

LETTER TO THE EDITOR

Structure of regular representation of quantum universal enveloping algebra  $SL_q(2)$  with  $q^p = 1$ \*

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**Abstract.** In this paper we construct the regular representation quantum Lie algebra of  $SL_q(2)$  and its quotient representations and then analyse the reducible structures of these representations when  $q^p = 1$  for an integer  $p$ . We show how the indecomposable representations of  $SL_q(2)$  are obtained by a purely algebraic method.

Due to the development of nonlinear physics relating the Yang-Baxter equation (YBE), quantum group, quantum universal enveloping algebras (QUEA) and their representation theory have drawn considerable attention from both mathematicians and physicists [1, 2]. Besides the standard representation theory of QUEA, [3, 4] the  $q$ -deformed boson (oscillator) realization has been proposed independently by different authors [5-7]. Another important development of the representation theory is the discussion of the case with  $q^p = 1$  ( $p \in \mathbb{D} = \{3, 4, 5, \dots\}$ ) [8, 9]. Lusztig first showed that a rich reducible structure will occur in representations when  $q^p = 1$  [8], and some indecomposable (reducible but not completely reducible) representations are obtained in this case [10].

In this letter, motivated by Lusztig's work, we construct and analyse some non-simple moduli (representations) of  $SL_q(2) \equiv U_q(\mathfrak{sl}(2))$  when  $q^p = 1$ . We first build the regular representation of  $SL_q(2)$  and then obtain some quotient representations induced by it. These quotient representations are usually indecomposable even if  $q^p \neq 1$  ( $\forall p \in \mathbb{D}$ ). In particular, we analyse the reducible structure of these representations when  $q^p = 1$  in detail and also show how the finite-dimensional representations are derived from the infinite-dimensional quotient representations. We also suppose that  $p$  is an odd integer  $\geq 3$  without loss of generality [8].

Since  $SL_q(2)$  is an associative algebra generated by  $J_{\pm}$  and  $K^{\pm 1}$  that satisfies

$$\begin{aligned} [J_3, J_{\pm}] &= \pm J_{\pm} & [J_+, J_-] &= [2J_3] \\ [f] &\equiv (q^f - q^{-f})(q - q^{-1})^{-1} & q &\in \mathbb{C} \text{ (complex field)} \end{aligned} \tag{1}$$

we can construct its regular representation on its own linear space by the left transformation action  $T: SL_q(2) \rightarrow \text{End}(SL_q(2))$ :  $T(x) \cdot u = x \cdot u$ ,  $\forall u, x \in SL_q(2)$ .

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Using the following relation resulting from (1)

$$\begin{aligned}
 K^\pm J_\pm^n &= q^{\pm 2n} J_\pm^n K^\pm & K^\pm J_\mp^n &= q^{\mp 2n} J_\mp^n K^\pm \\
 J_\pm K^n &= q^{\mp 2n} K^n J_\pm & J_\pm (K^{-1})^n &= q^{\pm 2n} (K^{-1})^n J_\pm \\
 J_- J_+^n &= J_+ J_- + [n] J_+^{n-1} (q^{-n+1} K^{-1} - q^{n-1} K) (q - q^{-1})^{-1} \\
 J_+ J_-^n &= J_- J_+ + [n] J_-^{n-1} (q^{-n+1} K - q^{n-1} K^{-1}) (q - q^{-1})^{-1}
 \end{aligned} \tag{2}$$

we explicitly obtain the regular representation

$$T(J_+)X(m, n, r) = X(m+1, n, r)$$

$$T(J_-)X(m, n, r)$$

$$\begin{aligned}
 &= X(m, n+1, r) + \{[m]/(q - q^{-1})\} \\
 &\quad \times \{q^{-m+1+2n} X(m-1, n, r-1) + q^{m-1-2n} X(m-1, n, r+1)\}
 \end{aligned} \tag{3}$$

$$T(K^{\pm 1})X(m, n, r) = q^{\pm 2(m-n)} X(m, n, r \pm 1).$$

On the basis

$$X(m, n, r) = J_+^m J_-^n K^r = \begin{cases} J_+^m J_-^n K^r & r = 1, 2, \dots \\ 1 & r = 0 \\ J_+^m J_-^n (K^{-1})^{-r} & r = -1, -2 \end{cases}$$

$(m, n \in \mathbb{N})$  for  $SL_q(2)$  where  $\mathbb{N} = \{0, 1, 2, \dots\}$ .

When  $q \rightarrow 1$ , this representation becomes the master representation of Lie algebra  $SL(2)$  [11]. This representation is quite general and some new representations besides the standard ones can be induced on certain quotient spaces by it. We discuss this representation in two cases as follows.

*Case 1.*  $q^p \neq 1$  ( $\forall p \in \mathbb{D}$ ). In this case, because the index  $n$  of the vector  $X(m, n, r)$  can only be increased by the action of  $T$ , all the vectors  $X(m, n+N, r)$  for a given  $N \in \mathbb{N}$  span an invariant subspace  $V_N$  and so the representation  $T$  is reducible. However, there does not exist an invariant complementary space for  $V_N$ , i.e. the representation  $T$  is indecomposable (reducible but not completely reducible). For the sequence of  $T$ -invariant subspaces

$$SL_q(2) = V_0 \supset V_1 \supset V_2 \supset \dots \supset V_N \supset V_{N+1} \supset \dots$$

the representation  $T$  induces various quotient representations of  $SL_q(2)$   $T^{[N, N']}$  on each quotient space  $V_N/V_{N'}$  ( $N' > N$ ). Let  $T^{[N]} = T^{[N, N+1]}$ , then  $T$  has a semidirect sum structure,  $T = T^{[0]} \oplus \dots \oplus T^{[1]} \oplus \dots \oplus T^{[N]} \oplus \dots$ .

*Case 2.*  $q^p = 1$  ( $p \in \mathbb{D}$ ),  $[ap] = 0$  for  $\alpha \in \mathbb{N}$

$$T(J_-)X(\alpha p, n, r) = X(\alpha p, n+1, r)$$

i.e.  $X(\alpha p, n, r)$  ( $n, r \in \mathbb{N}$ ) are extreme vectors and they define an  $T$ -invariant subspace  $V_0^\alpha: \{X(\alpha p + m, n, r) | m, n, r \in \mathbb{N}\}$ . Then, we have some 'smaller'  $T$ -invariant subspaces

$$V_N^\alpha = V_0^\alpha \cap V_N = \{X(\alpha p + m, N + n, r) | m, n, r \in \mathbb{N}\}$$

and 'bigger' ones

$$B_N^\alpha = V_0^\alpha \cup V_N = \{X(\alpha p + m, n, r), X(m', n' + N, r') | m, n, r, m', n', r' \in \mathbb{N}\}.$$

From these subspaces, we naturally obtain some new subrepresentations.

Now we consider a quotient representation  $\Gamma$  of  $SL_q(2)$  with definite weight induced by the regular representation  $T$ .

Corresponding to a left ideal  $I = \{X(K - q^{2\lambda}) \mid \text{all } x \in SL_q(2)\}$  for  $\lambda \in \mathbb{C}$ , the quotient space  $Q = SL_q(2)/I$  has a basis

$$X(m, n) = X(m, n, 0) \text{ Mod } I \quad m, n \in \mathbb{N}.$$

An easy calculation allows us to obtain a quotient representation on the quotient space

$$\begin{aligned} \Gamma(J_+)X(m, n) &= X(m + 1, n) \\ \Gamma(J_-)X(m, n) &= X(m, n + 1) + [m](2n - m + 1 - 2\lambda)X(m - 1, n) \\ \Gamma(K^\pm)X(m, n) &= q^{\pm 2(m-n+\lambda)}X(m, n). \end{aligned} \tag{4}$$

Like the regular representation  $T$  on  $V_0 \equiv SL_q(2)$ , this representation is also indecomposable. Its reducible structure can be induced by the reducible structure of  $T$ . Consider a projection mapping  $\varphi: V_0 = SL_q(2) \rightarrow Q$  such that

$$\varphi(X(m, n, r)) = q^{2r\lambda} X(m, n) = q^{2r\lambda} X(m, n, 0) \text{ Mod } L.$$

This mapping defines the quotient representation (4):  $\Gamma = \varphi \cdot T \cdot \varphi^{-1}$ . If  $S$  is a  $T$ -invariant subspace of  $V_0$ , then  $\varphi(S) = \{\varphi(x) \mid \text{all } x \in V_0\}$  is a  $\Gamma$ -invariant subspace of  $Q$ . Therefore, we have  $\Gamma$ -invariant subspaces of  $Q$  corresponding to  $V_N, V_0^\alpha, V_N^\alpha$  and  $B_\alpha^N$  respectively

$$\begin{aligned} Q_N &= \varphi(V_N): \{X(m, n + N) \mid m, n \in \mathbb{N}\} \\ Q_0^\alpha &= \varphi(V_0^\alpha): \{X(\alpha p + m, n) \mid m, n \in \mathbb{N}\} \\ Q_N^\alpha &= \varphi(V_N^\alpha): \{X(\alpha p + m, n + N) \mid m, n \in \mathbb{N}\} \\ W_N^\alpha &= \varphi(B_\alpha^N): \{X(m, n + N), X(\alpha p + m', n') \mid m, n, m', n' \in \mathbb{N}\}. \end{aligned}$$

There is also a sequence of  $\Gamma$ -invariant subspaces

$$Q = Q_0 \supset Q_1 \supset Q_2 \supset \dots \supset Q_N \supset Q_{N+1} \supset \dots$$

and the quotient representations  $\Gamma^{[N, N']}$  on  $Q_N/Q_{N'}$  ( $N' > N$ ).

Although all the subrepresentations on the subspaces  $Q_N, Q_0^\alpha, Q_N^\alpha$  and  $W_N^\alpha$  are infinite dimensional, a class of quotient spaces  $Q_N^\alpha = Q/W_N^\alpha: \{\tilde{X}(m, n) = X(m, n) \text{ Mod } W_N^\alpha \mid n \in \mathbb{N}\}$  is finite dimensional and the dimension is

$$\dim \cdot Q_{R\alpha} = \alpha N p.$$

On the quotient space  $Q_N^\alpha$ , the representation  $\Gamma^{[N, \alpha]}$  induced by  $\Gamma$  is formally defined by (4) and the additional requirements

$$\begin{aligned} \Gamma^{[N, \alpha]}(J_+)X(\alpha p - 1, n) &= 0 \\ \Gamma^{[N, \alpha]}(J_-)X(m, N - 1) &= 0. \end{aligned} \tag{5}$$

For example, when  $\alpha = 1$  and  $N = 2$ , we have a  $2p - D$  representation

$$\begin{aligned} \Gamma^{[2, 1]}(J_+)\tilde{X}(m, n) &= \tilde{X}(m + 1, n) \quad m = 0, 1, 2, \dots, p - 2 \\ \Gamma^{[2, 1]}(J_+)\tilde{X}(p - 1, n) &= 0 \\ \Gamma \in \zeta \phi \zeta^{[2, 1]}(J_-)\tilde{X}(m, 0) &= X(m, 1) + [m][1 - m - 2\lambda]X(m - 1, 0) \\ \Gamma^{[2, 1]}(J_+)\tilde{X}(m, 1) &= [m][3 - m - 2\lambda]\tilde{X}(m - 1, n) \\ \Gamma^{[2, 1]}(K^\pm)\tilde{X}(m, n) &= q^{\pm 2(m-n+\lambda)}\tilde{X}(m, n). \end{aligned}$$

It is easy to check that  $\Gamma^{[2, 1]}$  as well as  $\Gamma$  and  $T$  forms a representation of  $SL_q(2)$ .

In the last paragraph the representations discussed were defined on the basis  $X(m, n)$  or  $\tilde{X}(m, n)$  with two indices  $m$  and  $n$ . Now we discuss the representations whose bases have one index.

Let  $L = \{x(J_+ - \mu \mathbb{1}) \mid \text{all } x \in Q\}$  be a left ideal of  $Q$  for  $\mu \in \mathbb{C}$  and  $R = Q/L: \{x(n) = X(n, 0) \text{ Mod } L \mid n \in \mathbb{N}\}$  be the corresponding quotient space. On the quotient space  $R$ , the representation  $\rho$  induced by  $\Gamma$  is

$$\begin{aligned} \rho(J_+)X(n) &= \mu X(n) + [n][2\lambda - n + 1]X(n-1) \\ \rho(J_-)X(n) &= X(n+1) \\ \rho(K^{\pm 1})X(n) &= q^{\pm 2(\lambda - n)}X(n). \end{aligned} \tag{7}$$

Because the basic relations (1) allow a difference of  $J_+$  by a  $\mathbb{C}$ -number  $\mu$ , i.e.  $J'_+ = J_+ - \mu$ ;  $J_-$  and  $J_3$  still satisfy the relations (1), we can let  $\mu = 0$  without loss of generality and

$$\begin{aligned} \rho(J_+)X(n) &= [n][2\lambda - n + 1]X(n-1) \\ \rho(J_-)X(n) &= X(n+1) \\ \rho(K^{\pm 1})X(n) &= q^{\pm 2(\lambda - n)}X(n) \end{aligned} \tag{8}$$

is also a representation of  $SL_q(2)$ . In fact, the above representation is of a Verma module with highest weight  $2\lambda$ . The simple modulus relating to this Verma module has been well classified in [3, 4] and [8, 9] for both cases,  $q^p \neq 1$  ( $\forall p \in \mathbb{D}$ ) and  $qp' = 1$  ( $P' \in \mathbb{D}$ ). Here, we will pay attention to the case of non-simple modulus with  $q^p = 1$  for a given  $p$ .

(1) In the case with non-integer  $2\lambda \in \mathbb{C}$ , the representation (8) is irreducible unless  $q^p = 1$  ( $p \in \mathbb{D}$ ). When  $q^p = 1$ ,  $X(\alpha p)$  for  $\alpha \in \mathbb{N}$  is an extreme vector such that  $\rho(J_+)X(\alpha p) = 0$ . Then, we have a  $\rho$ -invariant subspace  $W_\alpha: \{X(\alpha p + n) \mid n \in \mathbb{N}\}$ . On the quotient space  $R_\alpha = R/W_\alpha: \{\tilde{X}(n) = X(n) \text{ Mod } W_\alpha \mid n \in \mathbb{N}\}$ ,  $\rho$  induces a  $\alpha p$ -dimensional representation  $\rho^{[\alpha]}$ :

$$\begin{aligned} \rho^{[\alpha]}(J_+)\tilde{X}(n) &= [n][2\lambda - n + 1]\tilde{X}(n-1) \\ \rho^{[\alpha]}(J_-)\tilde{X}(n) &= \tilde{X}(n+1) \quad n = 0, 1, 2, \dots, \alpha p - 1 \\ \rho^{[\alpha]}(J_-)\tilde{X}(n = \alpha p - 1) &= 0 \\ \rho^{[\alpha]}(K^{\pm 1})\tilde{X}(n) &= q^{\pm 2(\lambda - n)}\tilde{X}(n). \end{aligned} \tag{9}$$

Because  $W_\alpha$  is an invariant subspace of  $W_\alpha$  when  $\alpha' > \alpha$ , the representation  $\rho^{[\alpha]}(\alpha > 1)$  is still reducible. As Roche and Araudo have proved [9], the quotient representation  $\rho^{[\alpha, \alpha + 1]}$ :

$$\begin{aligned} \rho^{[\alpha, \alpha + 1]}(J_+)Y(n) &= [n][2\lambda - n + 1]Y(n-1) \\ \rho^{[\alpha, \alpha + 1]}(J_-)Y(n) &= Y(n+1) \quad n = 0, 1, 2, \dots, p^{-1} \\ \rho^{[\alpha, \alpha + 1]}(J_-)Y(p-1) &= 0 \\ \rho^{[\alpha, \alpha + 1]}(K^{\pm 1})Y(n) &= q^{\pm 2(\lambda - n)}Y(n) \end{aligned} \tag{10}$$

induced by  $\rho^{[\alpha]}$  on the quotient space  $R_\alpha/R_{\alpha+1} = \{Y(n) = \tilde{X}(n) \text{ Mod } R_{\alpha+1} \mid n = 0, 1, 2, \dots, p-1\}$  is irreducible.

(2) In the case where  $2\lambda \in \mathbb{N}$  and  $q^p = 1$  ( $p \in \mathbb{D}$ ), besides the invariant subspaces  $W_\alpha$ , there are  $\rho$ -invariant subspaces  $S_\lambda: \{X(2\lambda + 1 + n) \mid n \in \mathbb{N}\}$  determined by the vectors

$X(2\lambda + 1)$  such that  $\rho(J_+)X(2\lambda + 1) = 0$ . On the quotient space  $R/S_\lambda \cong R^\lambda: \{u(n) = X(n) \bmod S_\lambda | n \in \mathbb{N}\}$  a representation  $\rho_\lambda$  with dimension  $(2\lambda + 1)$  is induced by (8)

$$\begin{aligned} \rho_\lambda(J_+)u(n) &= [n][2\lambda + n + 1]u(n - 1) \\ \rho_\lambda(J_-)u(n) &= u(n + 1) \quad n = 0, 1, 2, \dots, 2\lambda - 1 \\ \rho_\lambda(J_-)u(2\lambda) &= 0 \\ \rho_\lambda(K^\pm)u(2\lambda) &= q^{\pm 2(\lambda - n)}u(2\lambda). \end{aligned} \tag{11}$$

(a) When  $2\lambda + 1 \leq p$ , the representation defined by (11) is irreducible. This case of simple modulus has been discussed in [9].

(b) When  $2\lambda + 1 > \alpha p$  (for a  $\alpha \in \mathbb{N}$ ), the representation (11) is not irreducible. In fact,  $S_\lambda$  is a  $\rho$ -invariant subspace of  $W_\alpha$  and  $R^\lambda = R/S_\lambda$  has  $\rho_\lambda$ -invariant subspaces  $R^\lambda: \{u(\beta p + k) = \tilde{u}(k) | k = 0, 1, 2, \dots, 2\lambda - \beta p\}$  ( $\beta = 1, 2, \dots, \alpha$ ). On a subspace  $R_\beta^\lambda$ , we have a subrepresentation  $\rho_\beta^\lambda$ :

$$\begin{aligned} \rho_\beta^\lambda(J_+)\tilde{u}(k) &= [\beta p + k][2\lambda - \beta p - k + 1]\tilde{u}(k - 1) \\ \rho_\beta^\lambda(J_-)\tilde{u}(k) &= \tilde{u}(k + 1) \\ \rho_\beta^\lambda(J_-)\tilde{u}(2\lambda - \beta k) &= 0 \\ \rho_\beta^\lambda(K^\pm)\tilde{u}(k) &= q^{\pm 2(\lambda - k)}\tilde{u}(k). \end{aligned} \tag{12}$$

However,  $R_\beta^\lambda$  has no invariant complementary subspace and the representation  $\rho_\lambda$  given by (11) is indecomposable.

It is easy from (2) to observe that  $(K^{\pm\rho}, X) = [J_\pm^\rho, X] = 0, \forall x \in \text{SL}_q(2)$ , that is to say,  $J_\pm^\rho$  and  $K^{\pm\rho}$  are the elements in the centre of  $\text{SL}_q(2)$ . On a certain space  $R$ , we also prove that  $J_+^\rho$  actually vanishes. In fact, when  $n < p, \Gamma(J_+^\rho)X(n) = 0$ ; when  $n = \alpha p + n' > p$ : where  $0 \leq n' < p, \alpha \geq 1$

$$\begin{aligned} \Gamma(J_+^\rho)X(n) &= [n][n - 1][n - 2] \dots [n - p + 1][n - p] \\ &\quad \times [2\lambda - n + 1] \dots [2\lambda - n + p]X(n - p) \\ &= [\alpha\rho][\rho - 1]![2\lambda - n + p]!([2\lambda - n]!)^{-1}X(n - \rho). \end{aligned}$$

Then, we can introduce Lusztig operator

$$\hat{L} = \lim_{q^\rho \rightarrow 1} \frac{J_+^\rho}{[p]!} \tag{13}$$

that acts on  $X(n)$  as

$$\hat{L}X(n) = \begin{cases} 0 & \text{if } n \leq p \\ \alpha \left( \frac{[2\lambda - n + p]!}{[2\lambda - n]!} \right) X_{[n - \rho]} & \text{if } n > p. \end{cases} \tag{14}$$

Now we consider the representations of an extension  $\hat{\text{SL}}_q(2)$  of  $\text{SL}_q(2)$  by adding the Lusztig element  $\hat{L}$ . The representations (8) and (11) of  $\text{SL}_q(2)$  are still the representations of  $\hat{\text{SL}}_q(2)$ . However, because  $W_\alpha$  and  $R_\beta^\lambda$  are not invariant under the action of  $\hat{L}$ , the representation (8) is irreducible when  $2\lambda \neq \text{integer}$  and the representation (11)  $\rho_\lambda$  is also irreducible for given  $\lambda \in \mathbb{N}$ .

Finally, it should be pointed out that the main results of this letter are not covered by previous work [8, 9] because we were mainly involved with the cases of non-simple moduli relating to the regular representation.

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