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The q-deformed Lipkin–Meshkov–Glick model for many-fermion systems with the quantum symmetry of $SU_q(2)$

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By building a multi-fermion realization of the quantum algebra $SU_q(2)$, we generalized the Lipkin–Meshkov–Glick (LMG) model to test the validity of many-body approximation methods. Our general LMG model incorporates the quantum group symmetry of $SU_q(2)$ and the many-body force besides the two-body one in many-fermions systems, which give a probe of the dynamic process that the quantum group symmetry should describe. The exact solvability is still guaranteed by the finite dimensional representation of $SU_q(2)$. The deformation parameter q is introduced in the generalized model to characterize the difference between this model and the original one. Especially, it leads to the crossing of different energy levels.

1. Introduction

Though the concept of quantum group and algebra originates from the mathematical abstraction of many non-linear physical problems [1–3], its role incorporating the new symmetry, the quantum group symmetry (QGS), has been discussed in many real physical problems such as the fitting of the rotational spectra of nuclei [4,5] and the description of diatomic molecules [6,7]. It should be pointed out that there are some discussions in connection with the concept of a q-deformed boson (or oscillator) [8–10].

Although these discussions about the QGS have had a certain success, they are phenomenological in our opinion. In fact, since the blocks constituting the real systems such as nuclei and diatomic molecules are fermions, the generators of the quantum group symmetry in them must be written in terms of ordinary fermion creation and annihilation operators. Hence, the fermion realization of a quantum algebra should be used in the microscopic theory to describe the quantum symmetry of a fermion system. However, it has not been used in previous discussions and thus there is not a microscopic theory based on the fermion Hamiltonian. In this paper, we will build the multi-fermion realization of a quantum algebra and then use it to generalize the Lipkin–Meshkov–Glick (LMG) model [11].

The original LMG model was built to test the validity of many-body approximation methods. Its Hamiltonian

$$H=\epsilon J_z+\tfrac{1}{2}V(J_+J_-+J_-J_+) \tag{1.1}$$

is expressed in terms of the generators of the Lie group SU(2) in the form of the fermion realization

$$J_+=\sum_{p=1}^{\Omega} a^+_{p+}a_{p-}\,,\quad J_-=\sum_{p=1}^{\Omega} a^+_{p-}a_{p+}\,,$$

$$J_z=\tfrac{1}{2}\sum_{p=1}^{\Omega}(a^+_{p+}a_{p+}-a^+_{p-}a_{p-})\,, \tag{1.2}$$

where a^+_{p+}, a_{p+}, a^+_{p-}, a_{p-} are the fermion operators. Obviously, the Hamiltonian includes a two-body force. It is a simple yet non-trivial model. Since the Casimir

$$J^2=J_+J_-+J_z(J_z-1)$$

commutes with H, the matrix of H will break up into submatrices in each carrier space of each representation $D[j]$ of SU(2), labelled by the so-called quasi-spin J, thus it is an exactly solvable model. Our generalization is to replace $J_\pm$, J_z by the generators $\hat{J}_\pm$, $\hat{J}_z$ of the quantum algebra $\mathrm{SU}_q(2)$ satisfying

$$[\hat{J}_+, \hat{J}_-] = [2\hat{J}_z]_q, \quad [\hat{J}_z, \hat{J}_\pm] = \pm \hat{J}_\pm , \qquad (1.3)$$

where $q \in \mathbb{R}$ is the real deformation parameter and $[x]_q = (q^x - q^{-x})/(q - q^{-1})$. This generalization incorporates the quantum group symmetry into the model system. In the Hamiltonian of the generalized LMG model, the many-body correlation is introduced and the exact solvability is still guaranteed. In subsequent works, we will compare the results obtained by the various formulations with exact results obtained from the q-deformed LMG model.

2. A multi-fermion realization of $\mathrm{SU}_q(2)$

Recently, a general method building a non-trivial fermion realization of a quantum algebra was presented by the authors [12]. For convenience, in the following discussion, we reformulate the central idea with a new example, which will be used in this paper. We begin with the simple but crucial observation that the following operators [13–15],

$$\hat{J}_+ = a^+ b, \quad \hat{J}_- = b^+ a,$$
$$\hat{J}_z = \tfrac{1}{2}(a^+ a - b^+ b) , \qquad (2.1)$$

not only form a Lie algebra, but also are the generators of the quantum algebra $\mathrm{SU}_q(2)$, where a^+, a, b^+, b are fermion operators. This is because

$$q^{\hat{N}} = 1 + (q-1)\hat{N}, \quad \hat{N} = f^+ f,$$
$$f = a, b, \quad [2\hat{J}_z] = 2J_z . \qquad (2.2)$$

However, for the fermion operators

$$ff^+ + f^+ f = 1, \quad (f^\pm)^2 = 0 , \qquad (2.3)$$

we have $(\hat{J}_\pm)^2 = 0$. This means that the fermion realization (2.1) leads to the fundamental representation for the quantum algebra $\mathrm{SU}_q(2)$, which is also a half-spin representation of SU(2). Therefore, we say that the fermion realization is trivial. To obtain the non-trivial fermion realization, we must use the multi-fermion operator

$$\hat{f}_i^\pm = f_i^\pm \exp\left(\mathrm{i}\pi \sum_{k=1}^{i-1} \hat{N}_k \right), \quad \hat{N}_k = f_k^+ f_k = \hat{f}_k^+ \hat{f}_k , \qquad (2.4)$$

where

$$f_i = 1 \otimes 1 \otimes \ldots \otimes 1 \otimes \underset{i}{f} \otimes 1 \otimes \ldots \otimes 1$$

and consider a property of the coproduct of quantum algebra. If $\hat{J}_\pm$ and $\hat{J}_z$ are generators for $\mathrm{SU}_q(2)$, then

$$L_\pm = q^{\hat{J}_z} \otimes \hat{J}_\pm + \hat{J}_\pm \otimes q^{-\hat{J}_z} ,$$
$$L_z = \hat{J}_z \otimes 1 + 1 \otimes \hat{J}_z \qquad (2.5)$$

also generate $\mathrm{SU}_q(2)$. Using (2.1), (2.4) and the multi-extension of eq. (2.5),

$$L_\pm = \sum_i q^{\hat{J}_z} \otimes q^{\hat{J}_z} \otimes \ldots \otimes \underset{i}{\hat{J}_\pm} \otimes q^{-\hat{J}_z} \otimes \ldots \otimes q^{-\hat{J}_z} ,$$
$$L_z = \sum_i 1 \otimes 1 \otimes \ldots \otimes \underset{i}{\hat{J}_z} \otimes 1 \otimes \ldots \otimes 1, \qquad (2.6)$$

we immediately write down a non-trivial fermion realization of $\mathrm{SU}_q(2)$ where

$$L_+ = \sum_{i=1}^{\Omega} C_i \hat{a}_i^+ \hat{b}_i, \quad L_- = \sum_{i=1}^{\Omega} C_i \hat{b}_i^+ \hat{a}_i ,$$
$$L_z = \tfrac{1}{2} \sum_{i=1}^{\Omega} (\hat{a}_i^+ \hat{a}_i - \hat{b}_i^+ \hat{b}_i) , \qquad (2.7)$$

where

$$C_i = q^{\sum_{k=1}^{\Omega} \frac{1}{2}\zeta(i-k)(\hat{a}_k^+ \hat{a}_k - \hat{b}_k^+ \hat{b}_k)} ,$$
$$\zeta(x) = 1, \quad x > 0 ,$$
$$= 0, \quad x = 0 ,$$
$$= -1, \quad x < 0 . \qquad (2.8)$$

This is the key to the generalization of the LMG model. Here the function C_i is an operator only containing finite terms,

$$C_i = \prod_{k=1}^{\Omega} \{1 + (q^{\zeta(i-k)/2} - 1)\hat{a}_k^+ \hat{a}_k\}$$
$$\times \{1 + (q^{-\zeta(i-k)/2} - 1)\hat{b}_k^+ b_k\} . \qquad (2.9)$$

3. Generalized LMG model with q-deformed symmetry

We consider a system of N fermions distributed in two levels each having an Ω-fold degeneracy and separated by an energy ϵ. Each state is described by two quantum numbers p and σ, where σ has two values: $+1$ if the particle is in the upper level; -1 is it is in the lower level. The Hamiltonian for the system is as follows,

$$H=\tfrac{1}{2}\epsilon \sum_{p\sigma} \sigma a^{+}_{p\sigma} a_{p\sigma} + \tfrac{1}{2} V \sum_{pp'\sigma} C_p C_{p'} q^{-\zeta(p-p')} a^{+}_{p\sigma} a^{+}_{p'\sigma} a_{p'-\sigma} a_{p-\sigma} + \tfrac{1}{2} W \sum_{pp'\sigma} C_p C_{p'} q^{-\zeta(p-p')} a^{+}_{p\sigma} a^{+}_{p'-\sigma} a_{p'\sigma} a_{p-\sigma} + \tfrac{1}{2} W \sum_{p\sigma} C^2_p a^{+}_{p\sigma} a_{p\sigma} , \tag{3.1}$$

where $a^{+}_{p\sigma}$ and $a_{p\sigma}$ are respectively the creation and annihilation operators acting on a particle in the p, σ state. The fourth term cannot be omitted except when $q=1$. By introducing the q-deformed quasi-spin operators

$$\hat{J}_{+} = \sum_p C_p a^{+}_{p+1} a_{p-1}, \quad \hat{J}_{-} = \sum_p C_p a^{+}_{p-1} a_{p+1},$$

$$\hat{J}_z = \tfrac{1}{2} \sum_{p\sigma} \sigma a^{+}_{p\sigma} a_{p\sigma} , \tag{3.2}$$

the Hamiltonian can be rewritten in terms of the deformed quasi-spin operators as

$$H=\epsilon \hat{J}_z + \tfrac{1}{2} V(\hat{J}^2_{+} + \hat{J}^2_{-}) + \tfrac{1}{2} W(\hat{J}_{+}\hat{J}_{-} + \hat{J}_{-}\hat{J}_{+}) , \tag{3.3}$$

where the operators $\hat{J}_{+}$, $\hat{J}_{-}$ and $\hat{J}_z$ are the generators of $SU_q(2)$. Obviously, H commutes with the q-Casimir operator

$$C=\hat{J}_{+}\hat{J}_{-} + [\hat{J}_z][\hat{J}_z - 1] .$$

Now we observe that the quantum group symmetry of $SU_q(2)$ is incorporated in the generalized model. The Hamiltonian matrix will break up into diagonal block matrices, each of whose submatrices is associated with different irreducible representations $D[j]$ of $SU_q(2)$. Because the Hamiltonian becomes the original LMG model when $q \to 1$, we can regard the generalized model as a q-deformation of the original one. When $q-1$ is very small the system is incorporated with an approximate SU(2) symmetry. Such an approximate symmetry model is of much interest.

In comparison with the SU(2) symmetry model (1.1), the physical meaning of the Hamiltonian is considered to be as follows. In the original model, any two fermion pairs interact with the same strength. However, in the generalized model, the interaction strength depends on the particle density of other fermion pairs and thus the many-body correlations are considered.

Now we illustrate the above description with the interaction matrix, in the original model,

$$\langle Q_1(N-2), p'\sigma, p\sigma | H | p-\sigma, p'-\sigma, Q_1(N-2) \rangle = \tfrac{1}{2} V , \tag{3.4}$$

$$\langle Q_2(N-2), p'-\sigma, p\sigma | H | p-\sigma, p'\sigma, Q_2(N-2) \rangle = \tfrac{1}{2} W , \tag{3.5}$$

where $Q_1(N-2)$ and $Q_2(N-2)$ represent all the unchanged states; but in the generalized model

$$\langle Q_1(N-2), p'\sigma, p\sigma | H | p-\sigma, p'-\sigma, Q_1(N-2) \rangle = \tfrac{1}{2} V C_p C_{p'} q^{-\zeta(p-p')} , \tag{3.6}$$

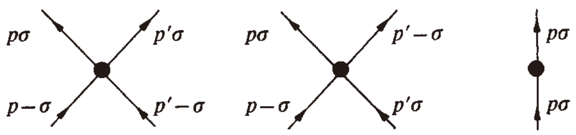
$$\langle Q_2(N-2), p'-\sigma, p\sigma | H | p-\sigma, p'\sigma, Q_2(N-2) \rangle = \tfrac{1}{2} W C_p C_{p'} q^{-\zeta(p-p')} , \tag{3.7}$$

where

$$C_p = q^{\sum_{k=1}^{\Omega} \zeta(p-k)/2 (N_{k,+1} - N_{k,-1})} , \tag{3.8}$$

which depends on the unchanged states. For the kth degenerate states, if both upper (+) and lower (−) states are occupied by particles, they will have no contribution to the other scattering; otherwise, they will.

A diagrammatic representation of the interaction is



Actually, the loop diagrams are considered in the vertexes.

In order to find the eigenvalues of H exactly, we build the carrier space of the irreducible representation D[j] from the lowest weight state space $|0\rangle_j$, satisfying $\hat{J}_-|0\rangle=0$. Obviously

$$|0\rangle=a^+_{1,-1}a^+_{2,-1}\dots a^+_{N,-1}|0\rangle\ . \tag{3.9}$$

4. Exact solutions of the generalized LMG model

The interaction term proportional to W in Hamiltonian (3.1) does not mix the configurations and is diagonalized exactly by the $SU_q(2)$ representation. Then we solve the generalized LMG model exactly for $N=2$, 3, 4, 6, 8 by setting W equal to zero. For $N=2$

$$\frac{E}{\epsilon}=\pm\{1+\tfrac{1}{4}[2]^2(V/\epsilon)^2\}\ ,\quad 0\ . \tag{4.1}$$

For $N=3$

$$\frac{E}{\epsilon}=\pm(\tfrac{1}{2}\pm\{1+\tfrac{1}{4}[2]^2[3](V/\epsilon)^2\}^{1/2})\ . \tag{4.2}$$

For $N=4$

$$\frac{E}{\epsilon}=\pm 2\{1+\tfrac{1}{8}[4]!(V/\epsilon)^2\}^{1/2},\quad 0\ , \tag{4.3}$$

$$\frac{E}{\epsilon}=\pm\{1+\tfrac{1}{4}[2]^2[3]^2(V/\epsilon)^2\}\ . \tag{4.4}$$

For $N=6$

$$\frac{E}{\epsilon}=\pm 2\{1+\tfrac{1}{8}[5]!(V/\epsilon)^2\}^{1/2},\quad 0\ , \tag{4.5}$$

$$\frac{E}{\epsilon}=\pm(5+\tfrac{1}{2}(k_2/\epsilon)^2+(k_1/\epsilon)^2$$
$$\pm\{16+\tfrac{1}{4}(k_2/\epsilon)^4+(k_1/\epsilon)^2(k_2/\epsilon)^2$$
$$+16(k_1/\epsilon)^2-4(k_2/\epsilon)^2\}^{1/2})^{1/2}\ ,$$
$$k_1=\tfrac{1}{2}V\sqrt{[6][5][2]}\ ,$$
$$k_2=\tfrac{1}{2}V[3][4]\ . \tag{4.6}$$

For $N=8$

$$\frac{E}{\epsilon}=\pm(5+\tfrac{1}{2}(k_4/\epsilon)^2+(k_3/\epsilon)^2$$
$$\pm\{16+\tfrac{1}{4}(k_4/\epsilon)^4+(k_3/\epsilon)^2(k_4/\epsilon)^2$$
$$+16(k_3/\epsilon)^2-4(k_4/\epsilon)^2\}^{1/2})^{1/2}\ , \tag{4.7}$$

$$\frac{E}{\epsilon}=\pm(10+(k_1/\epsilon)^2+(k_2/\epsilon)^2$$
$$\pm\{36+36(k_1/\epsilon)^2-12(k_2/\epsilon)^2+(k_2/\epsilon)^4\}^{1/2})^{1/2}\ ,$$
$$k_1=\tfrac{1}{2}V\sqrt{[8][7][2]},\quad k_2=\tfrac{1}{2}V\sqrt{\tfrac{1}{2}[6]!},$$
$$k_3=\tfrac{1}{2}V\sqrt{[7][6][3][2]},\quad k_4=\tfrac{1}{2}V[4][5]\ . \tag{4.8}$$

The energy levels of the systems are characterized by three parameters, the number of particles N, the interaction strength V/ϵ and the deformation parameter q. The above results are just the same as the original ones in ref. [11] when the deformation parameter $q\to 1$.

In order to compare the eigenvalues as a function of the interaction parameter NV/ϵ with different deformation parameters q, the positive energy values for $N=8$ with $q=1$ and 1.2 are plotted in fig. 1.

From the analytical expression of the energy levels for eight particles, we know that in the original case ($q=1$) crossing between different energy levels never occurs, but fig. 1 shows that it is found in the deformed case ($q=1.2$). This phenomenon also is found in fig. 2, where the energy levels are functions of deformation parameter q with fixed interaction parameter NV/ϵ.

In some problems, the excitation energy of the first excited state above the ground state shall be involved. The exact results for the excitation energy as functions of the interaction parameter NV/ϵ with

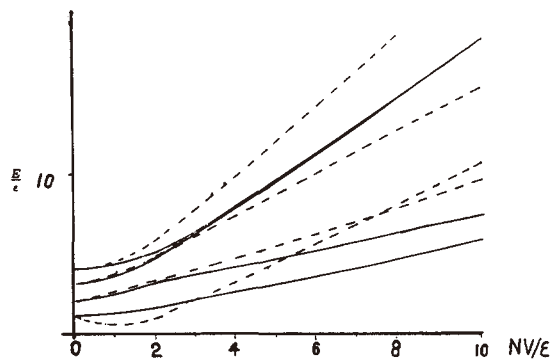


Fig. 1. Positive energy eigenvalue plotted versus the interaction parameter NV/ϵ for eight particles. The solid lines show the original case ($q=1$), the short dashed lines show the deformed case ($q=1.2$).

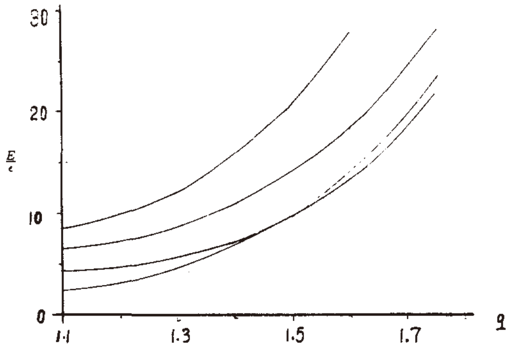


Fig. 2. Positive energy eigenvalue plotted versus the deformation parameter q for eight particles when the interaction parameter is $4NV/\epsilon$.

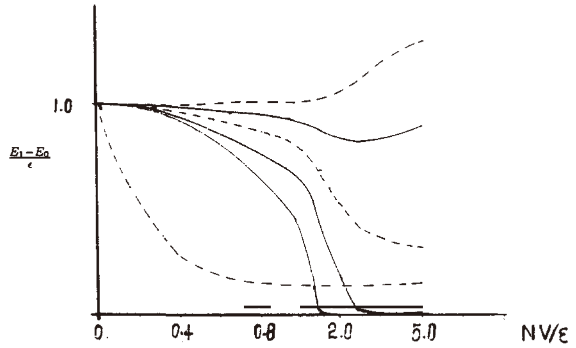


Fig. 3. Exact results for the excitation energy of the first excited state above the ground state plotted versus the interaction parameter NV/ϵ for $N=4$, 14 and 50 particles. The solid lines show the original case ($q=1$), the short dashed lines represent the deformation case.

different deformation parameters q are shown in fig. 3.

5. Perturbation theory

Using the generalized LMG model, we study the range of validity of perturbation theory, with special emphasis on the many-body interaction besides the two-body one. The non-perturbed energy levels are non-degenerate, therefore the non-degenerate perturbation theory can be used. The resulting excitation energy above the ground state is given by

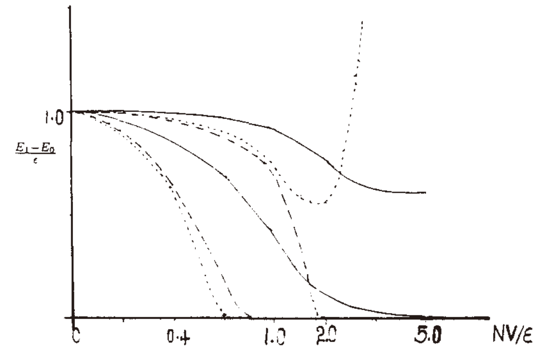


Fig. 4. Excitation energy of the first excited state above the ground state for the deformed case ($q=1.05$). The upper three lines refer to the eight-particle system and the lower three to $N=50$. The solid lines show the exact excitation energy, the short dashed lines represent the second order perturbation theory results, and the dotted lines show the results of perturbation theory to the fourth order.

$$\frac{E_1-E_2}{\epsilon}=1+\Delta^{(2)}+\Delta^{(4)},$$

$$\Delta^{(2)}=\tfrac{1}{8}[2][N-1]\{[N]-[3][N-2]\}(V/\epsilon)^2,$$

$$\Delta^{(4)}=-\tfrac{1}{256}([N-1][N-2][N-3]$$
$$\times\{[N-4][5]!-[N][4]!)$$
$$-2[2]^2[N-1]^2([3]^2[N-2]^2-[N]^2\})(V/\epsilon)^4, \quad (5.1)$$

where the three terms correspond the zeroth, second and fourth order. The results to the second and the fourth order are shown in fig. 4, together with the exact excitation energy for eight and 50 particles.

In the following papers, the generalized LMG model will be employed to test the validity of other approximation methods. Especially, we shall consider the many-parameter perturbation theory with r ($=q-1$), r^2, r^3, ... where r is small. More significantly, we shall extend the ideas in this paper to the generalized three-level LMG model, which is still extensively used.

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