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# General solutions of $R^{j\frac{1}{2}}(x)$ and quantum group structure

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**Abstract.** A general form of solutions of braid relations associated with $V^j \otimes V^{1/2}$ for Su(2) is calculated. A Yang–Baxterization prescription is presented to generate the corresponding solutions of Yang–Baxter equations. We show that this general form of solutions gives rise to a new construction of quantum group structure even though the $q$-commutation relations are kept standard.

## 1. Introduction

It is known that for spin models the solutions of braid relations

$$S_{12}^{j_1j_2}S_{23}^{j_1j_3}S_{12}^{j_2j_3} = S_{23}^{j_2j_3}S_{12}^{j_1j_3}S_{23}^{j_1j_2} \tag{1.1}$$

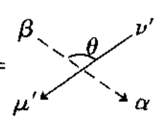
can best be explicitly constructed by the quantum groups (QA) through the standard theory of Drinfeld and Reshetikhin [1–3], where $j_1, j_2$ and $j_3$ are the weight labels associated with the corresponding spin spaces. In the following, for convenience, we will call the solutions that can be obtained by the QG (QG) in the usual way the 'standard' ones, which can also be worked with the help of the fusion rule [4, 5].

In comparison to the statistical model for a vertex model the Yang–Baxter equation (YBE) can be written in the form [6, 7]

$$\sum_{\gamma,\mu''\nu''} W(\mu, \alpha\,|\,\gamma, \mu')\,W'(\nu, \delta\,|\,\beta, \nu'')\,W''(\nu, \mu''\,|\,\nu', \mu')$$
$$= \sum_{\gamma,\mu'',\nu''} W''(\nu, \mu\,|\,\nu'', \mu'')\,W'(\mu'', \alpha\,|\,\gamma, \mu')\,W(\nu'', \gamma\,|\,\beta, \nu') \tag{1.2}$$

where $W$, $W'$ and $W''$ are Boltzmann weights. It is well known that if equation (1.2) can be additively parametrized, then one can introduce

$$T_{\mu'\alpha}^{\beta\nu'}(x) = W(\mu',\alpha\,|\,\beta, \nu') = \tag{1.3}$$




where $x = \exp(-\mathrm{i}\theta) = \exp(-u)$ and matrices $T$ satisfy

$$\sum_{\gamma,\mu'',\nu''} T_{\nu''\gamma}^{\beta\nu'}(x)\,T'^{\gamma\mu'}_{\mu''\alpha}(xy)\,T''^{\nu''\mu''}_{\nu\mu}(y) = \sum_{\gamma,\mu'',\nu''} T''^{\nu'\mu'}_{\nu''\mu''}(y)\,T'^{\beta\nu''}_{\nu\gamma}(xy)\,T^{\gamma\mu''}_{\mu d}(x) \tag{1.4}$$

which by redefining

$$(\check{R}^{j_1j_2}(x))^{c(j_2)d(j_1)}_{a(j_1)b(j_2)} = T^{\alpha\alpha'}_{\beta'\beta}(x) \tag{1.5}$$

is referred to as the incomplete YBE:

$$\check{R}_{12}^{j_1j_2}(x)\check{R}_{23}^{j_1j_3}(xy)\check{R}_{12}^{j_2j_3}(y) = \check{R}_{23}^{j_2j_3}(y)\check{R}_{12}^{j_1j_3}(xy)\check{R}_{23}^{j_1j_2}(x). \tag{1.6}$$

Noting that by considering the asymptotic behaviour

$$S^{j_1 j_2} = \lim_{x \to 0} \check{R}^{j_1 j_2}(x) \tag{1.7}$$

the YBE is reduced to the braid relation (1.1).

If we consider the model shown in figure 1, where the full lines denote spin $j$ and the broken lines denote spin $\frac{1}{2}$, then equation (1.1) is reduced to

$$\begin{aligned} S^{j\frac{1}{2}}_{12} S^{j\frac{1}{2}}_{23} S^{\frac{1}{2}\frac{1}{2}}_{12} &= S^{\frac{1}{2}\frac{1}{2}}_{23} S^{j\frac{1}{2}}_{12} S^{j\frac{1}{2}}_{23} \\ S^{\frac{1}{2}\frac{1}{2}}_{12} S^{\frac{1}{2}j}_{23} S^{\frac{1}{2}j}_{12} &= S^{\frac{1}{2}j}_{23} S^{\frac{1}{2}j}_{12} S^{\frac{1}{2}\frac{1}{2}}_{23} \\ S^{\frac{1}{2}j}_{12} S^{\frac{1}{2}\frac{1}{2}}_{23} S^{j\frac{1}{2}}_{12} &= S^{j\frac{1}{2}}_{23} S^{\frac{1}{2}\frac{1}{2}}_{12} S^{\frac{1}{2}j}_{23} \end{aligned} \tag{1.8}$$

which satisfies the graph in figure 2. As usual the CP invariance should be satisfied,

$$(S^{j\frac{1}{2}})^{a(j)b(\frac{1}{2})}_{c(\frac{1}{2})d(j)} = (S^{\frac{1}{2}j})^{-b(\frac{1}{2})-a(j)}_{-d(j)-c(\frac{1}{2})} \tag{1.9}$$

*under which only two equations in equation (1.8) are independent:*

$$\begin{aligned} S^{j\frac{1}{2}}_{12} S^{j\frac{1}{2}}_{23} S^{\frac{1}{2}\frac{1}{2}}_{12} &= S^{\frac{1}{2}\frac{1}{2}}_{23} S^{j\frac{1}{2}}_{12} S^{j\frac{1}{2}}_{23} \\ S^{\frac{1}{2}j}_{12} S^{\frac{1}{2}\frac{1}{2}}_{23} S^{j\frac{1}{2}}_{12} &= S^{j\frac{1}{2}}_{23} S^{\frac{1}{2}\frac{1}{2}}_{12} S^{\frac{1}{2}j}_{23}. \end{aligned} \tag{1.10}$$

Hence, solving the model in figure 1 consists of solving equation (1.10). So far no general solutions of equation (1.10) have been derived. All of the known solutions are based on the standard construction with usual treatment by QG or fusion rule.

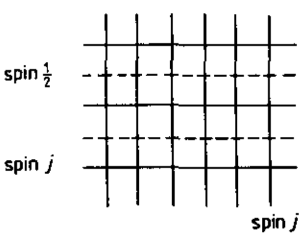



**Figure 1.**

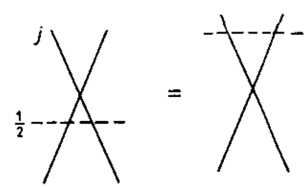



**Figure 2.**

Motivated by the results in [8–10] where the solutions are beyond the standard construction, for the braid relation with the same spin spaces

$$S_{12} S_{23} S_{12} = S_{23} S_{12} S_{23}$$

we want to find a general form of solutions for equation (1.10) which contains the standard ones as special case.

After lengthy computation we reach the following results:

(i) General solutions of equation (1.1) can be explicitly derived.
In the general case they depend on $j+2$ (for $j = 1, 2, \ldots$) or $j+\frac{3}{2}$ (for $j = \frac{3}{2}, \frac{5}{2}, \ldots$) free parameters, respectively, and cannot be related to the standard ones by a similar transformation.

(ii) The Yang–Baxterization prescription [10] can be performed to generate solutions of the YBE [11]:

$$\begin{aligned} \check{R}^{j\frac{1}{2}}_{12}(x) \check{R}^{j\frac{1}{2}}_{23}(xy) \check{R}^{\frac{1}{2}\frac{1}{2}}_{12}(y) &= \check{R}^{\frac{1}{2}\frac{1}{2}}_{23}(y) \check{R}^{j\frac{1}{2}}_{12}(xy) \check{R}^{j\frac{1}{2}}_{23}(x) \\ \check{R}^{\frac{1}{2}\frac{1}{2}}_{12}(x) \check{R}^{\frac{1}{2}j}_{23}(xy) \check{R}^{\frac{1}{2}j}_{12}(y) &= \check{R}^{\frac{1}{2}j}_{23}(y) \check{R}^{\frac{1}{2}j}_{12}(xy) \check{R}^{\frac{1}{2}\frac{1}{2}}_{23}(x) \\ \check{R}^{\frac{1}{2}j}_{12}(x) \check{R}^{\frac{1}{2}\frac{1}{2}}_{23}(xy) \check{R}^{j\frac{1}{2}}_{12}(y) &= \check{R}^{j\frac{1}{2}}_{23}(y) \check{R}^{\frac{1}{2}\frac{1}{2}}_{12}(xy) \check{R}^{\frac{1}{2}}_{23}(x). \end{aligned} \tag{1.11}$$

The procedure of Yang–Baxterization is the consequence of CP invariance and is based on the Reshetikhin approach [2] and cannot be made by simply following [12].

(iii) The FRT approach [13] is used to give QG structure for a given $S^{j_1j_2}$. The general solutions of equation (1.1) give rise to the standard structure of the QG, which is not surprising. However, the centralizer representation (in this paper only $j_2=\frac{1}{2}$ is taken into account)

$$R^{j_1j_2}=P^{j_1j_2}S^{j_1j_2} \tag{1.12}$$

where $P^{j_1j_2}(V^{j_1}\otimes V^{j_2})=V^{j_2}\otimes V^{j_1}$, possess a different structure from the standard one, namely

$$R^{j_1j_2}=R^{j_1j_2}_{\text{standard}}+\Delta^{j_1j_2} \tag{1.13}$$

where $R^{j_1j_2}_{\text{standard}}$ can be expressed in terms of the basis $e_s(v^{j_1})$ and $e^s(v^{j_2})$ in the usual way [1-3, 12]. The additional terms are determined through recurrence relations. In other words, the 'exotic' property occurs in the different way of construction of $R^{j_1j_2}$ with the basis rather than in the commutation relations.

## 2. General solutions for $S^{j\frac{1}{2}}$

We are looking for general solutions satisfying equations (1.8) and (1.9) and the weight conservation

$$a(j)+b(\tfrac{1}{2})=c(\tfrac{1}{2})+d(j). \tag{2.1}$$

Then equations (1.8), (1.9) and (2.1) constrain $S^{\frac{1}{2}\frac{1}{2}}$ and $S^{j\frac{1}{2}}$ to the forms

$$(S^{\frac{1}{2}\frac{1}{2}})^{a(\frac{1}{2})b(\frac{1}{2})}_{c(\frac{1}{2})d(\frac{1}{2})}=q\big|_{a=b=c=d}+\delta^a_d\delta^b_c\big|_{a\neq b}+w\delta^a_c\delta^b_d\big|_{a<b} \tag{2.2}$$

$$(S^{j\frac{1}{2}})^{a(j)b(\frac{1}{2})}_{c(\frac{1}{2})d(j)}=p^{(a(j),b(\frac{1}{2}))}\delta^{a(j)}_{d(j)}\delta^{b(\frac{1}{2})}_{c(\frac{1}{2})}+q^{(a(j),b(\frac{1}{2}))}\delta^{a(j)}_{d(j)-2b(\frac{1}{2})}\delta^{b(\frac{1}{2})}_{-c(\frac{1}{2})} \tag{2.3}$$

where $w=q-q^{-1}$ and the parameters $p^{(a(j),b(\frac{1}{2}))}$ and $q^{(a(j),b(\frac{1}{2}))}$ are to be determined. Obviously, for $S^{\frac{1}{2}\frac{1}{2}}$ only the standard solution is allowed, as shown by equation (2.2).

Substituting equations (2.2) and (2.3) into equation (1.8) and taking equation (1.9) into account we obtain the relations that the unknown parameters must satisfy:

$$\begin{gathered}q^{(a(j),-\frac{1}{2})}=0 \qquad p^{(a(j),\frac{1}{2})}=qp^{(a(j)-1,\frac{1}{2})} \qquad p^{(a(j),-\frac{1}{2})}=q^{-1}p^{(a(j)-1,-\frac{1}{2})}\\ q^{(-a(j),\frac{1}{2})}q^{(a(j)-1,\frac{1}{2})}\big|_{a(j)>-j}-q^{(a(j),\frac{1}{2})}q^{(-a(j)-1,\frac{1}{2})}\big|_{a(j)<j}\\ =q^{-1}w(p^{(a(j),\frac{1}{2})}p^{(-a(j),-\frac{1}{2})}-p^{(-a(j),\frac{1}{2})}p^{(a(j),-\frac{1}{2})}).\end{gathered} \tag{2.4}$$

Without loss of generality, taking

$$p^{(j,\frac{1}{2})}=q \qquad p(-j,-\tfrac{1}{2})=Q \tag{2.5}$$

where $q$ and $Q$ are arbitrary parameters, we then obtain the general solutions of equation (1.8) under equation (1.9):

$$\begin{gathered}p^{(a(j),\frac{1}{2})}=q^{a(j)+1-j} \qquad p^{(a(j),-\frac{1}{2})}=q^{-a(j)-j}Q \qquad q^{(a(j),-\frac{1}{2})}=0\\ q^{(-a(j)-1,\frac{1}{2})}q^{(a(j),\frac{1}{2})}=q^{-2j}Qw^2[j-a(j)]_q[j+a(j)+1]_q \qquad (a(j)\neq j)\\ w=q-q^{-1}\end{gathered} \tag{2.6}$$

where

$$[n]_q=(q^n-q^{-n})/(q-q^{-1}). \tag{2.7}$$

Thus $q^{(-a(j)-1,\frac{1}{2})}$ can be determined if $q^{(a(j),\frac{1}{2})}$ are regarded as free parameters. In the general solutions there will be $j+2$ or $j+\frac{3}{2}$ free parameters for $j=1,2,\ldots$ or $j=\frac{3}{2},\frac{5}{2},\ldots$, respectively.

When

$$Q=q \qquad q^{(-a(j)-1,\frac{1}{2})}=q^{(a(j),\frac{1}{2})} \tag{2.8}$$

the solutions (2.6) are reduced to the standard solutions, under which

$$\begin{aligned} &q^{(a(j),-\frac{1}{2})}=0 \qquad p^{(a(j),\frac{1}{2})}=q^{a(j)+1-j} \\ &p^{(a(j),-\frac{1}{2})}=q^{-a(j)+1-j} \\ &q^{(a(j),\frac{1}{2})}=q^{-j+\frac{1}{2}}w\{[j-a(j)]_q[j+a(j)+1]_q\}^{\frac{1}{2}} \end{aligned} \tag{2.9}$$

which coincide with the results given in [1-3]. Actually, by direct calculation, up to an overall factor $q^{-j+1}$, equation (2.9) leads to the representation of the universal $R$-matrix:

$$\begin{aligned} (R^{\frac{1}{2}j})^{m_1'm_2'}_{m_1m_2} &= \delta^{m_1'+m_2'}_{m_1+m_2}\frac{(1-q)^{m_1'-m_1}}{[m_1'-m_1]_q!}q^{m_1m_2'+m_2m_1'} \\ &\times\left(\frac{[\frac{1}{2}+m_1']_q![\frac{1}{2}-m_1]_q![j-m_2']_q![j+m_2]_q!}{[\frac{1}{2}-m_1']_q![\frac{1}{2}+m_1]_q![j+m_2']_q![j-m_2]q!}\right)^{1/2} \end{aligned} \tag{2.10}$$

($a(j)$ is taken as $m$).

To illustrate we now give some examples. When $j=1$ we have the solutions

$$S^{1\frac{1}{2}}=\begin{bmatrix} q & & & & & \\ & 0 & q^{-2}Q & & & \\ & 1 & q_1 & & & \\ & & & 0 & q^{-1}Q & \\ & & & q^{-1} & q_2 & \end{bmatrix} \tag{2.11}$$

with

$$q_1q_2=Qw(1-q^{-4}) \tag{2.12}$$

and

$$\tilde{S}^{1\frac{1}{2}}=\begin{bmatrix} q & & & & & \\ & 0 & \tilde{Q} & & & \\ & 1 & q_1 & & & \\ & & & \tilde{q}_2 & q\tilde{Q} & \\ & & & q & 0 & \\ & & & & & \tilde{Q} \end{bmatrix} \tag{2.13}$$

with

$$\tilde{q}_2=-q\tilde{q}_1. \tag{2.14}$$

Both solutions satisfy equations (1.8), (1.9) and (2.1).

We emphasize that the general solution possesses six distinct eigenvalues for $j=1$; for example, $q$, $Q$, $\frac{1}{2}(q_i\pm(q_i^2+4q^{-2}Q)^{1/2}$, $i=1,2$. However, the standard one has only three distinct eigenvalues because of the reduction. Obviously they cannot be connected through a similar transformation. This feature holds for any $j$. The example with $j=\frac{3}{2}$ has been analysed explicitly. We omit this example in this paper.

## 3. Yang–Baxterization

As was pointed out in [10] the Yang–Baxterization prescription is developed on the basis of Jimbo's loop extension theory [11], which is based on projectors and corresponding eigenvalues. It is worth noting that Yang–Baxterization for the present solutions cannot simply be done by following either [10] or [11], because in the present case the numbers of distinct eigenvalues and decomposed subspaces are no longer the same. The number of such subspaces is two, resulting in the decomposition

$$j\otimes\tfrac12 = j+\tfrac12 \oplus j-\tfrac12. \tag{3.1}$$

Our scheme is very simple and is based on direct verification. For $S^{\frac12\frac12}$ it is well known that

$$\check{R}^{\frac12\frac12}(x) = x(S^{\frac12\frac12})^{-1} - S^{\frac12\frac12}. \tag{3.2}$$

Taking equation (3.1) into account we know Yang–Baxterization for $S^{j\frac12}$ takes the form

$$\check{R}^{j\frac12}(x) = A_1 x(S^{\frac12 j})^{-1} + A_2 S^{j\frac12} \tag{3.3}$$

$$\check{R}^{\frac12 j}(x) = A_1 x(S^{j\frac12})^{-1} + A_2 S^{\frac12 j} \tag{3.4}$$

where the CP invariance has been considered. Substituting equations (3.2)-(3.4) into equation (1.11) we find equation (1.11) is satisfied for any spectral parameter $x$ if

$$A_1 = -Qq^{-2j}A_2. \tag{3.5}$$

For convenience we take $A_1 = q$ then

$$A_2 = -q^{2j+1}Q^{-1}. \tag{3.6}$$

In this manner the Yang–Baxterization of the general solutions leads to

$$\begin{aligned}\check{R}^{j\frac12}(x) = &\sum_{a(j),b(\frac12)} p^{(a(j),b(\frac12))}(x) E_{a(j),b(\frac12)} \otimes E_{b(\frac12),a(j)} \\ &+ \sum_{a(j),b(\frac12)} q^{(a(j),b(\frac12))}(x) E_{a(j),-b(\frac12)} \otimes E_{b(\frac12),a(j)+2b(\frac12)}\end{aligned} \tag{3.7}$$

where $(E_{a,b})_{cd} = \delta_{ac}\delta_{bd}$ and

$$p^{(a(j),b(\frac12))}(x) = \begin{cases} q^{j+1}(xq^{-a(j)} - Q^{-1}q^{a(j)+1}) & \text{for } b(\frac12) = \frac12 \\ q^{j}(xq^{a(j)} - Q^{-i}q^{-a(j)+1}) & \text{for } b(\frac12) = -\frac12 \end{cases} \tag{3.8}$$

$$q^{(a(j),b(\frac12))} = \begin{cases} -q^{2j+1}Q^{-1}q^{(a(j),\frac12)} & \text{for } b(\frac12) = \frac12 \\ -q^{2j+1}xq^{(-a(j),\frac12)} & \text{for } b(\frac12) = -\frac12. \end{cases} \tag{3.9}$$

Equation (3.8) allows the standard solutions as a special case where

$$p^{(a(j),b(\frac12))}(x) = \begin{cases} q^{j+1}(xq^{-a(j)} - q^{a(j)}) & \text{for } b(\frac12) = \frac12 \\ q^{j}(xq^{a(j)} - q^{-a(j)}) & \text{for } b(\frac12) = -\frac12 \end{cases} \tag{3.10}$$

$$q^{(a(j),b(\frac12))}(x) = \begin{cases} -q^{j+\frac12}w\{[j-a(j)]_q[j+a(j)+1]_q\}^{1/2} & \text{for } b(\frac12) = \frac12 \\ -q^{j+\frac12}w\{[j+a(j)]_q[j-a(j)+1]_q\}^{\frac12}x & \text{for } b(\frac12) = -\frac12. \end{cases} \tag{3.11}$$

It is interesting to point out that by taking $x=-1$ the standard solutions of $R^{j\frac12}(x)|_{x=-1}$ given by equations (3.10) and (3.11) exactly coincide with the results given in [14, 15]. This particular example ($j=1$) had been given in [16].

## 4. Quantum group structure

Following the FRT approach [13] for a given $S^{j\frac{1}{2}}$ the corresponding quantum group can be set up. For convenience the indices $a(j)$ will henceforth be replaced by Greek letters and $b(\frac{1}{2})$ by Latin letters; we then have the solutions

$$(S^{j\frac{1}{2}})^{\mu l}_{m\nu} = p^{(\mu,l)}\delta^{\mu}_{\nu}\delta^{l}_{m} + q^{(\mu,l)}\delta^{\mu}_{\nu-2l}\delta^{l}_{-m} \tag{4.1}$$

$$(S^{\frac{1}{2}j})^{l\mu}_{\nu m} = p^{(-\mu,-l)}\delta^{-\mu}_{-\nu}\delta^{-l}_{-m} + q^{(-\mu,-l)}\delta^{-\mu}_{-\nu+2l}\delta^{-l}_{m} \tag{4.2}$$

$$\mu,\ \nu = j, j-1, \ldots, -j+1, -j \qquad l,\ m = \pm\tfrac{1}{2} \tag{4.3}$$

where

$$p^{(\mu,\frac{1}{2})} = q^{\mu+1-j} \qquad p^{(\mu,-\frac{1}{2})} = q^{-\mu-j}Q \qquad q^{(\mu,-\frac{1}{2})} = 0 \tag{4.4}$$

$$q^{(-\mu-1,\frac{1}{2})}q^{(\mu,\frac{1}{2})} = q^{-2j}Qw^2[j-\mu]_q[j+\mu+1]_q \qquad (\mu \neq \pm j). \tag{4.5}$$

The corresponding FRT relations read

$$\begin{aligned}
S^{j\frac{1}{2}}(L^{\frac{1}{2}}_{\pm}\otimes L^{j}_{\pm}) &= (L^{j}_{\pm}\otimes L^{\frac{1}{2}}_{\pm})S^{j\frac{1}{2}}\\
S^{\frac{1}{2}j}(L^{j}_{\pm}\otimes L^{\frac{1}{2}}_{\pm}) &= (L^{\frac{1}{2}}_{\pm}\otimes L^{j}_{\pm})S^{\frac{1}{2}j}\\
S^{j\frac{1}{2}}(L^{\frac{1}{2}}_{+}\otimes L^{j}_{-}) &= (L^{j}_{-}\otimes L^{\frac{1}{2}}_{+})S^{j\frac{1}{2}}\\
S^{\frac{1}{2}j}(L^{j}_{+}\otimes L^{\frac{1}{2}}_{-}) &= (L^{\frac{1}{2}}_{-}\otimes L^{j}_{+})S^{\frac{1}{2}j}.
\end{aligned} \tag{4.6}$$

After calculation we find

$$\begin{aligned}
&(L_+)^{\mu}_{\nu} = \alpha^{\mu}_{\nu}K_1^{-2\nu}K_2^{2(\mu-j)}(X^+)^{\mu-\nu}\\
&(L_-)^{\mu}_{\nu} = \beta^{\mu}_{\nu}K_1^{2\mu}K_2^{2(\nu-j-2\mu)}(X^-)^{\nu-\mu}\\
&(L_+)^{\frac{1}{2}}_{\frac{1}{2}} = K_1^{-1}K_2 \qquad (L_+)^{-\frac{1}{2}}_{-\frac{1}{2}} = K_1K_2^{-1} \qquad (L_+)^{\frac{1}{2}}_{-\frac{1}{2}} = K_1K_2X^+\\
&(L_-)^{\frac{1}{2}}_{\frac{1}{2}} = K_1K_2 \qquad (L_-)^{-\frac{1}{2}}_{-\frac{1}{2}} = K_1^{-1}K_2 \qquad (L_-)^{-\frac{1}{2}}_{\frac{1}{2}} = K_1^{-1}K_2^3X^-\\
&(L_+)^{-\frac{1}{2}}_{\frac{1}{2}} = (L_-)^{\frac{1}{2}}_{-\frac{1}{2}} = 0
\end{aligned} \tag{4.7}$$

where

$$\begin{aligned}
&\alpha^{\mu}_{\nu} = 0 \qquad \text{for } \mu < \nu\\
&\beta^{\mu}_{\nu} = 0 \qquad \text{for } \mu > \nu\\
&\alpha^{\mu}_{\mu} = \beta^{\mu}_{\mu} = 1\\
&\alpha^{\nu+n}_{\nu} = \prod_{l=1}^{n}(q^{(-\nu-l,\frac{1}{2})}q^{\nu+j}(Qq^{-1})^{-(l-1)/j})/(q^l-q^{-l}) \qquad (n = 1, 2, \ldots, j-\nu)\\
&\beta^{\nu-n}_{\nu} = \prod_{l=1}^{n}(q^{(\nu-l,\frac{1}{2})}g^{\nu+j+1}Q^{-1}q^{-2l+1}(Qq^{-1})^{(l-1)/j})/(q^l-q^{-l}) \qquad (n = 1, 2, \ldots, j+\nu)
\end{aligned} \tag{4.8}$$

and the operators satisfy the relations

$$\begin{aligned}
&[K_1, K_2] = 0 \qquad K_1X^{\pm}K_1^{-1} = (Qq^{2j-1})^{\pm 1/2j}X^{\pm}\\
&K_2X^{\pm}K_2^{-1} = (Qq^{-1})^{\pm 1/2j}X^{\pm}\\
&(Qq^{-1})^{-1/j}X^+X^- - (Qq^{-1})^{1/j}X^-X^+ = q^{-1}wK_2^{-4}(K_1^{-2}K_2^2 - K_1^2K_2^{-2}).
\end{aligned} \tag{4.9}$$

The coproduct $\Delta$, co-unit $\varepsilon$ and antipode $\gamma$ are given by the following:

$$\begin{aligned}
&\Delta K_1 = K_1 \otimes K_1 \qquad \Delta K_2 = K_2 \otimes K_2 \\
&\Delta X^+ = K_1^{-2} \otimes X^+ + X^+ \otimes K_2^{-2} \\
&\Delta X^- = X^- \otimes K_1^2 K_2^{-4} + K_2^{-2} \otimes X^- \\
&\varepsilon(K_1) = \varepsilon(K_2) = 1 \qquad \varepsilon(X^\pm) = 0 \\
&\gamma(X^+) = -Q^{-1} q K_1^2 K_2^2 X^+ \qquad \gamma(X^-) = -Q q^{-3} K_1^{-2} K_2^6 X^- \\
&\delta(K_1) = K_1^{-1} \qquad \gamma(K_2) = K_2^{-1}.
\end{aligned} \tag{4.10}$$

Similar to the discussion in [17] the commutation relations can be simplified by introducing

$$\tilde{X}^\pm = X^\pm K_2^2 \qquad \mathcal{K} = K_1 K_2^{-1}. \tag{4.11}$$

We then have the usual commutation relations except a centre

$$\begin{aligned}
&K_2 \tilde{X}^\pm K_2^{-1} = (Qq^{-1})^{\pm 1/2j} \tilde{X}^\pm \qquad [\mathcal{K}, K_2] = 0 \\
&\mathcal{K} \tilde{X}^\pm \mathcal{K}^{-1} = q^{\pm 1} \tilde{X}^\pm \\
&[\tilde{X}^+, \tilde{X}^-] = q^{-1} w (\mathcal{K}^{-2} - \mathcal{K}^2).
\end{aligned} \tag{4.12}$$

It is not surprising that the above result has been reached since we cannot go beyond the basic structure of the QG. Actually, this result checks the validity of our solutions. We would like to emphasize that the derivation of such commutation relations does not mean that the QG structure is 'trivial'. In the next section we will point out that the construction of the $R$-matrix is no longer standard, i.e. it cannot be formed in the usual way in terms of basis, nor can it be related to any similar transformation of the standard basis.

## 5. Relationship between general and standard solutions

According to the standard construction of the $R$-matrix [1-3] the centralizer can be represented in terms of the basis of universal enveloping algebra. The representation in this case is

$$S^{\frac{1}{2}j}_{\text{standard}} = q^{\rho^{(\frac{1}{2})}(J_3) \otimes \rho^{(j)}(J_3)/2} \sum_{n=0} \frac{(1-q^{-2})^n}{[n]_q!} \times q^{(\frac{1}{2})n(n-1)} (q^{\rho^{(\frac{1}{2})}(J_3)} \rho^{(\frac{1}{2})}(J_+) \otimes q^{-\rho^{(j)}(J_3)} \rho^{(j)}(J_-))^n \tag{5.1}$$

where $\rho^{(j)}$ stands for the representation of $\mathrm{SU}_q(2)$ and

$$\begin{aligned}
&\rho^{(j)}(J_\pm)_m^{m'} = ([j \mp m]_q [j \pm m + 1]_q)^{1/2} \delta_{m+1}^{m'} \\
&\rho^{(j)}(J_3)_m^{m'} = 2m \delta_m^{m'} \qquad m, m' \in (j, j-1, \ldots, -j).
\end{aligned} \tag{5.2}$$

Following the idea $q$-weight conservation in [18] we write our solutions as

$$\begin{aligned}
&S^{\frac{1}{2}j} = S^{\frac{1}{2}j}_{\text{standard}} + \Delta^{\frac{1}{2}j} \\
&S^{j\frac{1}{2}} = S^{j\frac{1}{2}}_{\text{standard}} + \Delta^{j\frac{1}{2}} \\
&S^{\frac{1}{2}\frac{1}{2}} = S^{\frac{1}{2}\frac{1}{2}}_{\text{standard}}.
\end{aligned} \tag{5.3}$$

Let us determine $\Delta^{\frac{1}{2}j}$ and $\Delta^{j\frac{1}{2}}$ by substituting equation (5.3) into equation (1.1). The constraint relations yield

$$S_{12}^{\frac{1}{2}\frac{1}{2}}(\Delta_{13}^{\frac{1}{2}j}S_{23}^{\frac{1}{2}j}+S_{13}^{\frac{1}{2}j}\Delta_{23}^{\frac{1}{2}j}+\Delta_{13}^{\frac{1}{2}j}\Delta_{23}^{\frac{1}{2}j})=(\Delta_{23}^{\frac{1}{2}j}S_{13}^{\frac{1}{2}j}+S_{23}^{\frac{1}{2}j}\Delta_{13}^{\frac{1}{2}j}+\Delta_{23}^{\frac{1}{2}j}\Delta_{13}^{\frac{1}{2}j})S_{12}^{\frac{1}{2}\frac{1}{2}} \tag{5.4}$$

and

$$\Delta_{12}^{\frac{1}{2}j}S_{13}^{\frac{1}{2}\frac{1}{2}}S_{23}^{j\frac{1}{2}}+S_{12}^{\frac{1}{2}j}S_{13}^{\frac{1}{2}\frac{1}{2}}\Delta_{23}^{j\frac{1}{2}}+\Delta_{12}^{\frac{1}{2}j}S_{13}^{\frac{1}{2}\frac{1}{2}}S_{23}^{j\frac{1}{2}}=\Delta_{23}^{j\frac{1}{2}}S_{13}^{\frac{1}{2}\frac{1}{2}}S_{12}^{\frac{1}{2}j}+S_{23}^{j\frac{1}{2}}S_{13}^{\frac{1}{2}\frac{1}{2}}\Delta_{12}^{\frac{1}{2}j}+\Delta_{23}^{j\frac{1}{2}}\Delta_{13}^{\frac{1}{2}\frac{1}{2}}\Delta_{12}^{\frac{1}{2}j}. \tag{5.5}$$

After calculation we find that $\Delta^{j\frac{1}{2}}$ and $\Delta^{\frac{1}{2}j}$ should be expressed through the $q$-generators as

$$\begin{aligned}\Delta^{\frac{1}{2}j}=&\sum_{k=0}^{2j}D_k\rho^{(\frac{1}{2})}(J_-\rho^{(\frac{1}{2})}(J_+)\otimes(\rho^{(j)}(J_-))^k(\rho^{(j)}(J_+))^k\\&+\sum_{k=1}^{2j}B_k\rho^{(\frac{1}{2})}(J_+)\otimes(\rho^{(j)}(J_+))^{k-1}(\rho^{(j)}(J_-))^K\\&+\sum_{k=1}^{2j}C_k\rho^{(\frac{1}{2})}(J_-)\otimes(\rho^{(j)}(J_+))^k(\rho^{(j)}(J_-))^{k-1}.\end{aligned} \tag{5.6}$$

Using

$$\begin{aligned}&[(\rho^{(j)}(J_-))^k(\rho^{(j)}(J_+))^k]_m^{m'}=\prod_{l=1}^{k}[j+m+l]_q[j-m+l+1]_q\delta_m^{m'}\\&[\rho^{(\frac{1}{2})}(J_-)\rho^{(\frac{1}{2})}(J_+)\otimes(\rho^{(j)}(J_-))^k(\rho^{(j)}(J_+))^kk]_{m_1m_2}^{m_1'm_2'}\\&\quad=\prod_{l=1}^{k}[j+m_2+l]_q[j-m_2-l+1]_q\delta_{-\frac{1}{2}}^{m_1'}\delta_{m_1}^{-\frac{1}{2}}\delta_{m_2}^{m_2'}\end{aligned} \tag{5.7}$$

we obtain the recurrence relations for the coefficients $D_k$ and $B_k$,

$$\begin{aligned}&D_0+\sum_{k=1}^{2j}D_k\left(\prod_{l=1}^{k}[j+m+l]_q[j-m-l+1]_q\right)=q^{m-1}(Q-q)\qquad(m\neq j)\\&D_0=q^{-j-1}(Q-q)\end{aligned} \tag{5.8}$$

and

$$\begin{aligned}&([j+m]_q[j-m+1]_q)^{\frac{1}{2}}\left[B_1+\sum_{k=2}^{2j}\left(\prod_{l=2}^{k}B_k[j+m-l+1]_q[j-m+l]_q\right)\right]\\&\quad=q_{m-1}-q^{-\frac{1}{2}}w([j+m]_q[j-m+1]_q)^{\frac{1}{2}}\qquad(m\neq -j)\end{aligned} \tag{5.9}$$

as well as a sufficient choice

$$C_k=0. \tag{5.10}$$

Equations (5.8)–(5.10) can be used to determine $\Delta^{j\frac{1}{2}}$ and $\Delta^{\frac{1}{2}j}$ in terms of the representations of the $q$-generators $J_\pm$ and $J_3$ satisfying the standard QG, as shown by equation (4.12) ($\tilde{X}^\pm$ identifies with $J_\pm$ after rescaling).

It should be emphasized that since the general solution cannot be related to the standard one by a similar transformation and by virtue of equation (5.3), in general, our solution cannot be expressed by a new basis which is a similar transformation of the standard basis in the same manner as for the standard case. We have met the same situation for a new solution of $S^{11}$ [17, 18].

If $C_k \neq 0$, the calculation is very complicated, and we give an example only. For equation (2.13) we find

$$D_0 = \tilde{Q} - q^{-1},\ D_1 = (1/[2]_q)(q-1)(\tilde{Q} - q^{-1})$$
$$D_2 = (1-q)(\tilde{Q}+1)$$
$$B_1 = (-1/[2]_q)[(1-q^{-2})(q^2-q^{-2})]^{1/2},\ B_2 = (1/[2]_q)\tilde{q}_1$$
$$C_1 = (1/[2]_q)\tilde{q}_2 \qquad C_2 = -([2]_q)^{3/2}\tilde{q}_2$$
$$\tilde{q}_2 = -q\tilde{q}_1 .$$

In conclusion we have verified all of the results listed in the introduction.

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