J. Phys. A: Math. Gen. 23 (1990) L645-L648. Printed in the UK

LETTER TO THE EDITOR

Weight conservation and quantum group construction of the braid group representation

Mo-Lin Ge†, Chang-Pu Sun†‡, Lu-Yu Wang† and Kang Xue†

† Theoretical Physics Division, Nankai Institute of Mathematics, Tianjin 300071, People's Republic of China

‡ CCAST (World Laboratory), PO Box 8730, Beijing, People's Republic of China

Received 3 May 1990

Abstract. It is proved that the braid group representation constructed in terms of the quantum group can be covered by that obtained from the proposal of weight conservation by the extended Kauffman diagram technique. The non-standard braid group representations associated with SU(2) are obtained by adding the terms of weight conservation to the standard universal R-matrix.

At present the theories of braid group representation (BGR) draw much attention in mathematical physics [1] because of their key roles for the solvable models and low dimensional field theories through the Yang-Baxter equation (YBE). The standard BGRs or the R-matrices have been constructed in terms of the quantum groups (QGs) as q-analogues of the universal enveloping algebras (QUEAs) of Lie algebras in mathematics [2-4]. Recently, the Kauffman diagram technique (KDT) [5] associated with SU(2) has been extended to the cases of arbitrary classical Lie algebra. The BGRs obtained by this extended KDT not only covered the standard BGRs, but also included some new BGRs that are called non-standard BGRs [6-10].

The key point for the above-mentioned extended approach is the proposal of weight conservation which is a generalisation of charge conservation [5, 11] (for the details see [12]). Let w_a $(a=1,2,\ldots,N)$ be the weights of an N-dimensional irreducible representation $\Gamma^{[\lambda]}$ of a classical Lie algebra and $V^{[\lambda]}$ the corresponding space. A BGR R associated with the Lie algebra is defined on the product space $V^{[\lambda]}\otimes V^{[\lambda]}$ and its matrix element is labelled by the weight pair (w_a, w_b), i.e.

$$R^{ab}_{cd} \equiv R^{w_a, w_b}_{w_c, w_d}$$

The weight conservation requires that

$$R^{ab}_{cd}=0 \qquad \text{if } w_a+w_b \neq w_c+w_d. \tag{1}$$

Then R can be written as a block diagonal structure

$$R=\text{block diag } (A_1, A_2, \ldots, A_k) \tag{2}$$

and the extended KDT can be used to determine each block A_i. In the following we prove that the above weight conservation structure (2) is satisfied by the BGRs constructed from QG.

From a QG $U(L)_q$ generated by h_{α_i}, E_{α_i} and $E_{-\alpha_i}$ that satisfy

$$[h_{\alpha_i}, h_{\alpha_j}]=0 \qquad [E_{\alpha_i}, E_{-\alpha_i}]=\delta_{ij} \qquad [h_{\alpha_i}], [h_{\alpha_i}, E_{\alpha_i}]=\pm A_{ij}E_{\alpha_i} \tag{3}$$

the universal R-matrix is defined as

$$\hat{R}=\sum_{a} e_a\otimes e^a \tag{4}$$

where α_is are the simple roots of a Lie algebra L with the Cartan matrix $A=(A_{ij})$ and $f=(q^f-q^{-f})/(q^1-q^{-1})$ for any operator or number f. e_as are the basis for the subalgebra $U(L_+)_q$ generated by h_{α_i} and $E_{\alpha_i}(i=1,2,\ldots,1)$ and e_a^a's are the dual basis of the dual $U(L_+)_q^*$ of $U(L_+)_q$, which is an isomorphism of the subalgebra $U(L_-)_q$ as a Hopf algebra generated by h_{α_i} and $E_{-\alpha_i}$ $(i=1,2,\ldots,1)$ [13]. For a given matrix representation $\Gamma^{[\lambda]}$ of $U(L)$, $R=\Gamma^{[\lambda]}\otimes\Gamma^{[\lambda]}(\hat{R})$ defines a matrix solution of the YBE:

$$R_{12}R_{23}R_{12}=R_{23}R_{12}R_{23}. \tag{5}$$

According to Rosso's discussion about the analogue of the PBW theorem and its generalisation [13], e_a and e^a take the forms

$$e_a=\ldots E_{\alpha_i}^{m_i}\ldots E_{\alpha_j}^{m_j}\ldots \qquad e^a=\ldots E_{-\alpha_i}^{m_i}\ldots E_{-\alpha_j}^{m_j}\ldots \qquad m_i, m_j=0,1,2,\ldots.$$

Then, the universal R-matrix (4) is rewritten as

$$\hat{R}=\sum_{\ldots m_i\ldots m_j\ldots}\ldots E_{\alpha_i}^{m_i}\ldots E_{\alpha_j}^{m_j}\ldots\otimes\ldots E_{-\alpha_i}^{m_i}\ldots E_{-\alpha_j}^{m_j}\ldots \tag{6}$$

in explicit form. It is easily observed from (6) that E_{α_i} and $E_{-\alpha_i}$ appear with the same power m_i. The matrix element of $\hat{R}$ satisfies

$$R_{cd}^{ab}=\langle w_a, w_b|\hat{R}|w_c w_d\rangle\begin{cases}=0 & \text{if } w_a+w_b\neq w_c+w_d\\ \neq 0 & \text{if } w_a+w_b=w_c+w_d\end{cases} \tag{7}$$

because of the actions of E_{α_i} and $E_{-\alpha_i}$ on the weight vectors $|w_a\rangle$:

$$E_{\pm\alpha_i}^{m_i}|w_a\rangle=\begin{cases}0 & \text{if } w_a\pm m_i\alpha_i \text{ is not a root}\\ |w_a\pm m_i\alpha_i\rangle & \text{if } w_a\pm m_i\alpha_i \text{ is a root}\end{cases} \tag{8}$$

where $|w_a, w_b\rangle=|w_a\rangle\otimes|w_b\rangle$. It is immediately obvious from the above discussion that the quantum group construction (4) indeed satisfies the weight conservation and the BGRs directly obtained from the proposal of weight conservation include the non-standard BGRs as well as covering the standard ones.

An example of su(2) can be used to illustrate the above general analysis. In this case the universal R-matrix is

$$\hat{R}=q^{2(h\otimes h)}\sum_i q^{3/2i(i-1)}\{(q-q^{-1})^i/[i]\}E_+^i q^{ih}\otimes E_-^i q^{-ih} \tag{9}$$

where the generators h and $E_\pm$ of $U(\mathrm{su}(2))_q$ satisfy $[h, E_\pm]=\pm E_\pm$ and $[E_+, E_-]=[2h]$. For the angular momentum representation with the basis $|j, m\rangle$ [15], we have

$$\langle m_1, m_2|\hat{R}|m_3, m_4\rangle=\begin{cases}0 & \text{if } m_1+m_2\neq m_3+m_4\\ \neq 0 & \text{if } m_1+m_2=m_3+m_4.\end{cases} \tag{10}$$

Now, a question naturally arises: can we add some terms $\Delta\hat{R}$ that still satisfy weight conservation to the standard R-matrix (4) such that $\hat{R}=\hat{R}+\Delta\hat{R}$ also give a solution of the YBE? The answer is positive. In fact, we let

$$\Delta\hat{R}=\sum_{m+m'=n+n'} C_{m'n'r'}^{mnr}E_+^m E_-^n h^r\otimes E_+^{m'}E_-^{n'}h^{r'} \tag{11}$$

where the coefficients $C^{mnr}_{m'n'r'}$ can be determined by substituting $R = R + \Delta R$ into the YBE (5) for a given irreducible representation of $U(su(2))_q$ [14]. For the case $j = \frac{1}{2}$,

$$E_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad E_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad h = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

we determine

$$\Delta R_{1/2} = -(q^{-3/2} + q^{1/2}) E_- E_+ \otimes E_- E_+ . \tag{12}$$

For the case $j = 1$,

$$E_+ = [2]^{1/2}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \qquad E_- = [2]^{1/2}\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \qquad h = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

we determine

$$\begin{aligned}
\Delta R_1 = {} & (\omega q^{-6} - q^2)[2]^{-3}\{[2]^{-1}(E_-^2 E_+^2 \otimes E_-^2 E_+^2) + E - E_+^2 \otimes E_- E_+^2\} + (\omega^2 q^{-4} - 1)[2]^{-3} \\
& \times \{(E_+ E_- - [2]^{-1} E_+^2 E_-^2) \otimes E_-^2 E_+^2 + E_-^2 E_+^2 \otimes (E_+ E_- - [2]^{-1} E_+^2 E_-^2)\} \\
& + q^{-2}(q^2 - q^{-2})(1 - q^{-2}\omega)[2]^{-2} E_+^2 \otimes E_-^2 + q^{-1}\{\{\omega(q^2 - q^{-2})(q^{-4}\omega - 1)\}^{1/2} \\
& - (q - q^{-2})\}[2]^{-3}(E_+^2 E_- \otimes E_-^2 E_+ + E_- E_+^2 \otimes E_+ E_-^2) \\
& + (\omega q^{-2} - 1)[2]^{-2}(E_+ E_- - [2]^{-1} E_+^2 E_-^2) \otimes (E_+ E_- - [2]^{-1} E_+^2 E_-^2) \\
& \omega^3 = 1.
\end{aligned} \tag{13}$$

Equations (12) and (13) lead to two non-standard BGRS

$$R'_{1/2} = R_{1/2} + \Delta R_{1/2} = q^{-1/2}\begin{bmatrix} q & 0 & 0 & 0 \\ 0 & 1 & q - q^{-1} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -q^{-1} \end{bmatrix} \tag{14a}$$

$$R'_1 = R_1 + \Delta R_1 = \text{block diag}\,(A_1, A_2, A_3, A'_2, A'_1) \tag{14b}$$

$$A_1 = q^2 \qquad A'_1 = \omega q^{-6} \qquad A_2 = \begin{bmatrix} 1 & q - q^{-2} \\ 0 & 1 \end{bmatrix}$$

$$A'_2 = \begin{bmatrix} \omega^2 q^{-4}, & q^{-2}(\omega q^{-6} - 1) \\ 0 & \omega^2 q^{-4} \end{bmatrix}$$

$$A_3 = \begin{bmatrix} q^{-2} & q^{-1}\omega\{(q^2 - q^{-2})(q^{-4}\omega - 1)\}^{1/2} & (q^2 - q^{-2})(1 - \omega q^{-4}) \\ 0 & \omega q^{-2} & q^{-1}\omega\{(q^2 - q^{-2})(q^{-4}\omega - 1)\}^{1/2} \\ 0 & 0 & q^{-2} \end{bmatrix} \tag{14c}$$

which has been given in [15] and can be verified by the extended KDT to satisfy (5).

The discussion about non-standard BGRs associated with arbitrary classical Lie algebra based on this letter will be published elsewhere in which the explicit construction of representation for QC through the q-deformed boson realisation [14, 16] has been applied.

The authors are grateful to professor C N Yang for bringing to our attention the topics in this letter. This work was supported in part by the National Foundation of Science in China.

References

[1] Yang C N and Ge M L (eds) 1989 *Braid Group, Knot Theory and Statistical Mechanics* (Singapore: World Scientific)
[2] Drinfel'd V G 1986 *Proc. ICM Berkeley* p 798
[3] Jimbo M 1985 *Lett. Math. Phys.* **10** 63
[4] Jimbo M 1986 *Commun. Math. Phys.* **102** 537
[5] Kauffman L H 1988 Knot Theory and Applications *Ann. Math. Studies* **115** 1
[6] Ge M L, Wang L Y, Xue K and Wu Y S 1989 *Int. J. Mod. Phys.* **4** 3351
[7] Ge M L, Li Y Q and Xue K 1990 *J. Phys. A: Math. Gen.* **23** 619
[8] Ge M L, Li Y Q, Wang L Y and Xue K 1990 *J. Phys. A: Math. Gen.* **23** 605
[9] Ge M L, Gwa L H, Piao F and Xue K 1990 *preprint* ITP-SB-90-01 Stony Brook
[10] Couture M, Cheng Y, Ge M L and Xue K 1990 *preprint* ITP-SB-90-05 Stony Brook
[11] Akutsu Y and Wadati M 1988 *Commun. Math. Phys.* **117** 243
[12] Li Y Q 1989 Weight Conservation and Braid Group Representations *PhD thesis* Lanzhou University
[13] Rosso M 1989 *Commun. Math. Phys.* **124** 307
[14] Sun C P and Fu H C 1989 *J. Phys. A: Math. Gen.* **22** L983
[15] Lee H C 1989 *NATO Advanced Research Workshop on Physics and Geometry, Lake Tahoe*, 3-8 July 1989 contributed lecture
[16] Sun C P and Fu H C 1990 *Commun. Theor. Phys.* **13** 217