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LETTER TO THE EDITOR

Boson realisation of Virasoro and Kac–Moody algebras and their indecomposable representations

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Abstract. In this letter we give the boson realisation of Virasoro and Kac-Moody algebras without central terms and construct their indecomposable representations from some representations of the Heisenberg-Weyl algebra. The boson realisation of the Kac-Moody algebra $\hat{\mathrm{su}}(2)$ associated to the Lie algebra su(2) is discussed in detail.

Virasoro and Kac-Moody algebras have appeared in many areas of physics [1]. In superstring theory the Virasoro algebra plays an important role [2]. The Kac-Moody symmetries are found in (1+1)-dimensional classical field theory, Yang-Mills theory and so on. More recently, the new Virasoro and Kac-Moody symmetries in the non-linear σ model [3] and the solution space of the Ernst equation [4] have been studied by Hou and Li.

Although the representation theory of Virasoro and Kac-Moody algebras have been studied by many authors (e.g. [5]), we will pay attention to their physical indecomposable representations. By our method used to study the Lie algebras [6] and Lie superalgebras [7], we discuss Virasoro and Kac-Moody algebras in this letter.

On the quotient space $\Omega = \bar{\Omega}/L$:

$$\left\{ f(k_i, S_i) = \prod_{i=1}^{N} (b_i^{+k_i} b_i^{S_i}) \bmod L \,\middle|\, S_i, k_i \in \mathbb{N}, i = 0, 1, 2, \ldots, N \right\} \tag{1}$$

of the universal enveloping algebra $\bar{\Omega}$ of the Heisenberg-Weyl algebra $H : \{b_i^+, b_i, e\}$, where L is a left ideal generated by the element $e - 1$, an indecomposable representation of H is obtained in [6] as

$$\begin{aligned} &\rho(b_i^+) f(k_i, S_i) = f(k_i + \delta_{il}, S_i) \\ &\rho(b_i) f(k_i, S_i) = f(k_i, S_i + \delta_{il}) + k_i f(k_i - \delta_{il}, S_i) \\ &\rho(e) f(k_i, S_i) = f(k_i, S_i). \end{aligned} \tag{2}$$

From the above representation, we will construct certain types of representation of Virasoro and Kac-Moody algebras.

Let g be a Lie algebra with generators $\{T^a, a = 1, 2, \ldots, M\}$ that satisfy the commutation relations

$$[T^a, T^b] = \sum_{c=1}^{M} \mathscr{S}_c^{ab} T^c \tag{3}$$

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where complex numbers $\mathscr{S}_c^{ab}$ are the structure constants of Lie algebra g. For each given N-dimensional representation $P=[P_{ij}]$, $(i=1,2,\ldots,N)$ of Lie algebra g, we define

$$T_l^a = b_0^{+l}[b_1^+, b_2^+, \ldots, b_N^+][P_{ij}]\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_N \end{bmatrix} \tag{4}$$

where because of the existence of the left inverse of b_0^+, l can be taken as a integer, i.e. $l\in\mathbb{Z}$.

These T_l^a form an untwisted affine Kac–Moody algebra $\hat{g}$ without a central term. In fact, considering the explicit expression of T_l^a

$$T_l^a = b_0^{+l}\sum_{m,n}^{N} P_{mn}(T^a)b_m^+ b_n \tag{5}$$

we easily prove

$$[T_l^a, T_h^b] = \sum_{c=1}^{M} \mathscr{S}_c^{ab} T_{l+h}^c. \tag{6}$$

Defining the operators

$$L_l = -b_0^{l+1} b_0 \tag{7}$$

we obtain a Virasoro algebra $\hat{v}:\{L_l | l\in\mathbb{Z}\}$ that satisfies

$$[L_l, L_h] = [l-h]L_{l+h} \tag{8}$$

$$[L_l, T_h^a] = -hT_{l+h}^a. \tag{9}$$

Then (4) and (7) give the boson realisation of the Kac–Moody and Virasoro algebras.

According to (2), (4) and (7), extending the quotient space Ω to $k_0\in\mathbb{Z}$, we obtain the representations

$$\Gamma(T_l^a)f(k_i,S_i) = \sum_{m,n} P_{mn}(T^a)f(k_i+l\delta_{i0}+\delta_{mi}, S_i+\delta_{in}) + \sum_{mn} P_{mn}(T^a)f(k_i-\delta_{in}+l\delta_{i0}+\delta_{im}, S_i) \tag{10}$$

$$\Gamma(L_l)f(k_i,S_i) = f(K_i+\delta_{i0}l, \delta_i+\delta_{i0}) + k_0 f(k_i+l\delta_{i0}, S_i) \tag{11}$$

of the Kac–Moody and Virasoro algebras. By the same analysis as that in [6, 7], we see that the above representations are indecomposable.

On the quotient space $V=\Omega/L_+$

$$\{F(k_i) = F(k_i, 0) \bmod L_+\}$$

where the left ideal L_+ is generated by elements $b_i-\Lambda_i$ $(i=1,2,\ldots,N;\ \Lambda_i\in\mathbb{C})$ the representations (10) and (11) induce the new indecomposable representation

$$\begin{aligned}\Gamma(T_l^a)F(k_i) &= \sum_{m,n} P_{mn}(T^a)\Lambda_n F(k_i+l\delta_{i0}+\delta_{mi}) + \sum_{mn} k_n P_{mn}(T^a)F(k_i+l\delta_{i0}-\delta_{in}+\delta_{im}) \\ \Gamma(L_l)\Gamma(k_i) &= \Lambda F(k_i+\delta_{i0}l) + k_0 F(k_i+l\delta_{i0})\end{aligned} \tag{12}$$

for $\Lambda_i \neq 0$.

For the case with $\Lambda_i = 0$ $(i = 0, 1, \ldots, N)$, the representations

$$\begin{aligned}\Gamma(T_l^a)F(k_i) &= \sum_{m,n} k_n P_{mn}(T^a)F(k_i + l\delta_{i0} - \delta_{mi} + \delta_{ni}) \\ \Gamma(L_l)F(k_i) &= \Lambda_0 F(k_i + l\delta_{i0})\end{aligned} \tag{13}$$

given by (12) are completely reducible. This is due to the invariance of the sum $\Sigma_{i=1}^{N} k_i$ under the action of the representation (13).

Following the above general discussion, we study the Kac–Moody algebra $\widehat{\mathrm{su}}(2)$ associated with the Lie algebra su(2). According to (4), we have the boson realisation of $\widehat{\mathrm{su}}(2)$

$$\begin{aligned}J_l^+ &= b_0^{+l} b_1^+ b_2 \qquad J_l^- = b_0^{+l} b_2^+ b_1 \\ J_l^3 &= \tfrac{1}{2} b_0^{+l}[b_1^+ b_1 - b_2^+ b_2]\end{aligned} \tag{14}$$

corresponding to the Pauli representation of su(2). The boson realisation of an indecomposable representation of $\widehat{\mathrm{su}}(2)$ is obtained as

$$\begin{aligned}\Gamma(J_l^+)F(k_0, k_1, k_2) &= \Lambda_2 F(k_0 + l, k_1 + 1, k_2) + k_2 F(k_0 + l, k_1 + 1, k_2 - 1) \\ \Gamma(J_l^-)F(k_0, k_1, k_2) &= \Lambda_1 F(k_0 + l, k_1, k_2 + 1) + k_1 F(k_0 + l, k_1 - 1, k_2 + 1) \\ \Gamma(J_l^3)F(k_0, k_1, k_2) &= \tfrac{1}{2}\Lambda_1 F(k_0 + l, k_1 + 1, k_2) - \tfrac{1}{2}\Lambda_2 F(k_0 + l, k_1, k_2 + 1) \\ &\quad + \tfrac{1}{2}(k_1 - k_2)F(k_0 + l, k_1, k_2).\end{aligned} \tag{15}$$

When $\Lambda_0 = \Lambda_1 = \Lambda_2 = 0$, define a new basis for the space V

$$\tilde{F}[M, n, k] = F(k, n, M - n) \qquad k \in \mathbb{Z};\ n, M \in \mathbb{N}.$$

On this space, we obtain a representation of $\widehat{\mathrm{su}}(2)$

$$\begin{aligned}\Gamma(J_l^+)\tilde{F}[M, n, k] &= (M - n)\tilde{F}[M, n + 1, k + l] \\ \Gamma(J_l^-)\tilde{F}[M, n, k] &= n\tilde{F}[M, n - 1, k + l] \\ \Gamma(J_3^+)\tilde{F}[M, n, k] &= (2n - M)\tilde{F}[M, n, k + l].\end{aligned} \tag{16}$$

Then, for fixed $M \in \mathbb{N}$, $\{\tilde{F}(M, n, k) \mid n \in \mathbb{N}, k \in \mathbb{Z}\}$ forms an invariant subspace $V^{[M]}$ of V, and V can be written as the direct sum of spaces, i.e.

$$V = V^{[0]} \oplus V^{[1]} \oplus V^{[2]} \oplus \ldots \oplus V^{[N]} \oplus .$$

Thus, for the case with $\Lambda_0 = \Lambda_1 = \Lambda_2 = 0$, the representation (16) is completely reducible. On each space $V^{[M]}$, (16) gives an irreducible representation of $\widehat{\mathrm{su}}(2)$.

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