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The canonical quantization in terms of quantum group and Yang–Baxter equation

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Abstract

In this Letter, it is shown that a quantum observable algebra, the Heisenberg–Weyl algebra, is just given as a Hopf algebraic dual to a central extension of the classical observable algebra and the Planck constant is included in this quantization scheme as a compatible parameter living in the quantum double theory. In this sense the quantum Yang–Baxter equation naturally appears as a necessary condition to be satisfied by canonical elements, the universal R-matrix, intertwining the quantum and this central extension. As a by-product, a new "quantum group" is obtained as the quantum double of the algebra of the central extension. The physical meaning of these constructions is also discussed in connection with coherent states.

The purpose of this Letter is to try to understand directly the physical meaning of the quantum group theory [1] in basic quantum mechanics. We formally reproduce the canonical quantization from the quantum double of a central extension of the classical observable algebra (COA) A for the classical phase space. It has to be pointed out that the quantum group theory originally comes from the quantization of some non-linear problems in physics (such as the S-matrix theory in low-dimensional quantum field theory [2] and quantum inverse scattering methods [3]) and exactly solvable models in statistical mechanics [4].

The usual classical observable algebra (COA) is an Abelian functional algebra spanned by the functions $F(P, Q)$ over the phase space for a classical particle where P and Q are the canonical coordinate and momentum. Let A be an extension of the COA (ECOA) with an additional central element N. Later on, the physical meaning of N will be discussed in connection with the Yang–Baxter equation. As an associative algebra, the algebra A possesses a quite simple structure with commuting generators. However, this algebra can be endowed with a "quantum group" structure (Δ, S, ϵ)

$$\begin{aligned} &\Delta(x)=x\otimes 1+1\otimes x, \qquad x=P, Q, \qquad \Delta(N)=N\otimes 1+1\otimes N+\mu P\otimes Q\,, \\ &S(x)=-x, \qquad S(N)=-N+\mu PQ, \qquad S(1)=1, \qquad x=P, Q \\ &\epsilon(y)=0, \qquad y=P, Q, N, \qquad \epsilon(1)=1\,, \end{aligned} \tag{1}$$

where Δ, ϵ are algebraic homomorphisms and S an algebraic antihomomorphism; μ is a compatible complex parameter for (Δ, S, ϵ) satisfying the axioms of the Hopf algebra. With the above structure (Δ, S, ϵ) the ECOA becomes a commutative, but non-cocommutative Hopf algebra if μ is not zero.

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Now we set to find the quantum dual (also called Hopf (algebraic) dual [1]) B of the COA A according to Drinfeld's quantum double theory (for reviews easy for physicists, see Refs. [5,6]). To this end we define the generators $\hat{P}$, $\hat{Q}$ and E by

$$\langle P^m Q^n N^l, \hat{P}\rangle = \delta_{m,1}\delta_{n,0}\delta_{l,0} \qquad \langle P^m Q^n N^l, \hat{Q}\rangle = \delta_{m,0}\delta_{n,1}\delta_{l,0}, \qquad \langle P^m Q^n N^l, E\rangle = \delta_{m,0}\delta_{n,0}\delta_{l,1}\,, \tag{2}$$

where the bilinear form $\langle\ ,\ \rangle$: $\mathrm{A}\times\mathrm{B}\to\mathbb{C}$ satisfies the following conditions resulting from the duality in the quantum double,

$$\begin{aligned}
&\langle a, b_1 b_2\rangle = \langle \Delta_{\mathrm{A}}(a), b_1\otimes b_2\rangle, \qquad a\in\mathrm{A}, \qquad b_1, b_2\in\mathrm{B}\,,\\
&\langle a_1 a_2, b\rangle = \langle a_2\otimes a_1, \Delta_{\mathrm{B}}(b)\rangle, \qquad a_1, a_2\in\mathrm{A}, \qquad b\in\mathrm{B}\,,\\
&\langle 1_{\mathrm{A}}, b\rangle = \epsilon_{\mathrm{B}}(b), \qquad b\in\mathrm{B}, \qquad \langle a, 1_{\mathrm{B}}\rangle = \epsilon_{\mathrm{A}}(a), \qquad a\in\mathrm{A}\,,\\
&\langle S_{\mathrm{A}}(a), S_{\mathrm{B}}(b)\rangle = \langle a, b\rangle, \qquad a\in\mathrm{A}, \qquad b\in\mathrm{B}\,.
\end{aligned} \tag{3}$$

In this Letter, without confusion, we do not specify the operation (Δ, S, ϵ) for A or B.

It follows from Eqs. (2) and (3) that

$$\begin{aligned}
&\langle P^m Q^n N^s, \hat{P}^k \hat{Q}^l E^r\rangle = m!n!s!\,\delta_{m,k}\delta_{n,l}\delta_{s,r}\,,\\
&\Delta(x) = x\otimes 1 + 1\otimes x, \qquad S(x) = -x, \qquad \epsilon(x) = 0, \qquad \epsilon(1) = 1, \qquad S(1) = 1, \qquad x = \hat{P}, \hat{Q}, E\,.
\end{aligned} \tag{4}$$

The key to our study is the commutation relations between $\hat{P}$ and $\hat{Q}$. From Eqs. (1), (2) and (3), we derive

$$\begin{aligned}
&\langle P^m Q^n N^l, \hat{P}\hat{Q}\rangle = \langle \Delta(P^m Q^n N^l), \hat{P}\otimes\hat{Q}\rangle\,,\\
&\sum_{k=1}^{m}\sum_{r=1}^{n}\sum_{s=1}^{l}\sum_{t}^{l-s} \frac{m!l!n!\mu^s}{k!r!s!(m-k)!(n-r)!(l-s-t)!t!} \langle P^{m-k+s}Q^{n-r}N^{l-s-t}\otimes P^k Q^{r+s} N^t, \hat{P}\otimes\hat{Q}\rangle\\
&\quad = \mu^{\delta_{l,1}}\delta_{m+l,1}\delta_{n+l,1}\,,\\
&\langle P^m Q^n N^l, \hat{Q}\hat{P}\rangle = \delta_{m,1}\delta_{n,1}\,,
\end{aligned}$$

that is $\langle P^m Q^n N^l, [\hat{P}, \hat{Q}]\rangle = \mu\delta_{l,1}\delta_{m,0}\delta_{n,0}$. Then, we obtain

$$[\hat{P}, \hat{Q}] = \mu E, \qquad [\hat{P}, E] = 0 = [\hat{Q}, E]\,, \tag{5}$$

Then we show that *the Hopf duals $\hat{P}$ and $\hat{Q}$ of the classical canonical coordinate Q and momentum P are just the quantum coordinate and momentum operators, respectively, if we can take $\mu = -i\hbar$. Therefore, Drinfeld's quantum double theory provides us with an algebraic scheme of the canonical quantization in basic quantum mechanics*! In fact, the parameter μ characterizes the degrees of non-cocommutation of the classical observable algebra A and the non-commutation of the quantum observable algebra (QOA) B generated by $\hat{P}$, $\hat{Q}$ and E (it is usually called Heisenberg–Weyl (HW) algebra). Since, when μ is zero, the algebra B becomes commutative, it is reasonable to take $\mu = -i\hbar$. In this way, the Planck constant $\hbar$ automatically enters the algebra B to realize the "algebraic" canonical quantization. Notice that with the Hopf algebraic structure (4), the QOA B is a cocommutative, but non-commutative Hopf algebra.

In order to investigate the quantum Yang–Baxter equation in connection with canonical quantization, we need to combine A with B to get a quasi-triangular Hopf algebra as the quantum double of the COA A. Thanks to the double multiplication rule given in Drinfeld's theory,

$$ba = \sum_{i,j}\langle a_i(1), S(b_j(1))\rangle\langle a_i(3), b_j(3)\rangle a_i(2)b_j(2), \qquad (id\otimes\Delta)\Delta(c) = \sum_i c_i(1)\otimes c_i(2)\otimes c_i(3)\,, \tag{6}$$

we obtain the only non-zero commutation relations between A and B,

$$[N, \hat{P}] = \mu Q, \qquad [N, \hat{Q}] = -\mu P\,. \tag{7}$$

Then we obtain the universal R-matrix

$$\hat{R}=\sum a_i\otimes b_i=\sum_{m,n,s}^{\infty}\frac{P^mQ^nH^s\otimes\hat{P}^m\hat{Q}^nE^s}{m!n!s!}=\exp(P\otimes\hat{P})\,\exp(Q\otimes\hat{Q})\,\exp(N\otimes E) \tag{8}$$

as canonical elements intertwining the QOA B and ECOA A.

Now, let us use Eq. (7), resulting from the central result (6) in Drinfeld's theorem, to prove that the universal R-matrix constructed above indeed satisfies the abstract Yang–Baxter equation,

$$\hat{R}_{12}\hat{R}_{13}\hat{R}_{23}=\hat{R}_{23}\hat{R}_{13}\hat{R}_{12}\,,$$
$$\hat{R}_{12}=\sum_m a_m\otimes b_m\otimes 1,\qquad \hat{R}_{13}=\sum_m a_m\otimes 1\otimes b_m,\qquad \hat{R}_{23}=\sum_m 1\otimes a_m\otimes b_m\,, \tag{9}$$

where a_m and b_m are the basis elements of A and B respectively, and they are dual to each other, $\langle a_m, b_n\rangle=\delta_{m,n}$.

In fact, from Glauber's formula

$$e^F e^G = e^{F+G+[F,G]/2}\,,$$

where $[F, G]$ commutes with both F and G, we can rewrite the universal R-matrix in the following form,

$$R=\exp(P\otimes\hat{P}+Q\otimes\hat{Q}+N\otimes E+\tfrac{1}{2}\mu PQ\otimes E)\,.$$

By Applying Glauber's formula and the above expression of the universal R-matrix repeatedly, a direct calculation proves that both sides of the Yang–Baxter equation (9) are

$$\exp[P\otimes\hat{P}\otimes 1+Q\otimes\hat{Q}\otimes 1+N\otimes E\otimes 1+P\otimes 1\otimes\hat{P}+Q\otimes 1\otimes\hat{Q}+N\otimes 1\otimes E$$
$$+1\otimes P\otimes\hat{P}+1\otimes Q\otimes\hat{Q}+1\otimes N\otimes E+\tfrac{1}{2}\mu(PQ\otimes E\otimes 1+1\otimes PQ\otimes E+PQ\otimes 1\otimes E)]\,.$$

The above proof directly verifies Drinfeld's theorem with these quite simple algebras A and B.

From the above analysis we can conclude that the classical observable algebra A can incorporate the quantum observable algebra B to form a new quantum group D – a quasi-triangular Hopf algebra, which is generated by P, Q, N, E, $\hat{P}$ and $\hat{Q}$ with relations (1), (4), (5) and (7); the quantum Yang–Baxter equation as a necessary condition is satisfied in this construction of the canonical quantization. In fact, the new quantum group D, as an associative algebra, is the universal enveloping algebra of the non-simple Lie algebra with the basis P, Q, N, E, $\hat{P}$ and $\hat{Q}$. This means that, besides so-called q-deformations of some "classical algebra", one not only endows some finite-dimensional Lie algebras (strictly speaking, their universal enveloping algebras) with a cocommutative, but also with a non-cocommutative Hopf algebra structure. The discussion of this paper is a typical example of exotic quantum doubles of "non-q-deformation" and another type of such an exotic quantum double has been constructed in connection with infinite-dimensional Lie algebra [7]. As for the quantum theory, the present studies show that Drinfeld's quantum double theory can provide us with an algebraicized scheme of quantization available for both basic quantum mechanics and nonlinear quantum systems such as in the quantum inverse scattering method.

Before concluding the discussion in this paper, we would like to understand the physical meaning of the operator N in connection with the universal R-matrix (8). Considering the commutation relations (6), we can give N a realization in terms of $\hat{P}$ and $\hat{Q}$,

$$N=P\hat{P}+Q\hat{Q}\,,$$

which preserves all the commutators of N with other generators. Here we have taken a representation with E as unity. Let $Z=Q+iP$, $a=\frac{1}{2}(\hat{Q}-i\hat{P})$, then $N=Za^{+}+Z^{*}a$ defines the generator $D(Z)=e^N$ of the coherent state. For a further consideration of the physical meaning of N in connection with the Yang–Baxter equation we need to know the representation theory of the quantum group D. Thanks to the Schur lemma, the central elements P, Q and E must be some scalars in an irreducible representation. In this sense, $P\otimes\hat{P}$, and $Q\otimes\hat{Q}$ act as $P\hat{P}$ and $Q\hat{Q}$

on the second component of the product space V⊗V where V is the representation space. Thus, the universal R-matrix has an equivalent form,

$$\hat{R}=e^{N}\otimes\exp(P\hat{P})\exp(Q\hat{Q})\,, \tag{10}$$

where we have renormalized the scalar E to be 1 and N just appears in $D(Z)$!

Using the Glauber formula again we prove that the universal R-matrix is just the generating operator of the two-model coherent state

$$|Z\rangle=\exp(\tfrac{1}{2}\mathrm{i}\hbar PQ)\exp(Za^{+}+Z^{*}a)\otimes\exp(Za^{+}+Z^{*}a)|0\rangle, \qquad Z=Q+\mathrm{i}P\,,$$

that has not been normalized. This not only shows the possible relations between the coherent state and the quantum Yang–Baxter equation, but also demonstrates the physical meaning of the operator N. We also notice that in some representations similar to that considered above, the R-matrices are completely factorized as the solutions for the Yang–Baxter equation.

Finally, we point out that a class of new R-matrices, as the matrix representations of the universal R-matrix for the new quantum group D, can follow from the finite-dimensional representations. To obtain them, we need detailed discussions for the general representation theory of D. The corresponding studies, with some mathematical interest, will be published elsewhere.

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