A new quasitriangular Hopf algebra as the nontrivial quantum double of a simplest $\mathbb{C}$ algebra and its universal R matrix for Yang–Baxter equation

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Endowing a simple $\mathbb{C}$ algebra generated by two commuting elements and a unit with a noncocommutative Hopf algebra structure, a new quantum double and the corresponding universal R matrix for the Yang–Baxter equation is constructed. The finite-dimensional representations of the quantum double is also studied and a concrete R matrix is presented as an example.

I. INTRODUCTION

The quantum double (QD) theory[1] is a quite powerful tool in constructing solutions (R matrices) of the quantum Yang–Baxter equation (QYBE). Actually, according to this theory, from a given Hopf algebra one can obtain a unique quasitriangular Hopf algebra (QTHA), which contains a universal R matrix. At present, most of the QD's explicitly built are some "q-deformed" algebras, such as quantum algebras, quantum superalgebras, and quantum affine algebras,[2–5] and more recent studies[6–8] show that new QD's can be obtained by parametrizing some kinds of quantum algebras. In this paper our purpose is to establish a new QD from a very simple $\mathbb{C}$ algebra.

II. THE HOPF ALGEBRA A AND ITS DUAL A^0

Let A be a $\mathbb{C}$ algebra generated by the elements X,H and the unit 1, satisfying the relations

$$[H,X]=0, \quad [1,X]=0, \quad [1,H]=0. \tag{1}$$

We define the following operations for A:

$$\begin{aligned} &\Delta H = H\otimes 1 + 1\otimes H,\\ &\Delta X = X\otimes 1 + q^{-H}\otimes X, \quad \Delta 1 = 1\otimes 1,\\ &\epsilon(X)=\epsilon(H)=0, \quad \epsilon(1)=1, \quad q\in\mathbb{C},\\ &S(X)=-Xq^{H}, \quad S(H)=-H, \quad S(1)=1. \end{aligned} \tag{2}$$

By a direct calculation we can prove the following.

Proposition 1: A is a Hopf algebra with the coproduct Δ, the antipode S, and the counit ϵ defined above.

Remark: For the same $\mathbb{C}$ algebra A, we can also endow it with a different but "trivial" (cocommutative) Hopf algebra structure:

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$$\Delta e = e \otimes 1 + 1 \otimes e; \quad \Delta 1 = 1 \otimes 1;$$
$$\epsilon(e) = 0, \quad \epsilon(1) = 1, \quad S(e) = -e, \quad S(1) = 1, \tag{3}$$

where $e = X,H$.

Now, we try to establish the Hopf algebraic dual A^0 of A and the corresponding QD $D(A)$. According to the QD theory, we have the following results: (1) A and A^0 are subalgebras of $D(A)$; (2) there exists a bijective linear mapping $A \otimes A^0 \to D(A)$; and (3) the multiplication of $D(A)$ is defined by

$$ba = \sum_{i,j} \langle a_1^i, S(b_1^j) \rangle \langle a_3^i, b_3^j \rangle a_2^i b_2^j, \quad a \in A, \quad b \in A^0. \tag{4}$$

Here, we have used the notation

$$(\Delta \otimes id) \cdot \Delta(c) = (id \otimes \Delta) \cdot \Delta(c) = \sum_i c_1{}^i \otimes c_2{}^i \otimes c_3{}^i,$$

where $c = a,b$. To deduce the dual structure of A^0 we choose

$$\{ e_{mn} = X^m H^n \,|\, m,n \in \mathbb{Z}^+ = \{0,1,2,\ldots\} \}, \tag{5}$$

to be the basis for A, and define the pairs

$$\langle e_{mn}, Y \rangle = \delta_{m1} \delta_{n0}, \quad \langle e_{mn}, N \rangle = \delta_{n0} \delta_{n1}, \tag{6}$$

for the basic elements Y and N in A^0. Then, from the definition of the dual Hopf algebra, we obtain the following.

Proposition 2:

$$\langle e_{mn}, Y^{m'} N^{n'} \rangle = m! n! \delta_{mm'} \delta_{nn'}. \tag{7}$$

Proof: The proof is easy. Actually, we have

$$\langle H^n, N^m \rangle = \langle \Delta H^n, N^{m-1} \otimes N \rangle = n \langle H^{n-1}, N^{m-1} \rangle = n! \delta_{nm};$$

$$\langle X^m H^n, Y^n N^{n'} \rangle = n! \delta_{nn'} \langle X^m, Y^n \rangle,$$

$$\langle X^m, Y^n \rangle = \langle \Delta X^m, Y^{n-1} \otimes Y \rangle = \sum_k \frac{m!}{(m-k)!k!} \langle X^{m-k} q^{-kH} \otimes X^k, Y^{n-1} \otimes Y \rangle$$
$$= m \langle X^{m-1} q^{-H} \otimes X, Y^{n-1} \otimes Y \rangle = m \langle X^{m-1}, Y^{n-1} \rangle = m! \delta_{mn}.$$

In order to establish the Hopf algebra structure of A^0 and combine A with A^0 to form a QD, we need to prove the following propositions.

Proposition 3:

$$N = N \otimes 1 + 1 \otimes N, \quad \Delta Y = Y \otimes 1 + 1 \otimes Y,$$
$$S(N) = -N, \quad S(Y) = -Y, \quad \epsilon(Y) = \epsilon(N) = 0. \tag{8}$$

Proof: The proof directly follows from the equations

$$\langle X^m H^n \otimes X^{m'} H^{n'}, \Delta N \rangle = \delta_{m0} \delta_{n1} \delta_{m'0} \delta_{n'0} + \delta_{m0} \delta_{n0} \delta_{m'0} \delta_{n'1},$$

$$\langle X^m H^n \otimes X^{m'} H^{n'} \Delta Y \rangle = \delta_{m1}\delta_{n0}\delta_{m'0}\delta_{n'0} + \delta_{m0}\delta_{n0}\delta_{m'1}\delta_{n'0},$$

$$\langle X^m H^n, S(Y) \rangle = -\delta_{m1}\delta_{n0},$$

$$\langle X^m H^n, S(N) \rangle = -\delta_{m0}\delta_{n1},$$

$$\langle 1, N \rangle = 0 = \epsilon(N),$$

$$\langle 1, Y \rangle = 0 = \epsilon(Y).$$

Proposition 4: Between A and A^0, we have the commutation relations

$$\text{(i)} \quad [N,Y] = -\ln q \cdot Y; \tag{9}$$

$$\begin{gathered} \text{(ii)} \quad [N,X] = \ln q \cdot X, \\ [N,H] = 0 = [H,Y] = [H,N], \\ [X,Y] = 1 - q^{-H}. \end{gathered} \tag{10}$$

Proof: A straightforward calculation shows that

$$\begin{aligned} \langle X^m H^n, [N,Y] \rangle &= \langle X^m H^n, NY \rangle - \langle X^m H^n, YN \rangle \\ &= (\delta_{m1}\delta_{n1} - \ln q \cdot \delta_{m1}\delta_{n0}) - \delta_{m1}\delta_{n1} \\ &= -\ln q \cdot \delta_{m1}\delta_{n0}. \end{aligned}$$

So we prove part (i). To prove part (ii), we use Proposition 3, obtaining

$$\Delta^{(2)}(X) \equiv (id \otimes \Delta) \cdot \Delta X = X \otimes 1 \otimes 1 + q^{-H} \otimes X \otimes 1 + q^{-H} \otimes q^{-H} \otimes X;$$

$$\Delta^{(2)}(\xi) = (id \otimes \Delta)\Delta(\xi) = \xi \otimes 1 \otimes 1 + 1 \otimes \xi \otimes 1 + 1 \otimes 1 \otimes \xi; \quad \xi = Y, N \in A^0.$$

Thus, it immediately follows from Eq. (4) that

$$YX = \langle X, S(Y) \rangle \cdot 1 + \langle q^{-H}, S(1) \rangle XY + \langle q^{-H}, S(1) \rangle q^{-H} = -1 + XY + q^{-H},$$

$$NX = \langle q^{-H}, S(N) \rangle X + \langle q^{-H}, S(1) \rangle XN = \ln q \cdot X + XN,$$

and the other relations are obvious.

III. NEW QUANTUM DOUBLE AND ITS UNIVERSAL R MATRIX

Summarizing the above results, we finally obtain the following theorems as the central results in this paper.

Theorem 1: The QD $D(A)$ is generated by the elements X, Y, H, N, and the unit 1, satisfying Eqs. (1) and (9) and (10), and it is a Hopf algebra with coproduct Δ, counit ϵ, and antipode S defined above.

Theorem 2: Furthermore, $D(A)$ is a quasitriangular Hopf algebra with the universal R matrix,

$$\widetilde{R} = \exp(X \otimes Y) \cdot \exp(H \otimes N). \tag{11}$$

Proof: One easily observes that

$$\{e^{mn}=(m!n!)^{-1}Y^mN^n\,|\,m,n\in\mathbb{Z}^+\}$$

is a basis of A^0 dual to the basis $\{e_{mn}\}$. Then, the universal R matrix (11) follows from the following equations:

$$\widetilde{R}=\sum_{m,n=0}^{\infty}e_{mn}\otimes e^{mn}=\sum_{m,n=0}^{\infty}(m!n!)^{-1}X^mH^n\otimes Y^mN^n.$$

Remark: As a $\mathbb{C}$ algebra, the new QD $D(A)$ can be regarded as the boson algebra B generated by elements b, $b^-\equiv b,\widetilde{N}$, and E with the following relations:

$$[b,b^+]=E,\quad [\widetilde{N},b^\pm]=\pm b^\pm,\quad [E,N]=[E,b^\pm]=0, \tag{12}$$

by introducing the correspondence

$$b\leftrightarrow X,\quad b^+\leftrightarrow Y,\quad E\leftrightarrow 1-q^{-H},\quad N\leftrightarrow -(\ln q)^{-1}N, \tag{13}$$

between $D(A)$ and B. But we would like to point out that as is seen above, $D(A)$ carries a noncocommutative Hopf algebra structure, which is essentially different from the cocommutative one usually defined on B:

$$\begin{gathered}\Delta\xi=\xi\otimes 1+1\otimes\xi,\quad S(\xi)=-\xi,\\ \epsilon(\xi)=0,\quad \xi=b^\pm,E,\widetilde{N}.\end{gathered} \tag{14}$$

It is also worth pointing out that a similar discussion has been given in Ref. 9, but the QD construction is not investigated there.

IV. FINITE dim. REPRESENTATIONS OF $D(A)$ AND NEW R MATRICES

The remaining part of this paper is devoted to obtaining finite-dimensional representations (FDR's) of $D(A)$ and R. Denoting

$$N'=(\ln q)^{-1}N,\quad K=1-q^{-H},\quad X^+=X,\quad X^-=Y, \tag{15}$$

we have

$$[K,\text{everything}]=0,\quad [X,Y]=K,\quad [N',X^\pm]=\pm X^\pm. \tag{16}$$

Let us define a vacuum state $|0\rangle$ by $\widetilde{N}|0\rangle=\mu|0\rangle$ and $Y|0\rangle=0(\mu\in\mathbb{C})$. Then, one can directly verify that the vector space $W=\text{span}\{x(m,n)=X^mK^n|0\rangle\,|\,m,n\in\mathbb{Z}^+\}$ carries an infinite-dimensional representation of $D(A)$:

$$\begin{gathered}Xx(m,n)=x(m+1,n),\\ Yx(m,n)=-mX(m-1,n+1),\\ N'x(m,n)=(m+\mu)x(m,n),\\ Kx(m,n)=x(m,n+1).\end{gathered} \tag{17}$$

One easily observes that $V^L = \text{span}\{x(m,n) = x(m,n) \mid m+n \geqslant L\}$ ($L \in \mathbb{Z}^+ - \{0\}$) is an invariant subspace. So on the quotient space $Q^L = \text{span}\{\bar{x}(m,n) = x(m,n) \text{ Mod} \cdot V^L \mid m+n \leqslant L-1\}$, one can get a FDR of $D(A)$.

For example, when $L=2$, we obtain a 3 dim. representation

$$X = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$N' = \begin{pmatrix} 1+\mu & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \mu \end{pmatrix}, \quad K = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

$$H = -\frac{\ln(1-K)}{\ln(q)} = -\frac{-K}{\ln(q)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & (\ln q)^{-1} \\ 0 & 0 & 0 \end{pmatrix}.$$

Substituting this representation into the universal R matrix (11), one obtains a new R matrix,

$$R(\mu) = \begin{bmatrix} I & \tilde{O} & Y \\ \tilde{O} & I & N' \\ \tilde{O} & \tilde{O} & I \end{bmatrix},$$

where I and $\tilde{O}$ are, respectively, the 3×3 unit matrix and 3×3 zero matrix.

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