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Boson–fermion realisation of indecomposable representations for Lie superalgebras

Chang-Pu Sun

Physics Department, Northeast Normal University, Changchun, Jilin Province, China

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Abstract. Indecomposable representations of Lie superalgebras are studied on quotient spaces of the universal enveloping algebra of the Heisenberg-Weyl superalgebra $\{b_i^+, b_i, f_u^+, f_u, e\}$ by boson-fermion realisation. These representations are constructed from certain types of indecomposable representations of the Heisenberg-Weyl superalgebra and induce usual irreducible representations on invariant subspaces of a quotient space. As a physically significant example, the explicit form of the boson-fermion realisation of indecomposable representations of the classical Lie superalgebra SU(2|1) are obtained and discussed in this paper.

1. Introduction

About a decade ago a new kind of symmetry principle appeared in physics, namely supersymmetry. The generators of supersymmetry transformations form a Lie superalgebra whose odd generators mix bosons and fermions. Supersymmetry has been applied to many areas of physics, such as field theory, nuclear physics and superstrings [1]. It was necessary to develop the representation theory of Lie superalgebras.

Recently, the further development of representation theory of Lie algebras and superalgebras has been undertaken by Gruber and his co-workers [2-6]. Making use of the pure algebraic method, they have investigated indecomposable representations of some Lie algebras and superalgebras on the spaces of their universal enveloping algebras, their induced representations on quotient spaces and their subduced representations on invariant subspaces.

In this paper, the physical basis will be adopted. Considering that any Lie superalgebra is only an isomorphism of a quotient subalgebra of the universal enveloping algebra of the Heisenberg-Weyl superalgebra, we can regard a representation of this quotient subalgebra as its representation, which is called the boson-fermion realisation of representations, of the Lie superalgebra. In fact, the boson-fermion realisation of Lie algebras and their irreducible representations have been discussed on Fock space [7-10]. Extending our discussions [11] about the boson realisation of indecomposable representations of SO(3), we will study the boson-fermion realisation of indecomposable representations of Lie superalgebras.

The plan of this paper is as follows. In § 2, by extending the discussion about the one-state Heisenberg-Weyl algebra given by Gruber *et al*, we obtain indecomposable representations of the Heisenberg-Weyl superalgebra. Using these representations and the boson-fermion representation realisation of Lie superalgebras shown in § 3, we construct some indecomposable representations of Lie superalgebras and analyse their properties in § 4. Finally in § 5 we discuss the boson–fermion realisation of indecomposable representations for the classical Lie superalgebra of SU(2|1) on both the natural basis and the coupling basis.

2. Heisenberg–Weyl superalgebra and representation

Creation and annihilation operators b_i^+ and b_i $(i=1,2,\ldots,m)$ for boson states, f_u^+ and f_u $(u=1,2,\ldots,n)$ for fermion states and the unit operator e span a Lie superalgebra with the non-vanishing (anti)commutation relations

$$[b_i, b_i^+]=e \qquad \{f_u^+, f_u\}=e \tag{1}$$

which is an extension of the quantum mechanical algebra, namely the Heisenberg–Weyl algebra [3]. According to the Poincaré–Birckhoff–Witt theorem, we choose for its universal enveloping algebra $\bar{\Omega}$ a basis

$$F(k_i, s_i, d_u, \beta_u, r)=e^r \prod_{i=1}^{m} (b_i^{+k} b_i^{s_i}) \prod_{u=1}^{n} f_u^{+\alpha_u} \prod_{i=1}^{n} f_u^{\beta_u} \tag{2}$$

where $k_i, s_i, r=0,1,2,\ldots,$ and $\alpha_u, \beta_u=0,1$. Each vector in the space of $\bar{\Omega}$ is a linear combination of the basis with complex coefficients. Then, we consider an extension $\tilde{\Omega}$ of the space $\bar{\Omega}$, in which each element is a linear combination of the basis whose coefficients are elements of the Grassmann algebra with generators $\xi_1, \xi_2, \ldots, \xi_n$. This approach is analogous to that used by Ohnuki and Kashiva [12] to study the coherent states of the fermion.

We will discuss this subject on the quotient space

$$\Omega=(\tilde{\Omega}/I)\colon \{F(k_i, s_i, \alpha_u, \beta_u)=F(k_i, s_i, \alpha_u, \beta_u, 0) \bmod I\}$$

corresponding to the two-sided ideal I generated by the element $e-1$. The generalised Fock space is defined as a quotient space of Ω

$$V=(\Omega/L)\colon \{F(k_i, \alpha_u)=F(k_1, \alpha_u, 0) \bmod L\}$$

where L is the left ideal generated by the elements $b_i-\Lambda_i$ and $f_u-\xi_u$ $(i=1,2,\ldots,m;$ $u=1,2,\ldots,n)$, Λ_i are m complex numbers and ξ_u are n generators of the Grassmann algebra). On this space, the indecomposable representations of the Heisenberg–Weyl superalgebra is

$$\begin{aligned}
&P(b_t^+)F(k_i, \alpha_u)=F(k_i+\delta_{it}, \alpha_u)\\
&P(b_t)F(k_i, \alpha_u)=\Lambda_t F(k_i, \alpha_u)+k_t F(k_i-\delta_{it}, \alpha_u)\\
&P(f_w^+)F(k_i, \alpha_u)=(-1)^{\Sigma_{u=1}^{w-1}\alpha_u}(1-\alpha_w)F(k_i, \alpha_u+\delta_{uw})\\
&P(f_w)F(k_i, \alpha_u)=(-1)^{\Sigma_{u=1}^{w-1}\alpha_u}\alpha_w F(k_i, \alpha_u-\delta_{uw})+\xi_u F(k_i, \alpha_u)
\end{aligned} \tag{3}$$

where $P(e)$ is a unit matrix. Equation (3) shows that V is the usual Fock space when $\Lambda_t=0=\xi_w$, $t=1,2,\ldots,m$, $w=1,2,\ldots,n$.

3. Boson–fermion realisation of Lie superalgebra

Let $G=G_0\oplus G_1$ be a finite-dimensional Lie superalgebra with the generators

$$x_1, x_2, \ldots, x_k \in G_0 \qquad y_1, y_2, \ldots, y_l \in G_1$$

which satisfy the following (anti)commutation relations

$$
\begin{aligned}
&[x_m, \chi_n] = \sum_P f^P_{mn} x_P \\
&[x_m, y_\alpha] = \sum_\beta F^\beta_{m\alpha} y_\beta \\
&\{y_\alpha, y_\beta\} = \sum_m A^m_{\alpha\beta} x_m
\end{aligned}
\tag{4}
$$

where f^P_{mn}, $F^\beta_{m\alpha}$ and $A^m_{\alpha\beta}$ are the structure constants of the Lie superalgebra G.

For each given $(m+n)$-dimensional representation

$$
\bar{P}(z) = \begin{array}{c} \begin{array}{cc} \quad\; m & \quad\;\; m+n \end{array} \\ \left[\begin{array}{c|c} A(z) & B(z) \\ \hline C(z) & D(z) \end{array}\right] \end{array} \begin{array}{l} \\ m \\ m+n \end{array} \qquad z \in G
$$

of a Lie superalgebra G, which satisfies

$$
B(x) = 0 \qquad C(x) = 0 \qquad x \in G_0 \qquad A(y) = 0 \qquad D(y) = 0 \qquad y \in G_1 \tag{5}
$$

we can construct an operator representation of G; $R: G \to \Omega$, i.e.

$$
R(z) = [b^+_1, b^+_2, \ldots, b^+_m, f^+_1, \ldots, f^+_n] \left[\begin{array}{c|c} A(z) & B(z) \\ \hline C(z) & D(z) \end{array}\right] \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \\ f_1 \\ \vdots \\ f_n \end{bmatrix} \tag{6}
$$

being the boson–fermion realisation of the Lie superalgebra G.

In fact, due to the properties (5) of the matrices $P(z)$, the explicit expression of $R(z)$ is given as

$$
\begin{aligned}
R(x) &= \sum_{k,l=1}^{m} A_{kl}(x) b^+_k b_l + \sum_{\alpha,\beta=1}^{n} D_{\alpha\beta}(x) f^+_\alpha f_\beta \qquad x \in G_0 \\
R(y) &= \sum_{k=1}^{m} \sum_{\alpha=1}^{n} (B_{k,\alpha}(y) b^+_k f_\alpha + C_{\alpha,k}(y) f^+_\alpha b_k) \qquad y \in G_1 .
\end{aligned}
\tag{7}
$$

We can easily verify that $R(x)$ and $R(y)$ satisfy

$$
\begin{aligned}
&R([x, z]) = [R(x), R(z)] \qquad z \in G \\
&R(\{y, y'\}) = \{R(y), R(y')\} \qquad y' \in G_1 .
\end{aligned}
$$

Thus (6) gives an operator representation of the Lie superalgebra G.

As an example, we study the fundamental representation of the classical Lie superalgebra SU(2|1) [13]. Its generators are

$$
\boldsymbol{L} = \begin{bmatrix} \boldsymbol{\sigma}/2 & & 0 \\ & & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad L_4 = \frac{1}{2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \qquad V_1 = \frac{1}{2} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}
$$

$$V_2 = \tfrac{1}{2}\begin{bmatrix} 0 & 0 & -\mathrm{i} \\ 0 & 0 & 0 \\ \mathrm{i} & 0 & 0 \end{bmatrix} \qquad T_1 = \tfrac{1}{2}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \qquad T_2 = \tfrac{1}{2}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -\mathrm{i} \\ 0 & \mathrm{i} & 0 \end{bmatrix}$$

where σ_i ($i = 1, 2, 3$) are the Pauli matrices, I_m ($m = 1, 2, 3, 4$) are the generators forming a Lie subalgebra $\mathrm{SU}(2)\otimes\mathrm{U}(1)$ of $\mathrm{SU}(2|1)$ and V_1, V_2, T_1 and T_2 are the supergenerators. We obtain the boson-fermion realisation of the Lie superalgebra of $\mathrm{SU}(2|1)$:

$$\begin{aligned} &L_+ = b_1^+ b_2 \qquad L_- = b_2^+ b_1 \\ &L_3 = \tfrac{1}{2}(b_1^+ b_1 - b_2^+ b_2) \qquad L_0 = b_1^+ b_1 + f^+ f \\ &V_+ = b_1^+ f \qquad V_- = f^+ b_1 \qquad T_+ = b_2^+ f \qquad T_1 = f^+ b_2 \end{aligned} \tag{8}$$

where

$$\begin{aligned} &L_\pm = L_1 \pm \mathrm{i} L_2 \qquad L_0 = L_3 + L_4 \\ &T_\pm = T_1 \pm \mathrm{i}\dot{T}_2 \qquad V_\pm = V_1 \pm \mathrm{i} V_2 . \end{aligned}$$

It is easy to see that the generators of a Lie superalgebra obtained by the boson-fermion realisation (7) do not change the particle number $N = \Sigma_{i=1}^m K_i + \Sigma_{u=1}^n \alpha_u$ in Fock states, for

$$[\hat{N}, R(z)] = 0 \qquad \hat{N} = \sum_{i=1}^m b_i^+ b_i + \sum_{u=1}^n f_u^+ f_u .$$

4. Representations of Lie superalgebras

By making use of the representation (3) of the Heisenberg-Weyl superalgebra, a representation $\bar{P}$ of the quotient algebra Ω as a Lie superalgebra, is given by

$$\Gamma(f(k_i, s_i, \alpha_u, \beta_u)) = \prod_{i=1}^m (P(b_i^+))^{k_i} (P(b_i))^{s_i} \prod_{u=1}^n (P(f_u^+))^{\alpha_u} \prod_{u=1}^n (P(f_u))^{\beta_u} . \tag{9}$$

Naturally, (9) gives a class of representations of Lie subalgebras of Ω.

We can regard the boson-fermion realisation of G: $\{R(z), z \in G\}$ as a Lie subalgebra of the quotient algebra Ω and its representation is given by

$$\Gamma(z) = \sum_{l,l'}^{m+n} \bar{P}_{ll'}(z) P[a_l^+] P[a_l] \tag{10}$$

where operators a_l are defined as

$$a_l = \begin{cases} b_l & 1 \leqslant l \leqslant m \\ f_{l-m} & m+1 \leqslant l \leqslant m+n . \end{cases} \tag{11}$$

On the space

$$V = (\Omega/L)\colon \{F(k_i, \alpha_u), k_i \in \mathbb{N}, \alpha_u = 0, 1)\} \qquad \mathbb{N} = \{0, 1, 2, \ldots\}$$

the representation (10) is explicitly written as

$$\begin{aligned} \Gamma(x) F(k_i, \alpha_u) = &\sum_{k,l=1}^m A_{k,l}(x)[\Lambda_l F(k_i + \delta_{ik}, \alpha_u) + k_l F(k_i - \delta_{il} + k_{ik}, \alpha_u)] \\ &+ \sum_{w,v=1}^n D_{uv}(x)[\alpha_v(1 - \alpha_v + \delta_{wv})(-1)^{\Sigma_{\min(w,v)-1}^{\max(w,v)-1} \alpha_u + \theta(v, w-1)} \\ &\times F(k_i, \alpha_u - \delta_{uv} + \delta_{uw}) + (1 - \alpha_w)\xi_v(-1)^{\Sigma_{u=1}^{w-1} \alpha_u} F(k_i, \alpha_u + \delta_{uw})] \end{aligned} \tag{11a}$$

$$\Gamma(y)F(k_i,\alpha_u)=\sum_{k=1}^{m}\sum_{w=1}^{n}\{B_{k,w}(y)[\xi_w F(k_i+\delta_{ki},\alpha_u)+\alpha_w(-1)^{\Sigma_{u=1}^{w-1}\alpha_u}F(k_i+\delta_{ki},\alpha_u-\delta_{uw})]$$
$$+(-1)^{\Sigma_{u=1}^{w-1}\alpha_u}(1-\alpha_w)C_{wk}(y)[\Lambda_k F(k_i,\alpha_u+\delta_{uw})+k_k F(k_i-\delta_{ik},\alpha_u+\delta_{uw})]\} \tag{11b}$$

by using (3) and (7), where

$$\min(w,v)=\begin{cases} w & w\leqslant v \\ v & w>v \end{cases} \qquad \max(w,v)=\begin{cases} w & w\geqslant v \\ v & w<v \end{cases}$$

$$\theta(v,w)=\begin{cases} 0 & v\geqslant w \\ 1 & v<w. \end{cases}$$

From (11a) and (11b), it follows that the sum $\Sigma_{i=1}^{m} k_i+\Sigma_{u=1}^{n}\alpha_u$ does not decrease under the action of the representation Γ and the subspace

$$V_N\colon \left\{F(k_i,\alpha_u)\,\middle|\,\sum_{i=1}^{m}k_i+\sum_{u=1}^{n}\alpha_u\geqslant N+1,\ N\in\mathbb{N}\right\}$$

is invariant, for which no invariant complementary subspace exists when $\Lambda_i\neq 0$ or $\xi_u\neq 0$. Thus, the representation given by (11a) and (11b) on the space V is reducible and indecomposable for the cases with $\Lambda_i\neq 0$ or $\xi_u\neq 0$.

It is easy to see that there exists an invariant subspace chain of the space V

$$V\supset V_1\supset V_2\supset\ldots\supset V_N\supset V_{N+1}\supset\ldots.$$

Correspondingly, there are some finite-dimensional quotient spaces

$$V_{(N,k)}=(V_N/V_{N+k})\colon \left\{F(k_i,\alpha_u)\bmod V_N \bmod V_{N+k}\,\middle|\, N+1\leqslant\sum_i k_i+\sum_u \alpha_u\leqslant N+k\right\}$$
$$N,k\in\mathbb{N} \tag{12}$$

with the dimensions

$$D_{(N,k)}=\sum_{\alpha=N+1}^{N+k}\sum_{t=0}^{n}\frac{(\alpha-t+m-1)!\,n!}{(\alpha-t)!\,(n-t)!\,(m-1)!\,t!}. \tag{13}$$

On quotient space $V(N,k)$, the infinite-dimensional indecomposable representation Γ induces a finite-dimensional representation.

In particular, when $\Lambda_i=0=\xi_u$, the sum $\Sigma_{i=1}^{m}k_i+\Sigma_{u=1}^{n}\alpha_u$ remains the same under the action of the representation Γ and V is a direct sum of all the invariant subspaces

$$V^{[N]}\colon \left\{F(k_i,\alpha_u)\,\middle|\,\sum_{i=1}^{m}k_i+\sum_{u=1}^{n}\alpha_u=N\right\} \tag{14}$$

or $V=\Sigma_{N=0}^{\infty}\oplus V^{[N]}$. Therefore, when $\Lambda_i=0=\xi_u$, the representation given by (11a) and (11b) is completely reducible.

On each invariant subspace $V^{[N]}$, an irreducible representation $\Gamma^{[N]}$ with the dimension

$$D_{(N)}=\sum_{t=0}^{N}\frac{(N-t+m-1)!\,n!}{(N-t)!\,(n-t)!\,(m-1)!\,t!} \tag{15}$$

is subduced and has obvious physical meaning: The symmetry generators in G_0 maintain boson number $\Sigma_{i=1}^{m}k_i$ and fermion number $\Sigma_{u=1}^{n}\alpha_u$ respectively, while the supersymmetry generators in G_1 mix bosons and fermions in the states spanning an invariant representation space for the Lie superalgebra G.

Restricting the representation $\Gamma^{(N)}$ of G to the symmetry subalgebra G_0, we have the branch law

$$\Gamma^{[N]}\Big|_{G_0} = \sum_{t=0}^{\min(N,n)} \oplus D^{[N-t]}$$

where $D^{[l]}$ is an irreducible representation of the symmetry subalgebra G_0 on the G_0 invariant subspace

$$S_{(l)}: \left\{ F(k_i, \alpha_u) \Big| \sum_{i=1}^{m} k_i = l \text{ or } \sum_{u=1}^{n} \alpha_u = N - l \right\}.$$

5. Representations of SU(2|1) as physical examples

As an example of the above general discussion, the classical Lie superalgebra SU(2|1) is studied in this section.

According to (11a), (11b) and (8), on the space $V = \Omega/L$ with the natural basis $F(k_1, k_2, \alpha)$, $(k_1, k_2 \in \mathbb{N}, \alpha = 0, 1)$, we obtain an infinite-dimensional indecomposable representation of SU(2|1) as

$$\begin{aligned}
\Gamma(L_+)F(k_1, k_2, \alpha) &= \Lambda_2 F(k_1+1, k_2, \alpha) + k_2 F(k_1+1, k_2-1, \alpha) \\
\Gamma(L_-)F(k_1, k_2, \alpha) &= \Lambda_1 F(k_1, k_2+1, \alpha) + k_1 F(k_1-1, k_2+1, \alpha) \\
\Gamma(L_3)F(k_1, k_2, \alpha) &= \tfrac{1}{2}(\Lambda_1 F(k_1+1, k_2, \alpha) - \Lambda_2 F(k_1, k_2+1, \alpha) + (k_1 - k_2)F(k_1, k_2)) \\
\Gamma(L_0)F(k_1, k_2, \alpha) &= \Lambda_1 F(k_1+1, k_2, \alpha) + (k_1+\alpha)F(k_1, k_2, \alpha) \\
&\quad + (\alpha - 1)\xi F(k_1, k_2, \alpha+1) \qquad (16) \\
\Gamma(V_+)F(k_1, k_2, \alpha) &= \alpha F(k_1+1, k_2, \alpha-1) + \xi F(k_1+1, k_2, \alpha) \\
\Gamma(V_-)F(k_1, k_2, \alpha) &= (1-\alpha)(\Lambda_1 F(k_1, k_2, \alpha+1) + k_1 F(k_1-1, k_2, \alpha+1)) \\
\Gamma(T_+)F(k_1, k_2, \alpha) &= \alpha F(k_1, k_2+1, \alpha-1) + \xi F(k_1, k_2+1, \alpha) \\
\Gamma(T_-)F(k_1, k_2, \alpha) &= (1-\alpha)(\Lambda_2 F(k_1, k_2, \alpha+1) + k_2 F(k_1, k_2-1, \alpha+1))
\end{aligned}$$

for the cases with $\Lambda_1 \neq 0$ or $\Lambda_2 \neq 0$ or $\xi \neq 0$.

Corresponding to an invariant subspace

$$V(N): \{F(k_1, k_2, \alpha) | k_1 + k_2 + \alpha \geqslant N+1\}$$

the quotient space $V/V(N)$ has a finite dimension $D(N) = (N+1)^2$, on which the above representation of SU(2|1) induces a finite-dimensional representation.

If we choose for the space V a coupling basis

$$|j, m, \alpha\rangle = \frac{F(j+m, j-m-\alpha, \alpha)}{[(j+m)!(j-m-\alpha)!]^{1/2}}$$

where $m = j - \alpha, j - \alpha - 1, \ldots, -(j-1), -j$ for fixed half-integer j, then the representation (16) is rewritten as

$$\begin{aligned}
\Gamma(L_+)|j, m, \alpha\rangle &= \Lambda_2 (j+m+1)^{1/2} |j+\tfrac{1}{2}, m+\tfrac{1}{2}, \alpha\rangle \\
&\quad + [(j+m+1)(j-m-\alpha)]^{1/2} |j, m+1, \alpha\rangle
\end{aligned}$$

$$
\begin{aligned}
\Gamma(L_-)|j, m, \alpha\rangle &= \Lambda_1(j-m-\alpha+1)^{1/2}|j+\tfrac{1}{2}, m-\tfrac{1}{2}, \alpha\rangle \\
&\quad +[(j+m)(j-m-\alpha+1)]^{1/2}|j, m-1, \alpha\rangle \\
\Gamma(L_3)|j, m, \alpha\rangle &= \tfrac{1}{2}\Lambda_1(j+m+1)^{1/2}|j+\tfrac{1}{2}, m+\tfrac{1}{2}, \alpha\rangle \\
&\quad -\tfrac{1}{2}\Lambda_2(j-m-\alpha+1)^{1/2}|j+\tfrac{1}{2}, m-\tfrac{1}{2}, \alpha\rangle+(m+\tfrac{1}{2}\alpha)|j, m, \alpha\rangle \\
\Gamma(L_0)|j, m, \alpha\rangle &= \Lambda_1(j+m+1)^{1/2}|j+\tfrac{1}{2}, m+\tfrac{1}{2}, \alpha\rangle \\
&\quad +[(j+m)^{1/2}+\alpha]|j, m, \alpha\rangle+(\alpha-1)\xi|j+\tfrac{1}{2}, m-\tfrac{1}{2}, \alpha+1\rangle \\
\Gamma(V_+)|j, m, \alpha\rangle &= \alpha(j+m+1)^{1/2}|j, m+1, \alpha-1\rangle+\xi(j+m+1)^{1/2}|j+\tfrac{1}{2}, m+\tfrac{1}{2}, \alpha\rangle \\
\Gamma(V_-)|j, m, \alpha\rangle &= (1-\alpha)[\Lambda_1|j+\tfrac{1}{2}, m-\tfrac{1}{2}, \alpha+1\rangle+(j+m)^{1/2}|j, m-1, \alpha+1\rangle] \\
\Gamma(T_+)|j, m, \alpha\rangle &= \alpha(j-m-\alpha)^{1/2}|j, m, \alpha-1\rangle+\xi(j-m-\alpha+1)^{1/2}|j+\tfrac{1}{2}, m-\tfrac{1}{2}, \alpha\rangle \\
\Gamma(T_-)||j, m, \alpha\rangle &= (1-\alpha)[\Lambda_2|j+\tfrac{1}{2}, m-\tfrac{1}{2}, \alpha+1\rangle+(j-m-\alpha)^{1/2}|j, m, \alpha+1\rangle].
\end{aligned}
\tag{17}
$$

It is pointed out that on the subspaces $\{|j, m, 0\rangle, j=m, m-1, \ldots, -m\}$ and $\{|j, m, 1\rangle, m=j-1, j-2, \ldots, -j\}$, we obtain respectively two indecomposable representations of the symmetry subalgebra SO(3) which can be regarded as the indecomposable generalisations of the standard angular momentum representations of SO(3) [11].

When $\Lambda_1=\Lambda_2=0=\xi$ for a fixed half-integer j

$$\tilde{V}^{[j]}: \{|j, m, \alpha\rangle | \alpha=0, 1; m=j-\alpha, j-\alpha-1, \ldots, -j\}$$

is invariant under the action of the representation Γ, on which Γ subduces a usual irreducible representation. Its restriction to SO(3) is a direct sum of two irreducible representations $D^{[j]}$ and $D^{[j+1/2]}$ of the Lie subalgebra SO(3) of SU(2|1).

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