonal elements as $N\to\infty$ where $N$ is the particle number of D. It will be shown that similar circumstances also occur in the generalized Cini model. Let the system D be initially prepared in a coherent superposition of $M$ levels,

$$|\Psi\rangle=\sum_{k=1}^{M}C_k|\Phi_k\rangle\,, \tag{4.1}$$

and the detector be adjusted in the initial state $|N,0\rangle$, then the density matrix for the initial state of total system is

$$\hat{\rho}_0=\sum_{k,k'}C_kC_{k'}^*|\Phi_k\rangle\langle\Phi_{k'}|\otimes|N,0\rangle\langle N,0|\,. \tag{4.2}$$

Using the evolution operator $U_{\rm I}(t)$ in the interaction representation, one formally writes down the density matrix for the total system at $t$,

$$\hat{\rho}(t) = U_{\rm I}(t)\rho(0)U_{\rm I}^{+}(t) = \sum_{k,k'} C_k C_{k'}^{*} |\Phi_k\rangle\langle\Phi_{k'}| \otimes U_k(t)|N,0\rangle\langle N,0|U_{k'}^{+}(t)$$

$$= \sum_{k,k'}\sum_{n,n'} C_k C_{k'}^{*} a_n(t,k) a_{n'}^{*}(t,k') |\Phi_k\rangle\langle\Phi_{k'}| \otimes |n, N-n\rangle\langle n', N-n'| , \tag{4.3}$$

where $a_n(t, k)$ are given explicitly in Eq. (3.9). Because we are only interested in the final state of the system, not in that of the detector for consideration of the wavefunction collapse, we must take the trace over the detector variable in the total density matrix to obtain a reduced density matrix for S,

$$\hat{\rho}_{\rm S}(t) = {\rm Tr}_{\rm D}\hat{\rho}(t) = \sum_{m,m'} \langle m, m'|\rho(t)|m, m'\rangle$$

$$= \sum_k |C_k|^2 |\Phi_k\rangle\langle\Phi_{k'}| + \sum_{k\neq k'} \exp[-{\rm i}t_\eta(E_k - E_{k'})/\hbar] \cos^N[g(W_k - W_{k'})t_\eta] C_k C_{k'}^{*} |\Phi_k\rangle\langle\Phi_{k'}| , \tag{4.4}$$

where we have used

$$\sum_{n=0}^{N} |a_n(t,k)|^2 = 1 \tag{4.5}$$

and

$$\sum_{n=0}^{N} a_n(t,k) a_n^{*}(t,k') = \cos^N[g(W_k - W_{k'})t_\eta] . \tag{4.6}$$

Note that each off-diagonal element in the density matrix is accompanied by a time-dependent factor

$$F^N(k,k') = \cos^N[g(W_k - W_{k'})t_\eta] , \tag{4.7}$$

which is an $N$-multiple product of factors $\cos[g(W_k - W_{k'})t_\eta]$. Recalling that due to the existence of the strong decay factor $e^{-\eta t}$, $\eta \geqslant g(W_k - W_{k'})/\pi$ holds for any $k \neq k'$, we observed that the deformed time $t_\eta$ changes from $t_\eta = 0$ to $t_\eta = 1/\eta$ as the real time changes from $t = 0$ to $t \to \infty$ respectively. In this sense,

$$g(W_k - W_{k'})t_\eta < \pi \tag{4.8}$$

and then

$$0 \leqslant |\cos[g(W_k - W_{k'})t_\eta]| < 1$$

or

$$|\cos[g(W_k - W_{k'})t_\eta]| = \exp[-f_k(t_\eta)], \qquad f_k(t_\eta) > 0 . \tag{4.9}$$

This observation leads to

$$F^N(k,k') = \exp[-Nf_k(t_\eta)] ,$$

which obviously approaches zero as $N \to \infty$. Therefore, when the detector is macroscopic ($N \to \infty$) the off-diagonal terms of the density matrix vanish and the wavefunction collapse is realized quantum dynamically.

To end this section, we present some comments on the above discussions. It is well known that the evolution of a closed system from an initial state to a final state governed by the Schrödinger equation is given by a unitary transformation. This unitary transformation must not change the rank of the density matrix. However, the initial state as a pure state has a density matrix with rank 1, and the collapsed final state as a mixed state has a density matrix with rank larger than 1. Thus, for a closed system S, its pure state cannot evolve into a mixed state. In terms of the definition of information entropy of a system, we can also describe this impossibility. In fact, in dynamical models such as Cini's model, the Hepp–Coleman model and their generalizations, S is considered as an open system, which is soaked in a large closed system C (S plus D). In this sense, though the unitary evolution of C cannot change its state from purity to mixture, the effective evolution of S induced by the dynamics of the total system C may be non-unitary and can change the rank of the reduced density matrix. Therefore, it is possible to realize the decoherence in a purely quantum mechanical process. It is also pointed out that the wavefunction collapse described in this paper is only of limited use to the measurement problem. In fact, if the total system undergoes a unitary time evolution, the subsystem S can also end up in a mixture trivially. A good aspect of our model is that it leads to a right mixture of the subsystem S. The present model is not complete because the detector itself has not reached the state indicating a definite outcome of measurement.

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