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Cyclic boson algebra and q-boson realizations of cyclic representations of the quantum algebra $sl_q(3)$

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Abstract. In this paper we introduce the concept of cyclic boson algebra and study its representations. Using this algebra to realize $sl_q(3)$, we construct the cyclic representation of the quantum universal enveloping algebra $U_q(sl(3)) \equiv sl_q(3)$ on the q-Fock space. Restricting this representation to the subalgebra $sl_q(2)$ ($\subset sl_q(3)$), we naturally obtain the irreducible cyclic representations of $sl_q(2)$.

1. Introduction

Quantum group, quantum algebra (q-analogue of a universal enveloping algebra) and their representation theory play a crucial role in the construction of solutions (R-matrices) to the quantum Yang–Baxter equation (QYBE) [1–5]. Recently, the representations of quantum algebras at roots of unity have attracted much attention in both the mathematical field [6–8] and the physical field [9–13]. Concini and Kac, especially, have made a systematical study on the representation theory of quantum algebras in the case that q is a root of unity, and Date, Jimbo, Mike and Miwa, motivated by the problems in the Potts model, have given some concrete cyclic representations studied theoretically by Concini and Kac.

The aim of this paper is to try to establish a procedure to construct explicitly the cyclic representations of a quantum algebra through the q-deformed boson realization [14–19]. The so-called realization theory originated from the Jordan–Schwinger mapping of Lie algebras [18] and was later generalized to associative algebras, including quantum algebras [14–17]. To get a clear picture of the realization method, let us review it. Suppose A and S are two associative algebras over the complex number field $\mathbb{C}$. If there exists a homomorphic mapping $\varphi: A \to S$ such that the image $\varphi(A)$ is a subalgebra of S, then $\varphi(A)$ is called an S-realization of A. In fact, $\varphi(A)$ defines an operator representation of A. As a result, a representation of S naturally subduces a representation of $A \simeq \varphi(A) \subset S$. We call this subduced representation an S-realization of the representation of A. In a practical problem, S is always chosen to be 'simpler' than A, by which we mean that it is easier to obtain the representations of S than to obtain those of A. About this realization method there are the following three cases worth mentioning:

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(i) A is a Lie algebra; S is the Heisenberg–Weyl algebra generated by the creation operators and annihilation operators or the differential algebra generated by the operators Z and $\mathrm{d}/\mathrm{d}Z$ on the Bargmann space [19–21]. Correspondingly, $\varphi(A)$ is called a boson realization or a differential realization.

(ii) A is a quantum algebra; S is the q-deformed boson algebra or the differential algebra. In this case, $\varphi(A)$ is called a q-deformed boson (or oscillator) realization and a differential realization respectively [9, 22].

(iii) A is a quantum algebra; S is an associative algebra generated by X, Z and 1 satisfying $XZ = qZX$ and $Z^N = X^N = 1$. So far as we know, this realization first appears in [23], where it is used to construct cyclic representations of some quantum algebras.

In this paper, we will introduce an associative algebra called cyclic boson algebra and choose it as the above-mentioned S. Thus, the realization to be obtained is what we call cyclic boson realization, which, as will be seen, is the key to all our discussion. This paper is constructed as follows. In section 2 we give the definition of the cyclic boson algebra and construct its representations; in section 3, some realizations of the quantum algebra $\mathrm{sl}_q(3)$ and its subalgebra $\mathrm{sl}_q(2)$ are listed; in section 4, the cyclic representation of $\mathrm{sl}_q(3)$ is constructed on the cyclic Fock space; in section 5, we discuss the representation of $\mathrm{sl}_q(2)$ resulting from that of $\mathrm{sl}_q(2)$ as a subalgebra.

2. Cyclic boson algebra and its representations

Let us first recall the definition of the q-deformed boson algebra B_q [15–18]. As is known, it is defined as an associative algebra over $\mathbb{C}$ generated by the q-deformed boson operators a_i^+, $a_i^- \equiv a_i$ and $Q_i^\pm = q^{\pm N_i}$ satisfying the relations

$$
\begin{aligned}
&a_i a_j^+ - q^{\pm\delta_{ij}} a_j^+ a_i = \delta_{ij} Q_j^\pm \qquad && [N_i, N_j] = 0 \\
&[N_i, a_j^\pm] = \pm\delta_{ij} a_j^\pm && i = 1, 2, \ldots, l
\end{aligned}
\tag{2.1}
$$

where $q \in \mathbb{C}$. Now, we have:

Proposition 1. If q is a primitive pth root of unity, i.e. $q^p = 1$, the elements $a_i^{\pm p}$ and $Q_i^{\pm p}$ $(i = 1, 2, \ldots, l)$ belong to the centre of B_q.

Proof. The proof follows from the equations

$$
\begin{aligned}
&a_i a_i^{+n} = [n] a_i^{n-1} Q_i + q^{-n} a_i^{+n} a_i \\
&a_i^\pm Q_i = q^{\mp 1} Q_i a_i^\pm \qquad && a_i^\pm Q_i^- = q_i^{\pm 1} Q_i^- a_i^\pm
\end{aligned}
\tag{2.2}
$$

where $[n] \doteq (q^n - q^{-n})/(q - q^{-1})$. Using equation (2.1), one can easily prove them by induction. □

Since $(a_i^\pm)^p$ are central elements of B_q, we can restrict them to be constants without causing contradictions in the algebraic structure.

Definition 1. A cyclic boson algebra is an associative algebra generated by $a_i^\pm$ and $Q_i^\pm$, which satisfy equation (2.1) and

$$
(a_i^\pm)^p = \xi_{i\pm} \in \mathbb{C}. \tag{2.3}
$$

Having given the definition, we now turn to consider its representation. In the following we denote the cyclic boson algebra by $B_c(l)$. Let

$$v_\mu = v(\mu_1, \mu_2, \ldots, \mu_l) \qquad (\mu_i \in \mathbb{C}, i = 1, 2, \ldots, l)$$

be such a common eigenstate of Q_i $(i = 1, 2, \ldots, l)$ that

$$Q_i v_\mu = q^{\mu_i} v_\mu. \tag{2.4}$$

As a result of equation (2.4) we call it a cyclic vacuum state.

Definition 2. The cyclic Fock space $\mathscr{F}_c(l)$ is a span of the linear-independent states

$$F\langle m_i\rangle = F(m_1, m_2, \ldots, m_l) \equiv a_1^{+m_1} a_2^{+m_2} \ldots a_l^{+m_l} v_\mu$$
$$m_i \in \{0, 1, 2, \ldots, p-1\} \qquad i = 1, 2, \ldots, l.$$

From the definition one can easily write down the action of $B_c(l)$ on $\mathscr{F}_c(l)$

$$\begin{aligned}
&a_j^+ F\langle m_i\rangle = F\langle m_i + \delta_{ij}\rangle \qquad 0 \leqslant m_j \leqslant p-2 \\
&a_j^+ F\langle m_i\rangle|_{m_j=p-1} = \xi_{j+} F\langle m_i\rangle|_{m_j=0} \\
&a_j F\langle m_i\rangle = [m_j + \mu_j] F\langle m_i - \delta_{ij}\rangle \qquad 1 \leqslant m_j \leqslant p-1 \\
&a_j F\langle m_i\rangle|_{m_j=0} = [\mu_j](\xi_{j+})^{-1} F\langle m_i\rangle|_{m_j=p-1} \\
&Q_j^{\pm} F\langle m_i\rangle = q^{\pm(m_j+\mu_j)} F\langle m_i\rangle.
\end{aligned} \tag{2.5}$$

Proposition 2. If the parameters μ_j satisfy

$$[\mu_j][\mu_j+1][\mu_j+2]\ldots[\mu_j+p-1] = \xi_{i+} \cdot \xi_{i-}$$

then equation (2.5) defines a p^l-dimensional irreducible representation ρ: $B_c(l) \to \mathrm{End}(\mathscr{F}_c(l))$ of $B_c(l)$.

Proof. For $l = 1$, we denote $\xi_{1\pm}$, $Q_1^{\pm}$, $a_1^{\pm}$ and $F\langle m_1\rangle$ by $\xi_\pm$, $Q^\pm$, $a^\pm$ and $F(m)$ respectively. Then, we rewrite equation (2.5) as

$$\begin{cases} a^+ F(m) = F(m+1) \qquad 0 \leqslant m \leqslant p-2 \\ a^+ F(p-1) = \xi_+ F(0) \end{cases} \tag{2.6a}$$

$$\begin{cases} aF(m) = [m+\mu] F(m-1) \qquad 1 \leqslant m \leqslant p-1 \\ aF(0) = [\mu](\xi_+)^{-1} F(p-1) \end{cases} \tag{2.6b}$$

$$Q^\pm F(m) = q^{\pm(m+\mu)} F(m). \tag{2.6c}$$

It follows from (2.6*b*) that

$$a^p F(0) = (\xi_+)^{-1} [\mu][\mu+1]\ldots[\mu+p-1] F(0). \tag{2.7}$$

On the other hand, from equation (2.3) one has

$$a^p F(0) = \xi_- F(0). \tag{2.8}$$

Comparing (2.7) with (2.8), we get

$$\xi_+ \cdot \xi_- = [\mu][\mu+1]\ldots[\mu+p-1]. \tag{2.9}$$

For an arbitrary l, the proof is the same. It is easy to check that when equations (2.6*a*–*c*) are satisfied all the relations in (2.2) and (2.3) will be kept on the cyclic Fock space $\mathscr{F}_c(l)$. In other words, (2.5) defines a representation of $B_c(l)$. The dimension and irreducibility of this representation follow from (2.5) directly. □

3. The cyclic boson realization of $\mathrm{sl}_q(3)$ and $\mathrm{sl}_q(2)$

The cyclic boson realization $\varphi(\mathrm{sl}_q(k))=\{\hat{g}=g(g)\,|\,g\in \mathrm{sl}_q(k)\}$ is determined by a homomorphic mapping $\varphi:\mathrm{sl}_q(k)\rightarrow B_c(l)$. In this section, we consider $\mathrm{sl}_q(2)$ and $\mathrm{sl}_q(3)$. We would like to point out that for a quantum algebra, there may exist many different realizations.

Proposition 3. For the quantum algebra $\mathrm{sl}_q(2)$, which is generated by $J_\pm$ and $K^\pm=q^{\pm J_3}$ satisfying

$$[J_+J_-]=[J_3] \qquad [J_3,J_\pm]=\pm 2J_\pm \tag{3.1}$$

the following mappings define three cyclic boson realizations:

(i) $\varphi:\mathrm{sl}_q(2)\rightarrow B_c(1)$:

$$\hat{J}_+=a\dagger \qquad \hat{J}_-=a[\lambda+1-N] \qquad \hat{K}^\pm=Q^{\pm 2}q^{\mp\lambda} \tag{3.2}$$

(ii) $\varphi:\mathrm{sl}_q(2)\rightarrow B_c(2)$:

$$\hat{J}_+=a_1^\dagger a_2 \qquad \hat{J}_-=a_2^\dagger a_1 \qquad \hat{K}^\pm=Q_1^{\mp 1}Q_2^{\pm 2} \tag{3.3}$$

(iii) $\varphi:\mathrm{sl}_q(2)\rightarrow B_c(2)$:

$$\hat{J}_+=a_1^\dagger \qquad \hat{J}_-=a_2^\dagger+a_1[2N_2-N_1+\lambda] \qquad \hat{K}^\pm=Q_1^{\pm 2}Q_2^{\mp 2}q^{\mp\lambda}. \tag{3.4}$$

Proof. The proof follows from the observation that $a_ia_i^\dagger=[N_i+1]$, $a_i^\dagger a_i=[N_i]$ and $a_i^\pm a_j^\pm=a_j^\pm a_i^\pm$ $(i\neq j)$, which is equivalent to (2.1). □

For the quantum algebra $\mathrm{sl}_q(3)$ generated by E_i, F_i and $K_i^\pm$ $(i=1,2)$ satisfying

$$[E_i,F_j]=\delta_{ij}\frac{K_i-K_i^{-1}}{q-q^{-1}}$$
$$K_i^\pm E_j=q^{\pm\alpha_{ij}}E_jK_i^\pm \qquad K_i^\pm F_j=q^{\mp\alpha_{ij}}F_jK_i^\pm \tag{3.5a}$$

$$G_i^2G_j-(q+q^{-1})G_iG_jG_i+G_jG_i^2=0$$
$$i\neq j \qquad i,j=1,2 \qquad G_i=E_i \text{ or } F_i \tag{3.5b}$$

we have the following proposition.

Proposition 4. There exists a realization of $\mathrm{sl}_q(3)$ determined by φ: $\mathrm{sl}_q(3)\rightarrow B_c(3)$

$$\hat{E}_1=a_1^\dagger$$
$$\hat{F}_1=q^{-\lambda_1}a_2^\dagger a_3-a_1[N_1-N_2+N_3-1-\lambda_1] \tag{3.6a}$$
$$K_1^\pm=q^{\mp\lambda_1}a_1^{\pm 2}Q_2^{\mp 1}Q_3^{\pm 1}$$

$$\hat{E}_2=a_2^\dagger Q_1^{-1}-q^{-1}Q_2^{-1}a_1a_3^\dagger$$
$$\hat{F}_2=Q_3a_2[1+\lambda_2-N_2]-q^{1+\lambda_2}Q_2^{-1}a_1^\dagger a_3 \tag{3.6b}$$
$$K_2^\pm=q^{\mp\lambda_2}Q_2^{\pm 2}Q_1^{\mp 1}Q_3^{\pm 1}$$

where λ_1 and λ_2 are complex parameters.

Proof. The proof follows from direct calculation. □

It is worth pointing out that on the usual Fock spaces $\mathscr{F}_1$

$$\{|m\rangle = a^{+m}|0\rangle \,|\, a|0\rangle = 0,\ Q|0\rangle = 0\}$$

and $\mathscr{F}_2$

$$\{|m_1 m_2\rangle = a_1^{+m_1} a_2^{+m_2}|0\rangle \,|\, a_1|0\rangle = a_2|0\rangle = 0,\ Q_1|0\rangle = Q_2|0\rangle = 0\}$$

where $a^{\pm}$ and $a_i^{\pm}$ $(i=1,2)$ only satisfy equation (2.1), the realizations (3.2) and (3.6) define the so-called Verma representation of $\mathrm{sl}_q(2)$ and the Verma representation of $\mathrm{sl}_q(3)$ respectively.

4. The cyclic representation of $\mathrm{sl}_q(3)$

We construct the representation of $\mathrm{sl}_q(3)$ from the realization (3.6). The action of $\mathrm{sl}_q(3)$ on $\mathscr{F}_c(3)$ is defined by

$$g \circ F(m_1, m_2, m_3) = \varphi(g) \circ a_1^{+m_1} a_2^{+m_2} a_3^{+m_3} v_\mu \qquad g \in \{F_i, F_i, Q_i^{\pm}\ i=1,2\}.$$

Proposition 5. The following equations define a p^3-dimensional representation of $\mathrm{sl}_q(3)$:

$$E_1 F(m_1, m_2, m_3) = F(m_1+1, m_2, m_3) \qquad (m_1 \neq p-1)$$

$$E_1 F(p-1, m_2, m_3) = \xi_{1+} F(0, m_2, m_3)$$

$$\begin{aligned} F_1 F(m_1, m_2, m_3) &= q^{-\lambda_1}[m_3+\mu_3] F(m_1, m_2+1, m_3-1) \\ &\quad - [m_1+\mu_1][(m_1+\mu_1)-(m_2+\mu_2)+(m_3+\mu_3)-1-\lambda_1] F(m_1-1, m_2, m_3) \\ &\quad (m_1 \neq 0,\ m_2 \neq p-1,\ m_3 \neq 0) \end{aligned}$$

$$\begin{aligned} F_1 F(m_1, p-1, m_3) &= q^{-\lambda_1}[m_3+\mu_3] \xi_{2+} F(m_1, 0, m_3-1) \\ &\quad - [m_1+\mu_1][m_1+m_3+\mu_1+\mu_3-\lambda_1] F(m_1-1, p-1, m_3) \end{aligned} \tag{4.1a}$$

$$\begin{aligned} F_1 F(0, m_2, m_3) &= q^{-\lambda_1}[m_3+\mu_3] F(0, m_2+1, m_3-1) \\ &\quad - \frac{[\mu_1]}{\xi_{1+}}[\mu_1-(m_2+\mu_2)+(m_3+\mu_3)-1-\lambda_1] F(p-1, m_2, m_3) \end{aligned}$$

$$\begin{aligned} F_1 F(m_1, m_2, 0) &= q^{-\lambda_1}\frac{[\mu_3]}{\xi_{3+}} F(m_1, m_2+1, p-1) \\ &\quad - [m_1+\mu_1][m_1+\mu_1-(m_2+\mu_2)+\mu_3-1-\lambda_1] F(m_1-1, m_2, 0) \end{aligned}$$

$$K_1^{\pm} F(m_1, m_2, m_3) = q^{\pm(2(m_1+\mu_1)-(m_2+\mu_2)+(m_3+\mu_3)-\lambda_1)} F(m_1, m_2, m_3).$$

$$E_2F(m_1, m_2, m_3)$$
$$= q^{-(m_1+\mu_1)}F(m_1, m_2+1, m_3) - q^{-(m_2+\mu_2)-1}[m_1+\mu_1]F(m_1-1, m_2, m_3+1)$$
$$(m_1 \neq 0, m_2 \neq p-1, m_3 \neq p-1)$$

$$E_2F(0, m_2, m_3) = q^{-\mu_1}F(0, m_2+1, m_3) - \frac{[\mu_1]}{\xi_{1+}} q^{-(m_2+\mu_2+1)}F(p-1, m_2, m_3+1)$$

$$E_2F(m_1, p-1, m_3) = q^{-(m_1+\mu_1)}\xi_{2+}F(m_1, 0, m_3) - q^{-\mu_2}[m_1+\mu_1]F(m_1-1, p-1, m_3+1)$$

$$E_2F(m_1, m_2, p-1)$$
$$= q^{-(m_1+\mu_1)}F(m_1, m_2+1, p-1) - q^{-(m_2+\mu_2+1)}[m_1+\mu_1]\xi_{3+}F(m_1-1, m_2, 0)$$

$$F_2F(m_1, m_2, m_3)$$
$$= q^{m_3+\mu_3}[1+\lambda_2-(m_2+\mu_2)][m_2+\mu_2]F(m_1, m_2-1, m_3)$$
$$- q^{1+\lambda_2-(m_2+\mu_2)}[m_3+\mu_3]F(m_1+1, m_2, m_3-1)$$
$$(m_2 \neq 0, m_1 \neq p-1, m_3 \neq 0) \tag{4.1b}$$

$$F_2F(p-1, m_2, m_3)$$
$$= q^{m_3+\mu_3}[1+\lambda_2-(m_2+\mu_2)][m_2+\mu_2]F(p-1, m_2-1, m_3)$$
$$- q^{1+\lambda_2-(m_2+\mu_2)}[m_3+\mu_3]\xi_{1+}F(0, m_2, m_3-1)$$

$$F_2F(m_1, 0, m_3)$$
$$= q^{m_3+\mu_3}[1+\lambda_2-(m_2+\mu_2)]\frac{[\mu_2]}{\xi_{2+}}F(m_1, p-1, m_3)$$
$$- q^{1+\lambda_2-\mu_2}[m_3+\mu_3]F(m_1+1, 0, m_3-1)$$

$$F_2F(m_1, m_2, 0)$$
$$= q^{\mu_3}[\lambda_2-(m_2+\mu_2)+1][m_2+\mu_2]F(m_1, m_2-1, 0)$$
$$- q^{1+\lambda_2-(m_2+\mu_2)}\frac{[\mu_3]}{\xi_{3+}}F(m_1+1, m_2, p-1)$$

$$K_2^{\pm}F(m_1, m_2, m_3) = q^{\pm(2(m_2+\mu_2)-(m_1+\mu_1)+(m_3+\mu_3)-\lambda_2)}F(m_1, m_2, m_3).$$

Proof. It is proved through lengthy but straightforward calculation. □

Remark 1. When μ_i, $\xi_{i+} \neq 0$ $(i=1,2)$, the representation (4.1) has neither the lowest nor the highest weight, and E_i^p and F_i^p are non-zero constants in this representation. So it is an irreducible cyclic representation. In fact, according to Kac and Concini, the dimension D of an irreducible representation of a quantum algebra G_q satisfies

$$D \leqslant p^m \qquad m = (\dim G - \operatorname{rank} G)/2 \tag{4.2}$$

where dim G and rank G are respectively the dimension and the rank of the Lie algebra G corresponding to the quantum algebra. For $\mathrm{sl}_q(3)$, dim $G=8$, rank $=2$ and $m=3$. Thus, the representation (4.1) is an irreducible cyclic representation with the maximum dimension.

Remark 2. If $[\mu_i]=0$ $(i=1,2)$, (4.1) determines a representation with the lowest weight, which can be obtained on the cyclic Verma module

$$\{f(m_1, m_2, m_3)=E_1^{m_1}E_2^{m_2}E_3^{m_3}v_\lambda \mid m_1, m_2, m_3 \in \mathbb{Z}^+\}$$

where $E_i^p=\xi_{i+}$ $(i=1,2,3)$, $F_1v_\lambda=f_2v_\lambda=0$, $K_i^{\pm}v_\lambda=q^{\mp\lambda_i}v_\lambda$ $(i=1,2)$ and $E_3=E_1E_2-qE_2E_1$ is the generator corresponding to the third root of Lie algebra A_2.

5. The boson representation of $\mathrm{sl}_q(2)$

Since $\mathrm{sl}_q(2)$ is a subalgebra of $\mathrm{sl}_q(3)$, the restriction of the representation (4.1) to it naturally defines a representation of it. This representation is given by (4.1*a*). According to Concini and Kac (see equation (4.2)), this p^3-dimensional representation of $\mathrm{sl}_q(2)$ is reducible. We are trying to find an irreducible cyclic representation from it.

In (4.1*a*), we let $\mu_i=0$ $(i=2,3)$. Then we obtain the representation of $\mathrm{sl}_q(2)$:

$$E_1F(m_1, m_2, m_3)=F(m_1+1, m_2, m_3) \qquad (m_1\neq p-1)$$

$$E_1F(p-1, m_2, m_3)=\xi_{1+}F(0, m_2, m_3)$$

$$\begin{aligned} F_1F(m_1, m_2, m_3) &= q^{-\lambda_1}[m_3]F(m_1, m_2+1, m_3-1)-[m_1+\mu_1][(m_1+\mu_1)-m_2+m_3-1-\lambda_1] \\ &\quad \times F(m_1-1, m_2, m_3) \qquad (m\neq 0,\ m_2\neq p-1,\ m_3\neq 0) \end{aligned}$$

$$F_1F(m_1, p-1, m_3)=-[m_1+\mu_1][m_1+m_3+\mu_1-\lambda_1]F(m_1-1, p-1, m_3) \tag{5.1}$$

$$\begin{aligned} F_1F(0, m_2, m_3) &= q^{-\lambda_1}[m_3]F(0, m_2+1, m_3-1) \\ &\quad -\frac{[\mu_1]}{\xi_{1+}}[\mu_1-m_2+m_3-1-\lambda_1]F(p-1, m_2, m_3) \end{aligned}$$

$$F_1F(m_1, m_2, 0)=-[m_1+\mu_1][m_1+\mu_1-m_2-1-\lambda_1]F(m_1-1, m_2, 0]$$

$$K_1^{\pm}F(m_1, m_2, m_3)=q^{\pm(2(m_1+\mu_1)-m_2+m_3-\lambda_1)}F(m_1, m_2, m_3).$$

We note that V_c

$$\{F(m,0,0)\equiv F(m) \mid m=0,1,2,\ldots,p-1\}$$

is a p-dimensional invariant subspace for the representation defined by (5.1), so we have:

Proposition 6. On the space V_c, the representation (5.1) subduces an irreducible cyclic representation of $\mathrm{sl}_q(2)$

$$E_1F(m)=F(m+1) \qquad (m\neq p-1)$$

$$E_1F(p-1)=\xi_{1+}F(0)$$

$$F_1F(m)=[m+\mu_1][1+\lambda_1-(m+\mu_1)]\cdot F(m-1) \qquad (m\neq 0)$$

$$F_1F(0)=[\lambda_1+1-\mu_1]\cdot\frac{[\mu_1]}{\xi_{1+}}F(p-1)$$

$$K_1^{\pm}F(m)=q^{\pm(2(m+\mu_1)-\lambda_1)}F(m).$$

Remark. When $\mu=0$, this representation can be directly obtained on the cyclic Verma module

$$\{f(m)=E_1^m v_{\lambda_1} \mid m\in\mathbb{Z}^+\}$$

where $E_1^p=\xi_{1+}$, $F_1 v_{\lambda_1}=0$ and $K_1^{\pm} v_{\lambda_1}=q^{\mp\lambda_1} v_{\lambda_1}$.

6. Short discussion

We have seen that the q-deformed boson realization is indeed a powerful method of constructing representations of a quantum algebra. In fact, not only can it greatly simplify the calculation made to obtain the explicit representations of a quantum algebra, but it can also stimulate one's imagination: the fact that the special cyclic boson (with $\xi_{i\pm}=0$ and $p=2$) satisfies $a_i^{\pm 2}=0$ may lead one to make a guess at the relation between a general cyclic boson and an anyon. Finally we point out that the method discussed in this paper can be generalized to other quantum algebras in a straightforward way.

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