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LETTER TO THE EDITOR

The q-deformed boson realisation of the quantum group $SU(n)_q$ and its representations

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Abstract. The q-deformed boson realisation of the quantum group $SU(n)_q$ $((A_{n-1})_q)$ is constructed and certain types of representations of $SU(n)_q$ are obtained in the q-deformed Fock space by this boson realisation. The Jimbo representations of the quantum group $SU(2)_q$ are given as an example in this letter.

Recently, as a new type of algebraic structure, the q-deformation of Lie algebras has been discovered by Jimbo [1-3] and Drinfeld [4]. It is mathematically a Hopf algebra and loosely called the quantum group. It appears when one tries to solve the famous quantum Yang-Baxter equation (QYBE) from different physical models, such as the exactly soluble models in statistical mechanics, integrable models in two-dimensional field theory, etc. The representations of the quantum group associated with its QYBE are of central importance in these problems. So it is important to study the representations of quantum groups.

The boson realisation method is a useful method for studying representations of groups. It has been used to construct indecomposable representations of Lie algebras [5], Lie superalgebras [6] and Loop algebras [7] and give inhomogeneous differential realisations of Lie algebras [8]. In this letter, by introducing the q-deformation of the Bose algebra, we generalise the usual boson realisation method to obtain explicit expressions for the representations of the quantum group $SU(n)_q$ $((A_{n-1})_q)$.

We first consider a q-deformation of the one-state Bose algebra (we call it q Bose algebra $\mathscr{B}_q$) generated by operators b, $b^\dagger$, $\mathbb{1}$ and N in Hilbert space that satisfy

$$bb^\dagger - q^{-1}b^\dagger b = q^N \qquad [B, b^\dagger] = b^\dagger \qquad [N, b] = -b$$
$$[\mathbb{1}, x] = 0 \qquad \forall \times \in \mathscr{B}_q. \tag{1}$$

When the parameter $q \to 1$ (or $q = e^\hbar$, $\hbar \to 0$)$\mathscr{B}_q$ becomes the usual Bose algebra. It is easy to prove that

$$Nb^{\dagger n} = b^\dagger N + nb^{\dagger n}$$
$$b^\dagger q^N = q^{-1}q^N b^\dagger \qquad bq^N = qq^N b \tag{2}$$
$$bb^{\dagger n} = q^{1-n}[n]q^N b^{\dagger n-1} + q^{-n}b^{\dagger n}b \qquad [n] \equiv (q^n - q^{-n})/(q - q^{-1})$$

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by induction from (1). The relations in (2) can naturally be extended to the case of the n-state Bose algebra $\{b_i, b_i^\dagger, \mathbb{1}, N_i | i = 1, 2, \ldots, n\}$. For the n-state Bose algebra we define the q-analogous Fock space F_q (also called q-Fock space) with the basis

$$|m_i\rangle = |m_1, m_2, \ldots, m_n\rangle = \left(\prod_{i=1}^{n} \frac{b_i^{\dagger m_i}}{\sqrt{[m_i]!}}\right)|0\rangle \qquad [n]! \equiv [n][n-1] \ldots [2][1] \tag{3}$$

where the vacuum state $|0\rangle$ satisfies $b_i|0\rangle = 0$, $N_i|0\rangle = 0$ ($i = 1, 2, \ldots, n$). Then, it follows from (2) that

$$\begin{aligned} &b_j^\dagger|m_i\rangle = \sqrt{[m_j+1]}|m_i + \delta_{ij}\rangle \qquad b_j|m_i\rangle = \sqrt{[m_j]}|m_i - \delta_i - \delta_{ij}\rangle \\ &N_j|m_j\rangle = m_j|m_i\rangle \qquad j = 1, 2, \ldots, n. \end{aligned} \tag{4}$$

For given classical Lie algebra $\mathscr{L}$ with the Chevalley basis $\{h_i, e_i, f_i, i = 1, 2, \ldots, l\}$ which satisfies

$$\begin{aligned} &[h_i, h_j] = 0 \qquad [e_i, f_j] = \delta_{ij} h_i \\ &[h_i, e_j] = A_{ij} e_j \qquad [h_i f_j] = -A_{ij} e_j \end{aligned} \tag{5}$$

($\mathbf{A}$ is the Cartan matrix of $\mathscr{L}$). A q-deformed boson realisation (also called a q-boson realisation) of the quantum group $\mathscr{L}_q$ associated with the Lie algebra $\mathscr{L}$ is a mapping B of $\mathscr{L}$ onto the operator algebra $\mathscr{O}$ on the q-Fock space F_q, which satisfies

$$\begin{aligned} &[\tilde{h}_i, \tilde{h}_j] = 0 \qquad [\tilde{e}_i, \tilde{f}_j] = \delta_{ij}[\tilde{h}_i] \\ &[\tilde{h}_i \tilde{e}_j] = A_{ij}\tilde{e}_j \qquad [\tilde{h}_i, \tilde{f}_j] = -A_{ij}\tilde{f}_j \\ &\tilde{x} \equiv B(x) \qquad \forall \times \in \mathscr{L} \end{aligned} \tag{6}$$

where we define

$$[\hat{0}] = \frac{q^{\hat{0}} - q^{-\hat{0}}}{q - q^{-1}}$$

for any operator $\hat{0}$. The q-Boson realisation of $\mathscr{L}_q$ generated by $\{\tilde{h}_i, \tilde{e}_i, \tilde{f}_i, i = 1, 2, \ldots, l\}$ can be regarded as a subalgebra of $\mathscr{O}$, i.e. $\mathscr{L}_q \subset \mathscr{O}$.

Consider the Lie algebra $A_{n-1} = \mathrm{SU}(n)$ with the Chevalley basis

$$h_j = E_{jj} - E_{j+1,j+1} \qquad e_j = E_{jj+1} \qquad f_j = E_{j+1,j} \qquad j = 1, 2, \ldots n-1 \tag{7}$$

where E is a $n \times n$ matrix with elements $(E_{ij})_{\alpha\beta} = \delta_{i\alpha}\delta_{j\beta}$ ($\alpha, \beta = 1, 2, \ldots n$). The q-boson realisation of its quantum group $(A_{n-1})_q$ is given by

$$\tilde{h}_j = N_j - N_{j+1} \qquad \tilde{e}_j = b_j^+ b_{j+1} \qquad \tilde{f}_j = b_{j+1}^+ b_j \qquad j = 1, 2, \ldots n-1. \tag{8}$$

By direct computation we check

$$\begin{aligned} &[\tilde{h}_j, \tilde{h}_i] = 0 \qquad [\tilde{e}_i, \tilde{f}_j] = \delta_{ij}[\tilde{h}_i] \\ &[\tilde{h}_i \tilde{e}_j] = A_{ij}\tilde{e}_j \qquad [\tilde{h}_i, \tilde{f}_j] = -A_{ij}\tilde{f}_j \end{aligned} \tag{9}$$

where $A_{ij} = 2\delta_{ij} - \delta_{ij+1} - \delta_{i,j-1}$ is a element of the cartan matrix $\mathbf{A}$ of $\mathrm{SU}(n)$.

Now, we construct the representations of $(A_{n-1})_q$ from the Boson realisation (8) by defining the action of $(A_{n-1})_q$ on F_q

$$\Gamma(x)|u\rangle = x|u\rangle \qquad \forall x \in (A_{n-1})_q \qquad |u\rangle \in F_q$$

obtaining

$$\begin{aligned}&\Gamma(\tilde{h}_j)|m_i\rangle=(m_j-m_{j+1})|m\rangle\\&\Gamma(\tilde{e}_j)|m\rangle=([m_{j+1}][m_j+1])^{1/2}|m_i+\delta_{ij}-\delta_{i,j+1}\rangle\\&\Gamma(\tilde{f}_j)|m\rangle=([m_j][m_{j+1}+1])^{1/2}|m_i-\delta_{ij}+\delta_{i,j+1}\rangle.\end{aligned}\tag{10}$$

Because the sum $\Sigma_{i=1}^{n} m_i$ remains the same under the action of Γ, F_q is decomposed into the direct sum of all the invariant subspaces $F_q^{[m]}$ ($m=0,1,2,\ldots$):

$$\left\{|m_i\rangle\middle|\sum_{i=1}^{n} m_i=m\right\}.$$

Each one of these subspaces $F_q^{[m]}$ ($m=0,1,\ldots$) carries an irreducible representation $\Gamma^{[m]}$ of $(A_{n-1})_q$ with dimension

$$d_n^m=\frac{(n+m-1)!}{m!(n-1)!}.\tag{11}$$

When $q\to 1(q=e,\hbar\to 0)$, the representation $\Gamma^{[m]}$ becomes a symmetrised representation of Lie algebra A_{n-1} labelled by the Young diagram

$$\boxed{}\boxed{}\boxed{\ldots}\boxed{}\boxed{}\boxed{}.$$

In fact, from (10) we can give explicit matrices of some representations of the quantum group $(A_{n-1})_q$. For example, when $n=3$ and $M=2$, the six-dimensional representation of $SU(3)_q=(A_2)_q$ on the basis $\{|2,0,0\rangle, |0,2,0\rangle, |0,0,2\rangle, |1,1,0\rangle, |0,11\rangle, |101\rangle\}$ is

$$\begin{aligned}&\Gamma^{[2]}(\tilde{h}_1)=\mathrm{diag}[2,-2,0,0,-1,1]\\&\Gamma^{[2]}(\tilde{h}_2)=\mathrm{diag}[0,2,-2,1,0,-1]\end{aligned}$$

$$\Gamma^{[2]}(\tilde{e}_1)=\begin{pmatrix}0&0&0&\sqrt{[2]}&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&\sqrt{[2]}&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&1&0\end{pmatrix}\qquad\Gamma^{[2]}(\tilde{e}_2)=\begin{pmatrix}0&0&0&0&0&0\\0&0&0&0&\sqrt{[2]}&0\\0&0&0&0&0&0\\0&0&0&0&0&1\\0&0&\sqrt{[2]}&0&0&0\\0&0&0&0&0&0\end{pmatrix}$$

$$\Gamma^{[2]}(f_1)=\begin{pmatrix}0&0&0&0&0&0\\0&0&0&\sqrt{[2]}&0&0\\0&0&0&0&0&0\\\sqrt{[2]}&0&0&0&0&0\\0&0&0&0&0&1\\0&0&0&0&0&0\end{pmatrix}\qquad\Gamma^{(2)}(\tilde{f}_2)=\begin{pmatrix}0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&\sqrt{[2]}&0\\0&0&0&0&0&0\\0&\sqrt{[2]}&0&0&0&0\\0&0&0&1&0&0\end{pmatrix}\tag{12}$$

The Jimbo standard angular momentum representation of the quantum group $SU(2)_q$ is only a special example of the above discussion for $n=1$. Let $J_+=\tilde{e}_1$, $J-=\tilde{fi}$ and $J_3=\frac{1}{2}h_1$, it follows from (10) that

$$\begin{aligned}&\Gamma(J+)|n_1n_2\rangle=([n_1+1][n_2]^{1/2}|n_1+1,n_2-1\rangle\\&\Gamma(J-)|n_1n_2\rangle=([n_1][n_2+1]^{1/2}|n_1-1n_2+1\rangle\\&\Gamma'(J_3]\gamma n_1n_2\rangle=\tfrac{1}{2}(n_1-n_2)|n_1n_2\rangle.\end{aligned}\tag{13}$$

By defining angular momentum basis $|\Psi_{jm}\rangle = |j+m, j-m\rangle$ for the representation, (13) is rewritten as

$$\Gamma(J\pm)|\Psi_{jm}\rangle = (j\pm m+1][j\mp m])^{1/2}|\Psi_{jm\pm1}\rangle \tag{14}$$

$$\Gamma(J_3)|\Psi j_m\rangle = m|\Psi_{jm}\rangle.$$

This is just the Jimbo representation, which becomes the standard irreducible representation of SU(2) when $q\to1$.

The method used in this paper can be applied to C_{n-1} and also expected to study quantum groups of any classical Lie algebra by further generalisations.

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