Exotic quantum double, its universal R matrix, and their representations

Chang-Pu Sun

Institute for Theoretical Physics, State University of New York, Stony Brook, New York 11794-3840 and Department of Physics, Northeast Normal University, Changchun 130024, People's Republic of China

(Received 25 January 1993; accepted for publication 23 March 1993)

The exotic quantum double and its universal R matrix for the quantum Yang–Baxter equation are constructed in terms of Drinfeld's quantum double theory. As a new quasitriangular Hopf algebra, it is different from those standard quantum doubles that are the q deformations for Lie algebras or Lie superalgebras. By studying its representation theory, many-parameter representations of the exotic quantum double are obtained with an explicit example associated with Lie algebra A_2. The multiparameter R matrices for the quantum Yang–Baxter equation can result from the universal R matrix of this exotic quantum double and these representations.

I. INTRODUCTION

In recent years, the quantum Yang–Baxter equation (QYBE)[1,2] has become a focus of the attention from both theoretical physicists and mathematicians. This is because the QYBE is a key to the complete integrability of many physical systems appearing in the quantum inverse scattering methods,[3,4] the exactly solvable models in statistical mechanics,[5] and low-dimensional quantum field theory.[6] In solving the QYBE in a general way and classifying its solutions (R matrices) algebraically, a remarkable mathematical structure, the quasitriangular Hopf algebra (loosely called quantum group), is found in connection with the QYBE.[7–10] Among these developments, the Drinfeld's quantum double theory[7] provides one with a general construction to systematically obtain solutions of the QYBE in terms of the quantum doubles (QDs), which usually are the "q deformations" of certain algebras, and their representations. The recent studies show that, not only the standard R matrices,[11–13] but also the nonstandard ones,[14–16] such as the R matrices with nonadditive spectral parameters,[17–23] the colored R matrices,[24–29] can be obtained in the framework of Drinfeld's QD theory, but for the latter the cyclic representations, other nongeneric ones at roots of unity[30–38] and some parametrization of the quantum (universal eveloping) algebras[39,40] need to be considered. The purpose of the present paper is to search for the exotic quantum doubles, other than those "q deformations," so that the new universal R matrix can be obtained for the QYBE based on Drinfeld's quantum double theory.

For our discussion to proceed conveniently, we need to outline some basic ideas in Drinfeld's QD theory so that the notations used in this paper can be clarified. Suppose we are given two Hopf algebras A, B and a nondegenerate bilinear form $\langle,\rangle: A\times B\to C$ (the complex field) satisfying the following conditions:

$$\begin{gathered}
\langle a,b_1b_2\rangle=\langle\Delta_A(a),b_1\otimes b_2\rangle,\quad a\in A,b_1,b_2\in B,\\
\langle a_1a_2,b\rangle=\langle a_2\otimes a_1,\Delta_B(b)\rangle,\quad a_1,a_2\in A,b\in A,\\
\langle 1_A,b\rangle=\epsilon_B(b),\quad b\in B, \qquad (1.1)\\
\langle a,1_B\rangle=\epsilon_A(a),\quad a\in A,
\end{gathered}$$

0022-2488/93/34(8)/3440/13/$6.00

$$\langle S_A(a),S_B(b)\rangle=\langle a,b\rangle,\quad a\in A,b\in B,$$

where for $C=A$, B, Δ_C, ϵ_C, and S_C are the coproduct, counit, and antipode of C, respectively; 1_C is the unit of C. Drinfeld's QD theory (for a comprehensive reviews see the Refs. 41 and 42) states the central results in the QD theory as follows.

Theorem 1: There exists a Hopf algebra D satisfying the following conditions:

(1) D contains A and B as Hopf subalgebras.

(2) The mapping $A\times B\to D{:}a\otimes b\to ab$ is an isomorphism of vector space.

(3) For any $a\in A,b\in B$, we have multiplication

$$ba=\sum_{i,j}\langle a_i(1),S(b_j(1))\rangle\langle a_i(3),b_j(3)\rangle a_i(2)b_j(2), \tag{1.2}$$

where $c_i(k)$ ($k=1,2,3$; $c=a,b$) are defined by

$$\Delta^2(c)=(id\otimes\Delta)\Delta(c)=(\Delta\otimes id)\Delta(c)=\sum_i c_i(1)\otimes c_i(2)\otimes c_i(3).$$

Theorem 2: There exists a unique element

$$\hat{R}=\sum_m a_m\otimes b_m\in A\times B\subset D\times D$$

obeying the "abstract" QYBE

$$\hat{R}_{12}\hat{R}_{13}\hat{R}_{23}=\hat{R}_{23}\hat{R}_{13}\hat{R}_{12}, \tag{1.3}$$

where a_m and b_m are the basis vectors of A and B, respectively, and they are dual each other, i.e., $\langle a_m,b_n\rangle=\delta_{m,n}$;

$$\hat{R}_{12}=\sum_m a_m\otimes b_m\otimes 1,\quad \hat{R}_{13}=\sum_m a_m\otimes 1\otimes b_m,\quad \hat{R}_{23}=\sum_m 1\otimes a_m\otimes b_m,$$

where 1 is the unit of D.

Up to now, the QDs built explicitly are only the quantum (universal enveloping) algebras and superalgebras and their parametrizations. They are the q deformations of the universal algebras and possess a "standard" quantum double structure that both the subalgebras A and B are noncommutative and noncocommutative. This symmetric structure reflects the duality of A and B. Notice that these standard quantum doubles approach the usual universal enveloping algebras (UEA) in the classical limit $q\to 1$. In this paper, we will construct so-called exotic quantum doubles (EQD) that are not those q deformations and possess asymmetric dual structure that one of the subalgebras A and B is commutative but noncocommutative and another cocommutative but noncommutative. As new quasitriangular Hopf algebras, these EQDs naturally enjoy the QYBE, but they have not the usual classical limit.

This paper is arranged as follows. In Sec. II, we take the sub-Borel subalgebra of the UEA of the classical Lie algebra as the Hopf subalgebra A with cocommutative coproduct in the QD construction and then built its quantum dual as a noncocommutative but commutative Hopf subalgebra B. In Sec. III, we combine A and B to form the exotic quantum double and thereby obtain the new universal R matrix for the QYBE. In Sec. IV, we discuss an explicit example of EQDs, which is connected with the Lie algebra A_2 in details. In Sec. V, we study the representation theory of the EQD with the above example and construct a class of many-parameter representations to built the many-parameter R matrices for the QYBE. Finally, in Sec. VI, we give some remarks on the problems and the possible developments in the EQD.

II. QUANTUM DUAL FOR NONSIMPLE LIE ALGEBRA

Let $\phi^+:\{\alpha,\beta,\gamma,...\}$ be the system of all positive roots with respect to a simple root system of a classical Lie algebra L. Cartan elements $h\in H$ (the Cartan subalgebra) and all the positive root vectors $\{e_\alpha,|\alpha\in\phi^+\}$ generate an associative algebra A with the relations on the Cartan–Weyl basis

$$[h, e_\alpha]=\alpha(h)e_\alpha, \quad [e_\alpha, e_\beta]=N_{\alpha,\beta}e_{\alpha+\beta}, \tag{2.1}$$

where $\alpha\in H^*$ and the coefficients $N_{\alpha,\beta}$ enjoy the structure of the Lie algebra L. In fact, the algebra A is a subalgebra of the Borel subalgebra of the universal enveloping algebra (UEA) of the Lie algebra L. Defining the algebraic homomorphisms $\Delta:A\to A\otimes A,\epsilon:A\to C$, and the algebraic antihomomorphism $S:A\to A$ by

$$\Delta(x)=x\otimes 1+1\otimes x, \quad S(x)=-x, \quad \epsilon(x)=0, \tag{2.2}$$

where $x\in\{h,e_\alpha|\alpha\in\phi^+\}$, one gives the algebra a "trivial" (cocommutative) Hopf algebraic structure. It is a well-known fact in the theory of Hopf algebra since we can regard the algrbra A as a UEA of the nonsimple Lie algebra with basis $\{h,e_\alpha,\alpha\in\phi^+\}$. However, the Hopf algebraic dual (quantum dual) B of A is nontrivial (non-cocommutative) due to the duality of B to A. Now, we derive the structure of A in terms of this duality.

Because A is cocommutative, its dual is an Abelian algebra with commuting generators. So, the associative algebraic structure is quite simple. To consider the Hopf algebraic structure, we set an order for the basis of A: If $\alpha-\beta$ is a nonzero positive root, then we say $\alpha>\beta$; the basis for A is written down to enjoy this order as

$$\left\{a(m,m_\alpha)=h^m \prod_{\alpha\in\phi^+} e_\alpha^{m_\alpha}=h^m\cdots e_\beta^{m_\beta}\cdots e_\gamma^{m_\gamma}\cdots e_\delta^{m_\delta}\cdots \,\middle|\, \cdots\delta>\cdots\gamma\cdots>\cdots\beta,m_\alpha\in Z^+ =\{0,1,2,...\}\right\}.$$

Suppose that the dual Hopf algebra B to A is generated by the dual generators t and f_α ($\alpha\in\phi^+$) to h and e_α, respectively. They are defined by the following pairs in terms of a bilinear form $\langle,\rangle$

$$\langle h,t\rangle=1,\langle x,t\rangle=0,\langle e_\alpha,f_\alpha\rangle=1,\langle y,f_\alpha\rangle=0, \tag{2.3}$$

where x and y are the basis elements of A other than h and e_α, respectively.

Proposition 1: For $m_\alpha,n_\alpha,m\in Z^+$ ($\alpha\in\phi^+$),

$$\left\langle h^m \prod_{\alpha\in\phi^+} e_\alpha^{m_\alpha}, t^l \prod_{\alpha\in\phi^+} f_\alpha^{n_\alpha}\right\rangle=\delta_{m,l}\; m! \prod_{\alpha\in\phi^+} m_\alpha!\; \delta_{m_\alpha,n_\alpha}, \tag{2.4}$$

namely, the vectors

$$b(m,m_\alpha)=\frac{t^m}{m!}\prod_{\alpha\in\phi^+}\frac{f_\alpha^{m_\alpha}}{m_\alpha!}, \tag{2.5}$$

form a dual basis for B to $a(m,m_\alpha)$:

$$\langle a(m,m_\alpha),b(n,n_\alpha)\rangle=\delta_{m,n}\prod_{\alpha\in\phi^+}\delta_{m_\alpha,n_\alpha}.$$

Proof: Thanks to the duality between A and B, we have

$$\langle h^l,t^m\rangle=\langle\Delta(h^l),t^{m-1}\otimes t\rangle=\sum_{k=0}^{l}\frac{l!}{k!(l-k)!}\langle h^{l-k}\otimes h^k,t^{m-1}\otimes t\rangle=l\langle h^{l-1},t^{m-1}\rangle=\cdots=l!\delta_{l,m}.$$

Similarly, for $G=e_\alpha$, $F=f_\alpha$, $\alpha\in\phi^+$, respectively,

$$\langle G^m,F^n\rangle=m!\delta m,n.$$

Then,

$$\begin{aligned}\langle h^mG^n,F^s\rangle&=\sum_{k=0}^{m}\sum_{r=0}^{n}\frac{m!n!}{(m-k)!k!(n-r)!r!}\langle h^{m-k}G^{n-r}\otimes h^kG^r,F\otimes F^{s-1}\rangle\\&=\sum_{k=0}^{m}\sum_{r=0}^{n}\frac{m!n!}{(m-k)!k!(n-r)!r!}\delta_{n-r,1}\delta_{m-k,0}\langle h^kG^r,F^{s-1}\rangle\\&=n\langle h^mG^{n-1}.F^{s-1}\rangle\\&=n!\delta_{n,s}\delta_{m,0};\end{aligned}$$

$$\langle h^mG^n,t^lF^s\rangle=\sum_{k=0}^{m}\sum_{r=0}^{n}\frac{m!n!}{(m-k)!k!(n-r)!r!}\langle h^{m-k}G^{n-r}\otimes h^kG^r,t^l\otimes F^s\rangle=\langle h^m,t^l\rangle\langle G^n,F^s\rangle.$$

It follows from the above calculations that

$$\langle a(m,m^\alpha),b(n,n^\alpha)\rangle=\langle h^m,t^n\rangle\prod_{\alpha\in\phi^+}\langle e_\alpha^{m_\alpha},f_\alpha^{n_\alpha}\rangle=m!\delta_{m,n}\prod_{\alpha\in\phi^+}m_\alpha!\delta_{m_\alpha,n_\alpha}.$$

In this position, we can deduce the Hopf algebraic structure, i.e., $(\Delta=\Delta_B,\ \epsilon=\epsilon_B,\ S=S_B)$ of the algebra B. Let us first consider $\Delta(f_\gamma)$. Notice that the linear form $\langle\ ,\Delta(f_\gamma)\rangle$ is nonzero only on $e_\gamma\otimes 1,h^n\otimes e_\gamma,h^ne_\alpha\otimes e_\beta$ $(\beta>\alpha)$:

$$\langle e_\gamma\otimes 1,\Delta(f_\gamma)\rangle=\langle 1.e_\gamma,f_\gamma\rangle=1;$$

$$\langle h^n\otimes e_\gamma,\Delta(f_\gamma)=\langle e_\gamma h^n,f_\gamma\rangle=\langle(h-\gamma(h))^ne_\gamma,f_\gamma\rangle=(-\gamma(h))^n;$$

$$\begin{aligned}\langle h^ne_\alpha\otimes e_\beta,f_\gamma\rangle&=\langle(h-\beta(h))^ne_\beta e_\alpha,f_\gamma\rangle\\&=(-\beta(h))^n\langle e_\alpha e_\beta-N_{\alpha,\beta}\,e_{\alpha+\beta},f_\gamma\rangle\\&=-(-\beta(h))^nN_{\alpha,\beta}\delta(\alpha+\beta,\delta).\end{aligned}$$

Consequently,

$$\Delta(f_\gamma)=f_\gamma\otimes 1+\sum_{n=0}^{\infty}\frac{(-\gamma(h))^nt^n}{n!}\otimes f_\gamma-\sum_{\alpha,\beta\in\phi^+}N_{\alpha,\beta}\delta(\alpha+\beta,\gamma)\theta(\beta-\alpha)\sum_{n=0}^{\infty}\frac{(-\beta(h))^nt^n}{n!}$$
$$\times f_\alpha\otimes f_\beta,$$

where

$$\delta(\alpha,\beta)=\begin{cases}1, & \text{if } \alpha=\beta;\\ 0, & \text{if } \alpha\neq\beta\end{cases}$$

$$\theta(\alpha)=\begin{cases}1, & \text{if } \alpha(\neq 0)\in\phi^+;\\ 0, & \text{if } \alpha\notin\phi^+.\end{cases}$$

For $S(f_\gamma)$ and $\beta>\alpha$, the only nonzero pairs are

$$\begin{aligned}\langle h^n e_\alpha e_\beta, S(f_\gamma)\rangle &= \langle (-1)^n S(e_\beta e_\alpha h^n), S(f_\gamma)\rangle\\ &=(-1)^n\langle e_\beta e_\alpha h^n, f_\gamma\rangle\\ &=(-1)^n\langle (h-\beta(h)-\alpha(h))^n(e_\alpha e_\beta - N_{\alpha,\beta}e_{\alpha+\beta}), f_\gamma\rangle\\ &=-(\beta(h)+\alpha(h))^n N_{\alpha,\beta}\delta_{\alpha+\beta,\gamma};\end{aligned}$$

$$\begin{aligned}\langle h^n e_\gamma, S(f_\gamma)\rangle &= -(-1)^n\langle S(e_\gamma h^n), S(f_\gamma)\rangle\\ &=-(-1)^n\langle e_\gamma h^n, f_\gamma\rangle\\ &=-(-1)^n\langle (h-\gamma(h))^n e_\gamma, f_\gamma\rangle\\ &=-(\gamma(h))^n.\end{aligned}$$

Consequently,

$$S(f_\gamma)=-\sum_{n=0}^{\infty}\frac{\gamma(h)^n t^n}{n!}f_\gamma-\sum_{\alpha,\beta\in\phi^+}N_{\alpha,\beta}\theta(\beta-\alpha)\delta(\alpha+\beta,\gamma)\sum_{n=0}^{\infty}\frac{(\beta(h)+\alpha(h))^n t^n}{n!}f_\alpha f_\beta.$$

In the same way we derive $\Delta(h)$, $S(h)$, and $\epsilon(f_\alpha)$. These results are listed as follows.

Proposition 2: The duality between A and B results in the commutative associative algebraic structure and the non-cocomutative Hopf algebraic structure defined by

$$\Delta(t)=t\otimes 1+1\otimes t,$$

$$\Delta(f_\gamma)=f_\gamma\otimes 1+e^{-\gamma(h)t}\otimes f_\gamma-\sum_{\alpha,\beta\in\phi^+}C(\alpha,\beta,\gamma)e^{-\beta(h)t}f_\alpha\otimes f_\beta,$$

$$S(f_\gamma)=-e^{\gamma(h)}\Bigl(f_\gamma+\sum_{\alpha,\beta\in\phi^+}C(\alpha,\beta,\gamma)f_\alpha f_\beta\Bigr), \tag{2.6}$$

$$S(h)=-e^{-\gamma(h)t}h,\quad S(1)=1;\quad \epsilon(f_\gamma)=\epsilon(h)=0,\quad \epsilon(1)=1,$$

where

$$C(\alpha,\beta,\gamma)=N_{\alpha,\beta}\,\theta(\beta-\alpha)\delta(\alpha+\beta,\gamma).$$

III. THE QUANTUM DOUBLE AND ITS UNIVERSAL *R* MATRIX

In this section we show how the algebra A and its quantum dual B can be combined to form a quasitriangular Hopf algebra with the exotic structure. To define the multiplications between A and B, we need to use the following formula:

$$\Delta^2(x)=x\otimes 1\otimes 1+1\otimes x\otimes 1+1\otimes 1\otimes x,\quad x=h,t,e_\alpha,\alpha\in\phi^+,$$

$$\Delta^2(f_\gamma)=f_\gamma\otimes 1\otimes 1+e^{-\gamma(h)t}\otimes f_\gamma\otimes 1+e^{-\gamma(h)t}\otimes e^{-\gamma(h)t}\otimes f_\gamma-\sum_{\alpha,\beta\in\phi^+}C(\alpha,\beta,\gamma)e^{-\gamma(h)t}$$

$$\otimes e^{-\beta(h)t}f_\alpha\otimes f_\beta-\sum_{\alpha,\beta\in\phi^+}C(\alpha,\beta,\gamma)e^{-\beta(h)t}f_\alpha\otimes f_\beta\otimes 1-\sum_{\alpha,\beta\in\phi^+}C(\alpha,\beta,\gamma)e^{-\beta(h)t}f_\alpha$$

$$\otimes e^{-\beta(h)t}\otimes f_\beta-\sum_{\alpha,\beta,\sigma,\delta\in\phi^+}C(\alpha,\beta,\gamma)C(\sigma,\delta,\beta)e^{-\beta(h)t}f_\alpha\otimes e^{-\delta(h)t}f_\sigma\otimes f_\delta. \tag{3.1}$$

Using the above equations and the definition (1.2), we calculate the commutators $[e_\alpha, f_\gamma]$, $[h, f_\gamma]$, and $[t, x]$:

$$f_\gamma e_\gamma=\langle e_\gamma,S(f_\gamma)\rangle\langle 1,1\rangle 1.1+\langle 1,S(e^{-\gamma(h)t})\rangle\langle 1,1\rangle e_\gamma f_\gamma+\langle 1,S(e^{-\gamma(h)t})\rangle\langle e_\gamma f_\gamma\rangle 1.e^{-\gamma(h)t}$$

$$=-1+e_\gamma f_\gamma+e^{-\gamma(h)t};$$

$$f_\gamma e_\eta=-\sum_{\alpha,\beta\in\phi^+}C(\alpha,\beta,\gamma)\langle e_\eta,S(e^{-\beta(h)t}f_\alpha)\rangle\langle 1,1\rangle 1.f_\beta+\langle 1,S(e^{-\gamma(h)t})\rangle\langle 1,1\rangle e_\eta f_\gamma$$

$$-\sum_{\alpha,\beta\in\phi^+}C(\alpha,\beta,\gamma)\langle 1,S(e^{-\gamma(h)t})\rangle\langle e_\eta,f_\beta\rangle e^{-\beta(h)t}f_\alpha$$

$$=\sum_{\beta\in\phi^+}C(\eta,\beta,\gamma)f_\beta-\sum_{\alpha\in\phi^+}C(\alpha,\eta,\gamma)e^{-\eta(h)t}f_\alpha+e_\eta f_\gamma;$$

$$f_\gamma h=\langle h,S(e^{-\gamma(h)t})\rangle\langle 1,1\rangle f_\gamma+\langle 1,,S(e^{-\gamma(h)t})\rangle\langle 1,1\rangle hf_\gamma+\gamma(h)f_\gamma+hf_\gamma;$$

$$tx=\langle 1,1\rangle\langle 1,1\rangle xt. \tag{3.2}$$

The above results are rewritten as follows.

Proposition 4: The multiplication between A and B is defined by the following commutators:

$$[e_\alpha, f_\alpha]=1-e^{-\alpha(h)t},\quad [h, f_\alpha]=-\alpha(h)f_\alpha,$$

$$[e_\alpha, f_\beta]=\sum_{\gamma\in\phi^+}C(\gamma,\alpha,\beta)e^{-\alpha(h)t}f_\gamma-\sum_{\gamma\in\phi^+}C(\alpha,\gamma,\beta)f_\gamma,\quad \alpha\neq\beta, \tag{3.3}$$

$$[t, x]=0,x=h,e_\alpha.$$

The above commutators combine the algebra A with its quantum dual B to form a noncocommutative and noncommutative Hopf algebra $D(A)=D$ as the quantum double of A (or B). As an associative algebra, it is generated by h,t,e_α,f_α, $(\alpha\in\phi^+)$ and the unit 1 obeying Eqs. (2.1), and endowed with the Hopf algebraic structure by Eqs. (2.2) and (2.6). Now, let us show that this Hopf algebra D is also quasi-triangular. In fact, the construction of Drinfeld's QD theory automatically perseveres the existence of the quasitriangular structure. Intertwining A and B, the universal R matrix is a canonical element

$$\hat{R}=\sum_{m,m_\alpha=0(\alpha\in\phi^+)}^{\infty} a(m,m_\alpha)\otimes b(m,m_\alpha)=e^{h\otimes t}\prod_{\alpha\in\phi^+}\exp(e_\alpha\otimes f_\alpha). \tag{3.4}$$

This element $\hat{R}(\in D\otimes D)$ endows the Hopf algebra D with a quasi-triangular stucture enjoyed by the following relations

$$\begin{gathered}\hat{R}\Delta(x)=\sigma\Delta(x)\hat{R},\\ (\Delta\otimes id)\hat{R}=\hat{R}_{13}\hat{R}_{23},\\ (id\otimes\Delta)\hat{R}=\hat{R}_{13}\hat{R}_{12},\\ (\epsilon\otimes id)\hat{R}=1=(id\otimes\epsilon)\hat{R},\\ (S\otimes id)\hat{R}=\hat{R}^{-1}=(id\otimes S)\hat{R},\end{gathered} \tag{3.5}$$

where σ is such a permutation that $\sigma(x\otimes y)=y\otimes x,x,y\in D$. Equations (3.5) imply that the above-constructed universal R matrix satisfies the abstract QYBE. It is not too difficult to verify the above relations (3.5) by a straightforward calculation.

In the above discussion, we have constructed a new quantum "group" (quasitriangular Hopf algebra) D associated with an arbitrary classical Lie algebra in terms of Drinfeld's QD theory. In comparison with the "standard" quantum "groups" that are the q deformations of UEAs of classical Lie algbras and superalgebras, our quantum "group" D possesses some new features: (1) D has not the usual classical limit since it is not a q deformation of the QEA, (2) it has an exotic subalgebraic structure that the subalgebra A is cocomutative but not commutative and the subalgebra B is commutative but not cocommutative. This asymmetric structure is quite different from the symmetric structure that both A and B are noncommutative and non-cocommutative. We will call D exotic quantum double.

IV. EXAMPLE OF THE EXOTIC QUANTUM DOUBLE FOR A_2

In this section an explicit example of the exotic quantum double will be given in connection with the classical Lie algebra A_2. In this example, the subalgebra A is taken to be an associative algebra generated by h,a,b, and the relations

$$[h,a]=\mu a,[h,b]=b,\quad \mu\in C, \tag{4.1a}$$

$$[a,[a,b]]=0=[b,[b,a]]. \tag{4.1b}$$

The generators a and b can be regarded as the root vectors with respect to the simple roots α_1 and α_2, respectively, for A_2. The third positive root vector corresponding to $\alpha_1+\alpha_2$ is just the commutator of a and b, i.e.,

$$c=[a,b], \tag{4.2}$$

which satisfies

$$[c,a]=0=[c,b],\quad [h,c]=(\mu+1)c. \tag{4.3}$$

The first equation in Eqs. (4.3) results from the Serre relation (4.1b). If we take h_1 and h_2 as the Cartan elements in the Chevalley basis for A_2 and

$$[h_1,a]=2a,[h_1,b]=-b,$$
$$[h_2,a]=-a,[h_2,b]=2b, \tag{4.4}$$

then

$$h=\frac{2\mu+1}{3}h_1+\frac{\mu+2}{3}h_2. \tag{4.5}$$

The cocommutative Hopf algebraic structure of A is endowed with

$$\Delta(x)=x\otimes 1+1\otimes x,$$
$$S(x)=-x, S(1)=1, \quad \epsilon(x)=0, \quad \epsilon(1)=1. \tag{4.6}$$

Let B be the quantum dual to A; and t, d, f, and g be its dual generators to h, a, b, and c, respectively. According to the last section, a straightforward calculation gives the Hopf algebraic structure of B:

$$\Delta(d)=d\otimes 1+e^{-\mu t}\otimes d,$$
$$\Delta(f)=f\otimes 1+e^{-t}\otimes f,$$
$$\Delta(g)=g\otimes 1+e^{-(\mu+1)t}\otimes g-e^{-t}d\otimes f,$$
$$S(d)=-e^{\mu t}d, \tag{4.7}$$
$$S(g)=-e^{(\mu+1)t}(g+d\otimes f),$$
$$S(f)=e^{t}f,$$

and the multiplication relation between A and B,

$$[h,d]=-\mu d, \quad [h,f]=-f, \quad [h,g]=-(\mu+1)g,$$
$$[a,d]=1-e^{-\mu t}, \quad [b,f]=1-e^{-t}, \quad [c,g]=1-e^{-(\mu+1)t}, \tag{4.8}$$
$$[a,g]=-f, \quad [b,g]=e^{-t}d,$$
$$[a,f]=0=[b,d], \quad [c,d]=0=[c,f],$$
$$[t,x]=0, \quad x=a,b,c,h.$$

The quantum double $D(2)$ is generated by a, b, c, h, t, d, e, and f with the relations (4.1a), (4.2), (4.3), and (4.8) as an associative algebra. Its quasitriangular Hopf algebraic stucture is endowed with by Eqs. (4.7) and the universal R matrix

$$\hat{R}=e^{h\otimes t}e^{a\otimes d}e^{h\otimes f}e^{c\otimes g}. \tag{4.9}$$

V. THE REPRESENTATION THEORY AND MANY-PARAMETER R-MATRICES

One purpose of building quantum double is to obtain the solutions of the QYBE in terms of its universal R matrix and matrix representations. In order to find the solutions of QYBE associated with the exotic quantum double D, we should study its representation theory. In fact, for a given representation $T^{[x]}$ of D,

$$T^{[x]}:D \to \mathrm{End}(V),$$

on the linear space V where x is a continuous parameter, we can construct an R matrix

$$R(x,y) = T^{[x]} \otimes T^{[y]}(\hat{R})$$

satisfying the QYBE:

$$R_{1,2}(x,y)R_{1,3}(x,z)R_{2,3}(y,z) = R_{2,3}(y,z)R_{1,3}(x,z)R_{1,2}(x,y), \tag{5.1}$$

Here, x, y, and z appear as the color parameters[24] similar to the nonadditive spectrum parameters in QYBE. The R matrices without additivity were first found in Refs. 17 and 18 for the chiral Potts model in statistical mechanics. Thus, it is necessary to study the representation theory and construct the many-parameter representations for the exotic quantum double D. However, to write down an explicit representation of a general D is rather overelaborate. So, we only discuss the typical example $D(2)$ in this section, but the main ideas and method can be directly applied to the general case.

To simplify our discussion, we have to distinguish between the trivial and nontrivial D modules.

Definition 1: The action of an operator on the representation space V is called trivial if its kernal is the whole space that it acts on.

Definition 2: A $D(2)$ module V is called trivial if at least one of the generators of D acts trivially on V; otherwise, it is called a nontrivial module.

Before studying the representation theory of $D(2)$, we would like to give a remark on the above definitions. To study the trival $D(2)$ module is much easier than that of a nontrivial one. In fact, the structure of a nontrivial $D(2)$ module collapses into that of the module of a simpler algebra D'. For example, if the action of t in $D(2)$ is trivial, one need only to study the module of the associative algebra generated by h,a,b,c,d,f, and g with nonzero commutation relations

$$[a,b]=c, \quad [a,g]=-f, \quad [b,g]=d,$$

$$[h,a]=\mu a, \quad [h,b]=b,$$

$$[h,g]=(\mu+1)g, \quad [h,d]=-\mu d, \quad [h,f]=-f.$$

For this reason, we will mainly study the nontrivial $D(2)$-model.

Having the above description, we are now in the position to prove a proposition as a central result for the representation theory of $D(2)$.

Proposition 5: There does not exist a finite-dimensional irreducible $D(2)$ module.

Proof: Suppose there exists a finite-dimensional irreducible $D(2)$ model V and $T:D \to \mathrm{End}(V)$ is the corresponding finite-dimensional irreducible representation. For simplicity we by x denote $T(x)$ as follows for $x \in D(2)$. Since t belongs to the center of D, its representative must be a nonzero scalar in nontrivial finite-dimensional irreducible representation according to the Schur lemma. Otherwise, if t is zero, V is trivial. Because C is an algebraic closure, there must be an eigenvector v such that

$$hv = \xi v, \quad \xi \in C.$$

Noticing the vectors $v, av, a^2v, ..., a^nv, ...,$ correspond to the distinct eigenvalues $\xi, \xi+\mu, \xi+2\mu, ..., \xi + n\mu, ...,$ for $\mu \neq 0$, we come to the conclusion that there exists $r \in Z^+$ such that nonzero vectors $v, av, a^2v, ..., a^{r-2}v$ and $a^{r-1}v = u$ are linearly independent and $a^r v = 0$. Similarly, there are $s, q \in Z^+$ such that the nonzero $u, bu, b^2u, ...$ and $b^{s-1}u = w$ are linearly independent and $bw = b^s u = 0$; the nonzero vectors $w, cw, c^2w, ..., c^{q-1}w = z$ are linearly independent and $cz = c^q w = 0$. Then, we can prove that $az = bz = cz = 0$ and thus the vector z generates a $D(2)$ submodule

$$S = \mathrm{Span}\{F(m,n,l) = d^m f^n g^l z \mid m,n,l \in Z^+\}$$

under the action of $D(2)$. Thanks to the irreducibility of V and its finite dimension, we must have $S = V$ and conclude that there must exist m', n', l' so that

$$dF(m'-1,n,l) = 0, \quad fF(m,n'-1,l) = 0, \quad gF(m,n,l'-1) = 0, \tag{5.2}$$

that is to say, the dimension of V is $m'n'l'$ of S. However, it follows from Eq. (5.2) that

$$0 = adF(m'-1,0,0) = ad^{m'}z = [d^{m'}a + m'(1-e^{-\mu t})]z = m'(1-e^{-\mu t})z,$$

that is, $m' = 0$. Similarly, $n' = l' = 0$. This means the $D(2)$ module is trivial.

According to the above proposition, for the study of nontrivial representation, we only need to focus on two cases, the indecomposable (reducible, but not completely reducible) representations and the infinite-dimensional irreducible representations. Now, we only discuss the latter. To construct an infinite-dimensional irreducible representation explicitly, we define a Verma-like space

$$V(\eta,\pi) = \mathrm{Span}\{|M\rangle = |m,n,l\rangle = a^m b^n c^l |0(\eta,\pi)\rangle \mid m,n,l \in Z^+\}$$

based on the vacuumlike state $|0(\eta,\pi)\rangle$:

$$\begin{gathered} d|0(\eta,\pi)\rangle = f|0(\eta,\pi)\rangle = g|0(\eta,\pi)\rangle = 0, \\ h|0(\eta,\pi)\rangle = \eta|0(\eta,\pi)\rangle, t|0(\eta,\pi)\rangle = \pi|0(\eta,\pi)\rangle, \end{gathered} \tag{5.3}$$

where $\eta, \pi \in C$. The existence of the vacuumlike state $|0(\eta,\pi)\rangle$ is easily proved by considering that t and h commute with each other and (a,b,c), (d,f,g), and (h,t) act as the "lifting" operators (for the positive roots), the "lowering" operator (for the negative roots), and the Cartan operators, respectively, for a classical Lie algebra.

Proposition 6: On the Verma-like space, the infinite-dimensional representation $T^{[\eta,\pi]}$,

$$\begin{gathered} h|M\rangle = [\eta + m\mu + (1+\mu)l]|M\rangle, \quad a|M\rangle = |M+e^1\rangle, \\ b|M\rangle = |M+e^2\rangle - m|M-e^1+e^3\rangle, \quad c|M\rangle = |M+e^3\rangle, \\ d|M\rangle = m(e^{-\mu\pi}-1)|M-e^1\rangle, \quad f|M\rangle = n(e^{-\pi}-1)|M-e^2\rangle, \\ g|M\rangle = l(e^{-(\mu+1)\pi}-1)|M-e^3\rangle + mn(e^{-\pi}-1)|M-e^1-e^2\rangle \end{gathered} \tag{5.4}$$

is irreducible. Here $e^1 = (1,0,0)$, $e^2 = (0,1,0)$, $e^3 = (0,0,1)$ are the unit vectors in the lattice space Z^3:$\{M = (m,n,l) \mid m,n,l \in Z^+\}$.

Proof: Using the commutation relations of $D(2)$, we can first prove by induction for $n \in Z^+$:

$$
\begin{gathered}
da^n = a^n d + n(e^{-\mu t} - 1)a^{n-1}, \\
fb^n = b^n f + n(e^{-t} - 1)b^{n-1}, \\
gc^n = c^n g + n(e^{-(\mu+1)t} - 1)c^{n-1}, \\
ga^n = a^n g + nfa^{n-1}, \quad gb^n = b^n g - ne^{-t}db^{n-1}, \\
ba^n = a^n b - nca^{n-1}, \quad ab^n = b^n a + ncb^{n-1}, \\
ha^n = a^n h + n\mu a^n, \quad hb^n = b^n h + nb^n, \\
hc^n = c^n h + n(1+\mu)b^n.
\end{gathered} \tag{5.5}
$$

Equations (5.4) follows from Eqs. (5.5) and (5.3) immediately. It is not difficult to verify that Eqs. (5.4) indeed define a representation of $D(2)$. By considering that the indices m, n, and l not only decrease but also increase by unit 1, it can be proved that this representation is irreducible if it is nontrivial.

Let us make an observation that there exist many parameters μ, π, and η. Among them, μ and η are allowed by the quantum double structure and the representation theory respectively while π is due to the existence of the central element t. Since η and π can be used to distinguish the different representations, we can set $x=(\eta,\pi)$ and obtain the colored R matrices with two-dimensional color parameters x where the parameter μ is intrinsic and plays the similar role to that of the q in the standard quantum double quantum algebras. In fact, for a rank l classical Lie algebra, we can introduce $l-1$ independent intrinsic parameters to the corresponding exotic quantum double since its Cartan subalgebra is l dimensional.

VI. DISCUSSIONS

To conclude this paper, we should give some remarks on our exotic quantum double and its relations to the known results, such as the Hopf algebraic structure for the function algebra on the formal group,[42,43] the extended Heisenberg–Weyl algebra (the boson algebra) as a quantum double,[44,46] and so on.

(i) From the construction of the exotic quantum double in this paper, we can see that a commutative (Abelian) algebra, e.g., the subalgebra B, can be endowed with a non-cocommutative Hopf algebraic structure and its quantum dual and quantum double can be deduced as noncommutative algebras. Such a process can be regarded as the inversion of the construction in this paper and maybe provide us with a scheme of "quantization" from commutative object to noncommutative one. An example of this "quantization" was given[45] recently. A simplest associative algebra is generated by two commuting generators X and H. Its non-cocommutative Hopf algebraic structure is defined by

$$
\begin{gathered}
\Delta(H) = H \otimes 1 + 1 \otimes H, \quad \Delta(X) = X \otimes 1 + e^{-H} \otimes X, \\
S(H) = -H, \quad S(X) = -e^{H}X, \quad \epsilon(X) = 0 = \epsilon(H).
\end{gathered}
$$

Let Y and N be the dual generators to X and H, respectively. Its quantum dual has a cocommutative Hopf algebraic structure and the elements X, Y, H, and N generate a quantum double $D(1)$ with the only nonzero commutation relations

$$
[N, X] = X, [N, Y] = -Y, [X, Y] = 1 - e^{-H}.
$$

This quantum double $D(1)$ is a special example of the exotic quantum double where A is a "half" of UEA of A_1. There exists a homomorphisim

$$a \to X, \quad a^+ \to Y, \quad E \to 1-e^{-H}, \quad \hat{N} \to -N$$

from the boson algebra generated by the creation operator a^+, the annihilation operator a, the number operator $\hat{N}$, and the central operator E to this exotic quantum double where the only nonzero commutation relations for the boson algebra are

$$[a, a^+] = E, \quad [\hat{N}, a] = -a, \quad [\hat{N}, a^+] = a^+.$$

This example shows the so-called "quantization" from commutative object to noncommutative one in which the quantum Yang–Baxter equation is enjoyed by the universal R matrix

$$\hat{R} = \exp(X \otimes Y)\exp(N \otimes H).$$

(ii) It has to be pointed out that there are some difficulties in the futher development in constructing other exotic quantum doubles. When one takes the subalgebra B to be the whole UEA of a classical algebra, we hardly write down the dual basis explicitly and so the construction scheme of this paper cannot work well. The similar problem also appears in the discussion in terms of the formal group. How to generalize the method and ideas of this paper to work on the case of the whole UEA other than a Borel subalgebra is the first open question we should mention. The second open question is how to find a finite-dimensional representation for the exotic quantum double except the example of A_1 mentioned above. It is well known that the finite-dimensional R matrices usually make sense in the quantum inverse scattering method and even in the exactly solvable models in statistical mechanics. Thus, it is also expected that some new finite-dimensional R matrices can follow from the exotic quantum double through its universal R matrix where the finite-dimensional representations of the exotic quantum double must be used. However, although we can do it for the special case of A_1 by building the finite-dimensional indecomposable representation of $D(1)$ on certain quotient space of linear space $D(1)$, we cannot obtain a finite-dimensional representation for other higher-rank exotic quantum doubles by the same method due to the existence of the multicenter in certain subalgebras. Therefore, there needs to be futher works on finite-dimensional representation of the exotic quantum double

(iii) In the formal group theory of Lie algbera,[43] the bialgebra structure of the dual to the UEA of a classical Lie algebra can be given abstractly in terms of the formal group. It is not difficult to further define the antipode for this dual bialgebra. So, in this abstract way, the Hopf algebraic structure can be endowed with the dual Hopf algebra of the UEA. However, writing out the explicit Hopf algebraic structure, namely, the explicit multiplication relations, coproduct, antipode, and counit for the dual generators, completely depends on the explict evolution of the Baker–Comppell–Hausdorff formula for classical Lie algebra. However, it is much more difficult to do it even for the simple case, e.g., SU(2). The study in this paper avoids this evalution so that not only the dual Hopf algebraic structure is obtained, but also the corresponding quantum double—the exotic quantum double is built for the Borel subalgebra of the UEA of arbitrary classical Lie algebra by combining the two subalgebras dual to each other.

ACKNOWLEDGMENTS

The author would like to express his sincere thanks to Professor C. N. Yang for drawing his attentions to the research field related to the quantum Yang–Baxter equation and for giving him very kind help. He thanks Professor L. Tankhtajian for many useful and instructive discussions which point out the possible relation of this work to the relevant topics of formal group. He also thanks Professor B. McCoy for telling him about the works on the chiral Potts model and useful discussions.

He is supported by Cha Chi-Ming fellowship through the CEEC at the State University of New York at Stony Brook, and also is supported in part by the NFS of China and Fok Ying-Tung Education Foundation through Northeast Normal University.

[1] C. N. Yang, Phys. Rev. Lett. **19**, 1312 (1967).
[2] R. Baxter, Ann. Phys. **70**, 193 (1972).
[3] E. K. Sklyanin, L. A. Takhtajian, and L. D. Faddeev, Theor. Math. Fis. **40**, 194 (1979).
[4] P. P. Kulish and N. Y. Reshetikhin, J. Phys. A **16**, L591 (1983).
[5] R. J. Baxter, *Exactly-Solved Models in Statistical Mechanics* (Academic, New York, 1982).
[6] A. B. Zamolodchikov and Al. B. Zamolodchikov, Ann. Phys. **120**, 253 (1979).
[7] V. G. Drinfeld, Proc. ICM Berkeley, 1986, edited by A. Gleason (American Mathematical Society, Providence, 1987), p. 798.
[8] M. Jimbo, Lett. Math. Phys. **10**, 63 (1985).
[9] P. P. Kulish and N. Y. Reshetikhin, Zap. nauchn. Seminarod LOMI **101**, 101 (1981).
[10] L. D. Faddeev and L. A. Takhtajian, Lect. Notes Physics **246**, 166 (1986).
[11] N. Y. Reshetikhin, LOMI preprints E-4 and E-11 (1987).
[12] M. Rosso, Commun. Math. Phys. **117**, 307 (1989).
[13] A. Kirillov and N. Y. Reshetikhin, Commun. Math. Phys. **134**, 421 (1990).
[14] H. C. Lee, M. Couture, and N. C. Schmeing, Chalk River preprint CRNL-TP-1125 (1988).
[15] M. L. Ge, Y. S. Wu, and K. Xue, Int. J. Mod. Phys. A **6**, 1645 (1991).
[16] Y. Cheng, M. L. Ge, and K. Xue, Commun. Math. Phys. **136**, 196 (1991).
[17] H. Au-Yang, B. McCoy, J. Perk, S. Tan, and M. L. Yan, Phys. Lett. A **123**, 219 (1987).
[18] B. McCoy, J. Perk, S. Tan, and C. H. Sah, Phys. Lett. A **125**, 9 (1987).
[19] R. J. Baxter, J. Perk, and H. Au-Yang, Phys. Lett. A **128**, 138 (1988).
[20] V. V. Bazhanov and Y. G. Stroganov, J. Stat. Phys. **59**, 799 (1990).
[21] E. Date, M. Jimbo, K. Mike, and T. Miwa, Phys. Lett. A **148**, 45 (1990).
[22] E. Date, M. Jimbo, K. Mike, and T. Miwa, Commun. Math. Phys. **137**, 133 (1991).
[23] V. V. Bazhanov, R. M. Kashaev, V. V. Mangazeev, and Y. G. Stroganov, Commun. Math. Phys. **138**, 393 (1991).
[24] J. Murakami, Osaka preprints, 1990.
[25] C. P. Sun, X. F. Liu, and M. L. Ge, J. Math. Phys. **32**, 2409 (1991).
[26] M. L. Ge, X. F. Liu, and C. P. Sun, Phys. Lett. A **155**, 137 (1991).
[27] M. L. Ge, X. F. Liu, and C. P. Sun, Phys. Lett. A **160**, 433 (1991).
[28] M. L. Ge, C. P. Sun, and K. Xue, Int. J. Mod. Phys. A **7**, 6609 (1992).
[29] M. L. Ge, C. P. Sun, and K. Xue, Phys. Lett. A **163**, 176 (1992).
[30] G. Lusztig, Contemp. Math. **82**, 59 (1989).
[31] P. Roche and D. Arnaudon, Lett. Math. Phys. **17**, 295 (1989).
[32] C. P. Sun, J. F. Lu, and M. L. Ge, J. Phys. A **23**, L1199 (1990).
[33] L. Alvarez-Gaume, C. Gomez, and G. Sirra, Nucl. Phys. B **330**, 347 (1990).
[34] V. Pasquier and H. Saleur, Nucl. Phys. B **330**, 523 (1990).
[35] C. De Concini and V. G. Kac, *Representations of Quantun Group at roots of 1*, preprint (1990).
[36] C. P. Sun and M. L. Ge, J. Phys. A **24**, L969, 3265, 3731 (1991).
[37] C. P. Sun and M. L. Ge, J. Phys. A **25**, 19 (1992).
[38] C. P. Sun, H. C. Fu, and M. L. Ge, Lett. Math. Phys. **23**, 19 (1990).
[39] X. F. Liu and C. P. Sun, Sci. China A **35**, 73 (1992).
[40] X. F. Liu and M. L. Ge, Lett. Math. Phys. **24**, 197 (1992).
[41] M. Jimbo, *The Topics from the Representations of $U(g)_q$*, in *Nankai Lecture Series on Mathematical Physics*, edited by M. L. Ge (World Scientific, Singapore, 1992).
[42] L. A. Tankhtajian, *Quantum Groups*, in *Nankai Lecture Series In Mathematical Physics, 1989*, edited by M. L. Ge and B. H. Zhao (World Scientific, Singapore, 1991).
[43] J-P Serre, *Lie algebras and Lie Groups* (Benjamin, New York, 1965).
[44] C. P. Sun, X. F. Liu, and M. L. Ge, J. Math. Phys. **34**, 1218 (1993).
[45] C. P. Sun, preprint ITP.SB-92-61 (1992).
[46] W. Li, C. P. Sun, and M. L. Ge, ITP.SB-92-59 (1992).