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The q-boson realization of new solutions for the Yang–Baxter equation associated with the quantum algebra $U_q sl(3)$ at $q^p=1$

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In this Letter a series of new solutions for the quantum Yang–Baxter equation is obtained by constructing a new boson realization of λ-dependent representations of the quantum algebra $U_q sl(3)$ with q a root of unity.

At present, it has been recognized that the solutions for the quantum Yang–Baxter equation (QYBE) can be constructed in terms of the quantum (universal enveloping) algebra $U_q(L)$ of the classical Lie algebra L and its representations [1–3]. Especially, we found that the new solutions, beyond the standard and non-standard ones, for the QYBE without a spectral parameter, can be obtained from the non-generic representations (for q a root of unity, i.e., $q^p=1$, $p=3, 4, \ldots$) of $U_q sl(2)$ [4–6]. Both the results obtained and the method used by us are essentially different from those obtained by Date, Jimbo, Miki and Miwa [7].

Naturally, it is reasonable trying to find the new solutions for the QYBE associated with other quantum algebras or a superalgebra such as $U_q sl(3)$. We have done it for the quantum superalgebra $U_q osp(1,2)$ [8]. This paper will be devoted to studying the case of $U_q sl(3)$. To this end, we first generalize the universal R-matrix of $U_q sl(3)$ [9] in such a way that it still works for the non-generic case, in which q is a root of unity, i.e., $q^p=1$, $p=3, \ldots$.

To make this generalization, we add the new relations

$$(E_i)^p=(F_i)^p=0\,, \qquad i=1,2\,, \tag{1}$$

to the basic q-commutation relations

$$[H_i, E_i]=2E_i\,, \qquad [H_i, F_i]=-2F_i\,,$$

$$[H_i, E_j]=-E_j\,, \qquad [H_i, F_j]=F_j\,, \qquad i\neq j\,,$$

$$[E_i, F_j]=\delta_{ij}e^{h/2}\frac{\text{sh}(\frac{1}{2}hH_i)}{\text{sh}(\frac{1}{2}h)}\,, \qquad i=1,2\,, \tag{2}$$

with $h\in\mathbb{C}$ (the complex number field), $q=e^{h/2}$ and the Serre relations for $U_q sl(3)$. Here, F_i, E_i and H_i ($i=1$, 2) are the generators for $U_q sl(3)$. For the third kind of basis elements,

$$E_3=E_1E_2-qE_2E_1\,, \qquad F_3=F_1F_2-qF_2F_1\,, \tag{3}$$

we also require

$$(E_3)^p=(F_3)^p=0\,. \tag{4}$$

The generalized algebra $U_q sl(3)$ by adding relations (1) and (4) is called the non-generic quantum algebra and denoted by $\hat{U}_q sl(3)$.

Endowing $\hat{U}_q sl(3)$ with the same coproduct, counit and antipode as for $U_q sl(3)$, we obtain a Hopf algebra $\hat{U}_q sl(3)$. After a long derivation similar to that by Rosso in ref. [9], we obtain the non-generic universal R-matrix for $q^p=1$,

$$\mathscr{R}=\left(\sum_{m_1,m_2,m_3=0}^{p-1}\frac{(1-e^{-h})^{m_1}(1-e^{-h})^{m_3}(1-e^{-h})^{m_2}}{\phi_{m_1}(e^{-h})\phi_{m_3}(e^{-h})\phi_{m_2}(e^{-h})}E_2^{m_1}E_3^{m_3}E_1^{m_2}\otimes\eta_2^{m_1}\eta_3^{m_3}\eta_1^{m_2}\right)e^{h/4\sum_{i=1}^{2}H_i\otimes\xi_i}, \tag{5}$$

where

$$\phi_n(e^{-h})=\prod_{k=1}^{n}(1-e^{-kh}),\qquad \eta_i=(1-e^{-h})F_i,\qquad i=1,2, \tag{6}$$

$$\eta_3=(1-e^h)^{-1}(\eta_1\eta_2-q\eta_2\eta_1),\qquad \xi_1=\tfrac{1}{3}(4H_1+2H_2),\qquad \xi_2=\tfrac{1}{3}(2H_1+4H_2). \tag{7}$$

Now, we try to construct a representation ρ for $\hat{U}_q sl(3)$, which is also a representation of $U_q sl(3)$, but satisfies $(\rho(E_i))^p=(\rho(F_i))^p=0$, namely, $\rho(E_i)$ and $\rho(F_i)$ are nilpotent. It is just at this point that our study is essentially different from that made by Date et al., in which $\rho(E_i)$ and $\rho(F_i)$ are not nilpotent and the cyclic representation is used. To construct such representations, we use the new q-boson realization,

$$E_1=a_1^+,\qquad F_1=-e^{h/2}[N_1+N_2+\lambda]a_1,\qquad E_2=-[N_1+N_2+\lambda]a_2,\qquad F_2=a_2^+e^{h/2},$$

$$H_1=2N_1+N_2+\lambda,\qquad H_2=-(2N_2+N_1+\lambda), \tag{8}$$

where the q-boson operators $a_i=a_i^-$, a_i^+ and N_i $(i=1,2)$ satisfy [10–12]

$$a_ia_j^+-q^{\mp\delta_{ij}}a_j^+a_i=\delta_{ij}q^{\pm N_i},\qquad [N_i,a_j^{\pm}]=\pm\delta_{ij}a_j^{\pm},\qquad [N_i,N_j]=0,\qquad [a_i^{\pm},a_j^{\pm}]=0. \tag{9}$$

In fact, using eq. (3), we can verify that E_i, F_i and H_i $(i=1,2)$ defined by (8) satisfy eq. (2) and the Serre relations.

On the q-Fock space $\mathscr{F}_q(2)$,

$$\{f(m_1,m_2)=a_1^{+m_1}a_2^{+m_2}|0\rangle\ |a_i|0\rangle=N_i|0\rangle=0,\ i=1,2;\ m\in\mathbb{Z}^+=\{0,1,2,...\}\},$$

we obtain an infinite-dimensional representation Γ:

$$E_1f(m_1,m_2)=f(m_1+1,m_2),\qquad F_1f(m_1,m_2)=-e^{h/2}[m_1-1+m_2+\lambda][m_1]f(m_1-1,m_2),$$

$$E_2f(m_1,m_2)=-[m_1+m_2-1+\lambda][m_2]f(m_1,m_2-1),\qquad F_2f(m_1,m_2)=e^{h/2}f(m_1,m_2+1),$$

$$H_1f(m_1,m_2)=(2m_1+m_2+\lambda)f(m_1,m_2),\qquad H_2f(m_1,m_2)=-(2m_2+m_1+\lambda)f(m_1,m_2). \tag{10}$$

Because $q^p=\pm1$ leads to $[kp]=0$, where $[f]=(q^f-q^{-f})/(q-q^{-1})$, there exists an invariant subspace $V(\alpha_1,\alpha_2)$ $(\alpha_1,\alpha_2\in\mathbb{Z}^+)$,

$$\text{span}\{f(m_1,m_2)\,|\,m_i\geq\alpha_ip,\ i=1,2;\ \alpha_i\in\mathbb{Z}^+\},$$

and the representation (10) is reducible. Naturally, on the corresponding quotient space $\mathscr{F}_q(2)/V(\alpha_1,\alpha_2)$,

$$\text{span}\{\bar{f}(m_1,m_2)=f(m_1,m_2)\ \text{mod}\ V(\alpha_1,\alpha_2)\,|\,0\leq m_i\leq\alpha_ip-1,\ i=1,2;\ \alpha_i\in\mathbb{Z}^+\},$$

we get an $\alpha_1\alpha_2p^2$-dimensional representation $\rho^{[\alpha_1,\alpha_2]}$,

$$\rho^{[\alpha_1,\alpha_2]}(g)\bar{f}(m_1,m_2)=(\rho(g)f(m_1,m_2))\ \text{mod}\ V(\alpha_1,\alpha_2),\qquad g\in\{E_i,F_i,H_i\,|\,i=1,2\}, \tag{11}$$

for $U_q sl(3)$. In fact, eqs. (10) and (11) only define the representation for $U_q sl(3)$ when α_1 or $\alpha_2\neq1$. Because $(\rho(E_i))^p=(\rho(F_i))^p=0$ when $\alpha_1=\alpha_2=1$, eq. (11) defines a representation $\rho=\rho^{[1,1]}$ for $\hat{U}_q sl(3)$ in this case.

When $q^2=-1$, as an example, the lowest dimensional representation $\rho=\rho^{[\lambda]}$ of $\hat{U}_q\text{sl}(3)$ is given in an explicit form,

$$\rho(E_1)=\epsilon_{2,1}+\epsilon_{4,3}, \quad \rho(F_1)=-e^{-h/2}([\lambda]\epsilon_{1,2}+[\lambda+1]\epsilon_{3,4}),$$
$$\rho(E_2)=-[\lambda]\epsilon_{1,3}-[\lambda+1]\epsilon_{2,4}, \quad \rho(F_2)=e^{h/2}(\epsilon_{3,1}+\epsilon_{4,2}),$$
$$\rho(H_1)=\text{diag}(\lambda,\lambda+2,\lambda+1,\lambda+3), \quad \rho(H_2)=\text{diag}(\lambda,\lambda+1,\lambda+2,\lambda+3), \tag{12}$$

where ϵ_{ij} is the unit matrix such that $(\epsilon_{ij})_{k,l}=\delta_{ik}\delta_{jl}$.

Using the representation $\rho^{[\lambda]}$ of $\hat{U}_q\text{sl}(3)$, we immediately obtain the quantum R-matrix

$$R^{[\lambda]}=\rho^{[\lambda]}\otimes\rho^{[\lambda]}(\mathscr{R})=\text{block diag}(A_1,A_2,A_3,A_4,A'_1,A'_2,A'_3)\exp(\tfrac{1}{3}h\lambda^2), \tag{13}$$

where

$$A_1=1, \quad A'_1=-t^4, \quad t=q^\lambda,$$
$$A_2=\begin{pmatrix} t & 0 \\ 1-t^2 & t\end{pmatrix}, \quad A'_2=\begin{pmatrix} -qt^3 & 0 \\ -t^2(1+t^2) & -qt^3\end{pmatrix},$$
$$A_3=\begin{pmatrix} t & 0 & 1-t^2 \\ 0 & -t^2 & 0 \\ 0 & 0 & t\end{pmatrix}, \quad A'_3=\begin{pmatrix} -qt^3 & 0 & -t^2(1+t^2) \\ 0 & -t^2 & 0 \\ 0 & 0 & -qt^3\end{pmatrix},$$
$$A_4=\begin{pmatrix} t^2 & 0 & qt(1-t^2) & 0 \\ q^{-1}t(1+t^2) & qt^2 & 1-2t^2-t^4 & q^{-1}t(1+t^2) \\ 0 & 0 & qt^2 & 0 \\ 0 & 0 & qt(1-t^2) & t^2\end{pmatrix}.$$

We need to point out that although q is a cyclic parameter, the quantum R-matrix still contains the continuous parameter $t=q^\lambda$.

Another example of an R-matrix given by the non-generic universal R-matrix and the representation (11) for $\alpha_1=\alpha_2=1$ is

$$R^{\lambda_1\lambda_2}=R(\lambda_1,\lambda_2)=\rho^{[\lambda_1]}\otimes\rho^{[\lambda_2]}(\mathscr{R})=\text{block diag}(\bar{A}_1,\bar{A}_2,\bar{A}_3,\bar{A}_4,\bar{A}'_3,\bar{A}'_2,\bar{A}'_1)\exp(\tfrac{1}{3}h\lambda_1\lambda_2), \tag{14}$$

where

$$\bar{A}_1=1, \quad \bar{A}'_1=-t_1^2t_2^2, \quad t_1=q^{\lambda_1}, \quad t_2=q^{\lambda_2},$$
$$\bar{A}_2=\begin{pmatrix} t_1 & 0 \\ t_1(t_2^{-1}-t_2) & t_2\end{pmatrix}, \quad \bar{A}'_2=\begin{pmatrix} -qt_1^2t_2 & 0 \\ -t_1^2(1+t_2^2) & -qt_1t_2^2\end{pmatrix},$$
$$\bar{A}_3=\begin{pmatrix} t_1 & 0 & t_2(t_1^{-1}-t_1) \\ 0 & -t_1t_2 & 0 \\ 0 & 0 & t_2\end{pmatrix}, \quad \bar{A}'_3=\begin{pmatrix} -qt_1^2t_2 & 0 & -t_2^2(1+t_1^2) \\ 0 & -t_1t_2 & 0 \\ 0 & 0 & -qt_1t_2^2\end{pmatrix},$$
$$\bar{A}_4=\begin{pmatrix} 1 & 0 & qt_2(1-t_1^2) & 0 \\ t_1^2(q^{-1}t_2^{-1}-qt_2) & qt_1t_2 & -2t_1^2+(1-t_2^2)(t_1^2+1) & t_2^2(q^{-1}t_1^{-1}-qt_1) \\ 0 & 0 & qt_1t_2 & 0 \\ 0 & 0 & qt_1(1-t_2^2) & t_2^2\end{pmatrix}.$$

In fact, because the R-matrix $R(\lambda_1,\lambda_2)$ satisfies the QYBE

$$R_{12}(\lambda_1,\lambda_2)R_{13}(\lambda_1,\lambda_3)R_{23}(\lambda_2,\lambda_3)=R_{23}(\lambda_2,\lambda_3)R_{13}(\lambda_1,\lambda_3)R_{12}(\lambda_1,\lambda_2), \tag{15}$$

we formally regard the λ's as the spectral parameters. This point was just pointed out by Jimbo. This kind of parameter is also identified with the so-called colour, which distinguishes the different representations with the same dimension.

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