Non-standard R-matrices for the Yang–Baxter equation and boson representations of the quantum algebra $\mathrm{sl}_q(2)$ with $q^p = \pm 1$ ☆

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By constructing a new q-deformed boson realization of the quantum universal enveloping algebra $\mathrm{sl}_q(2)$ its finite dimensional representations with an arbitrary parameter are obtained in the case that q is a root of unity. With the obtained representations various non-standard $\mathrm{sl}_q(2)$ R-matrices are presented through the universal R-matrix.

It has become well known that the Yang–Baxter equation plays a very important role in non-linear physics as for instance exactly solvable statistical mechanics models and low dimensional integrable field theories, and it has also been realized that one can construct various R-matrices, which are the solutions of the Yang–Baxter equation, through quantum universal enveloping algebras (also called quantum algebras for short) and their representations [1,2]. When the representations are irreducible the R-matrices obtained in this way were called standard ones. By taking the weight conservation condition into account some non-standard R-matrices have been obtained with the extended Kauffman diagram technique [3], whose relation to quantum algebras has been discussed in some special cases [4]. In this paper we will probe this relation in a more general way and successfully put the non-standard R-matrices associated with $\mathrm{sl}_q(2)$ into the framework of quantum algebra. The conclusion obtained in this paper can be generalized to other quantum algebras. We should point out that so far as we know, Date et al. have investigated the standard R-matrices when q is a root of unity, and that our discussion is different from theirs in that we mainly consider non-standard cases [5].

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The q-deformed boson algebra $\mathscr{B}_q$ was first introduced by different authors independently [6] and then evolved into a well defined form [7]. As in ref. [7] we equivalently define $\mathscr{B}_q$ as an associative algebra over the complex number field $\mathbb{C}$ which is generated by elements N, a^+ and $a \equiv a^-$ satisfying

$$a^+a = [N], \quad aa^+ = [N+1],$$
$$[N, a^\pm] = \pm a^\pm, \tag{1}$$

where we have defined

$$[f] = \frac{q^f - q^{-f}}{q - \bar{q}^{-1}}$$

for any $f \in \mathscr{B}_q$. One can check that the above algebraic relations guarantee

$$J_+ = a_1^+ a_2, \quad J_- = a_2^+ a_1, \quad J_3 = N_1 - N_2, \tag{2}$$

to satisfy the basic commutation relations

$$[J_+, J_-] = [J_3], \quad [J_3, J_\pm] = \pm 2J_\pm \tag{3}$$

of the quantum algebra $\mathrm{sl}_q(2)$. it is worth pointing out that in ref. [6] only on the q-deformed Fock space $\mathscr{F}_q(2)$,

$$\{|m_1, m_2\rangle = a_2^{+m_2}|0\rangle \,|\, a_1|0\rangle = a_2|0\rangle = 0\,,$$
$$m_1, m_2 \in \mathbb{Z}^+ = \{0, 1, 2, ...\}\}\,,$$

relation (3) is satisfied, but in this paper (3) becomes an algebraic relation because of (1), in other words, it is satisfied on any representation space of $\mathscr{B}_q$.

Now, using the relation $[N]a^{\pm} = a^{\pm}[N\pm 1]$ which results from (1), we can easily prove that

$$J_+ = a^+[\lambda - N]\,, \quad J_- = a\,, \quad J_3 = 2N - \lambda \tag{4}$$

is a new boson realization of $\mathrm{sl}_q(2)$, i.e., (4) satisfies (3). Here $\lambda\in\mathbb{C}$ is an arbitrary complex parameter. It will be seen that the introduction of λ is a key to the whole discussion below. On the one-state q-deformed Fock space $\mathscr{F}_q$,

$$\{f(m) = |m\rangle = a^{+m}|0\rangle \,|\, a|0\rangle = N|0\rangle = 0\,, m\in\mathbb{Z}^+\}\,,$$

realization (4) gives an infinite dimensional representation of $\mathrm{sl}_q(2)$,

$$J_+ f(m) = [\lambda - m] f(m+1)\,,$$
$$J_- f(m) = [m] f(m)\,,$$
$$J_3 f(m) = (2m - \lambda) f(m)\,. \tag{5}$$

When λ is a non-negative integer, representation (5) is equivalent to that constructed from realization (2). In fact, when $\lambda\in\mathbb{Z}^+$, if we define

$$f'(m_1, m_2) = f(m_1)$$
$$(m_2 = \lambda - m_1,\ m_1, m_2, \lambda\in\mathbb{Z}^+)\,,$$

then we have

$$J_+ f'(m_1, m_2) = [m_2] f'(m_1+1, m_2-1)\,,$$
$$J_- f'(m_1, m_2) = [m_1] f'(m_1-1, m_2+1)\,,$$
$$J_3 f'_1(m_1, m'_2) = (m_1 - m_2) f'(m_1, m_2)\,. \tag{6}$$

This is nothing but the representation on $\mathscr{F}_q(2)$ given by realization (2), which has been discussed in detail in ref. [5].

When λ is not an integer and q is not a root of unity, all the results are the q-deformation of those about the Lie algebra su(2), so we will not consider this case. Instead, we will investigate the non-generic case that λ is not an integer and q is a root of unity [8]. In this case, $q^{p'} = 1$ for p' a positive integer larger than two. Because when $p' = 2p$ ($p\in\mathbb{Z}^+$) $q^{p'} = 1$ means $q^p = \pm 1$, in the non-generic case we can assume $q^p = 1$ for p an odd integer larger than one or $q^p = -1$ for p an even integer larger than one without losing generality.

When $q^p = \pm 1$, $[\alpha p] = 0$ for $\alpha\in\mathbb{Z}^+$. As a result, $J_- f(\alpha p) = 0$. So the extreme vector $f(\alpha p)$ defines an invariant subspace V_α,

$$\{f(\alpha p + m) \,|\, m\in\mathbb{Z}^+\}\,.$$

Since there is not an $\mathrm{sl}_q(2)$-invariant subspace $\bar{\mathrm{V}}_\alpha$ complementary to V_α, representation (5) is reducible but not completely reducible, i.e., indecomposable.

On the quotient space $\mathrm{Q}^{[J]} = \mathscr{F}_q/\mathrm{V}_\alpha$,

$$\{F(M) = f(m) \bmod \mathrm{V}_\alpha \,|\, m = 0, 1, 2, ..., \alpha p - 1 = 2J\}\,,$$

representation (5) induces a finite dimensional representation $\mathscr{J}^{[J]}$,

$$J_+\psi_J(M) = [\lambda - (J+M)]\psi_J(M+1)\theta(J-1-M)\,,$$
$$J_-\psi_J(M) = [J+M]\psi_J(M-1)\,,$$
$$J_3\psi_J(M) = (2(J+M) - \lambda)\psi_J(M)\,, \tag{7}$$

with the dimension $\dim \mathrm{Q}^{[J]} = 2J+1$ where we have defined $\psi_J(M) = F(J+M)$ ($M = J, J-1, ..., -J$) and

$$\theta(x) = 1\,, \quad x \geqslant 0\,,$$
$$= 0\,, \quad x < 0\,.$$

For $\alpha\geqslant 2$, this representation is also indecomposable.

Using representation (7), through the $\mathrm{sl}_q(2)$-universal R-matrix

$$\mathscr{R} = q^{J_3\otimes J_3/2} \sum_{n=0}^{\infty} \frac{(1-q^{-2})^n}{[n]!} \times q^{n(n-1)/2} (q^{J_3/2}J_+ \otimes q^{-J_3/2}J_-)^n\,, \tag{8}$$

we can now construct the R-matrix

$$R^{J_1J_2} = \mathscr{J}^{[J_1]}\otimes\mathscr{J}^{[J_2]}(\mathscr{R}) \in \mathrm{End}(\mathrm{Q}^{[J_1]}\otimes\mathrm{Q}^{[J_2]})$$

in an explicit form,

$$R^{J_1J_2}\psi_{J_1}(M_1)\otimes\psi_{J_2}(M_2)$$
$$= \sum (R^{J_1J_2})^{M'_1M'_2}_{M_1M_2} \psi_{J_1}(M'_1)\otimes\psi_{J_2}(M'_2)\,,$$

where we have chosen $\psi_{J_1}(M_1)\otimes\psi_{J_2}(M_2)$ as the basis of $\mathrm{Q}^{[J_1]}\otimes\mathrm{Q}^{[J_2]}$ as in (7) and $(R^{J_1J_2})^{M'_1M'_2}_{M_1M_2}$ can be written as

$$(R^{J_1J_2})^{M'_1M'_2}_{M_1M_2}=q^{(2J_1+2M'_1-\lambda)(2J_2+2M'_2-\lambda)/2}\delta^{M'_1}_{M_1}\delta^{M'_2}_{M_2}$$
$$+q^{(2J_1+2M'_1-\lambda)(2J_2+2M'_2-\lambda)/2}$$
$$\times\sum_{n=1}^{\infty}\frac{(1-q^{-2})^n}{[n]!}q^{-n(n-1)/2}q^{n(J_1-J_2+M'_1-M'_2)}/2$$
$$\times\prod_{l=1}^{n}[\lambda-J_1-M'_1+l][J_2+M'_2+l]\delta^{M'_1}_{M_1+n}\delta^{M'_2}_{M_2-n} \tag{9}$$

after a straightforward calculation.

Before concluding this rather short paper let us consider two examples. Taking $q^2=-1$, $\alpha=1$ and $J_1=J_2=1/2$, from eq. (9) we get a 4×4 R-matrix

$$R^{1/2,1/2}=q^{\lambda^2/2-\lambda}\begin{pmatrix} t & & & \\ & 1 & t-t^{-1} & \\ & & 1 & \\ & & & -t^{-1}\end{pmatrix}, \tag{10}$$

where $t=-q^{-\lambda}$ is an arbitrary parameter. If we choose $q^3=-1$, $\alpha=1$ and $J_1=J_2=1$, then (9) turns out to be a 9×9 R-matrix

$$R^{11}=\mathrm{diag}(A_1,A_2,A_3,A'_2,A'_1)\,. \tag{11}$$

Here,

$$A_1=-q^{-1}q^{\lambda^2/2-4\lambda}\,,\quad A'_1=q^{\lambda^2/2}=A_1q^2t^4\,,$$

$$A_2=A_1\begin{pmatrix}-t & 1-t^2\\ & -t\end{pmatrix},$$

$$A'_2=A_1\begin{pmatrix}qt^3 & t^2(q^2t^2-1)\\ & qt^3\end{pmatrix},$$

$$A_3=A_1\begin{pmatrix}t^2 & qt(t^2-1) & (t^2-1)(q^2t^2-1)\\ & q^2t^2 & t(q^2t^2-1)\\ & & t^2\end{pmatrix},$$

where $t=q^{\lambda-1}$. Using the extended Kauffman diagram technique, one finds no difficulty in checking that the general R-matrix given by (9), as well as its two examples (10) and (11), indeed satisfies the Yang–Baxter equation without a spectral parameter. We have seen that from our approach we can get the non-standard R-matrices, namely (10) and (11), in a very natural way. Moreover, we can obtain many new R-matrices in a similar way.

Finally, we point out that the R-matrices given by (9) can be Yang–Baxterized into a solution of the Yang–Baxter equation with a spectral parameter [9].

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