The q-analog of the boson algebra, its representation on the Fock space, and applications to the quantum group

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In this paper, a realization of the q-deformed boson operators on the Fock space from a generally algebraic point of view is given. The representations of the quantum group $(C_n)_q$ are thereby constructed in terms of this realization. Some infinite- and finite-dimensional representations of the q-analog of the Heisenberg–Weyl algebra are obtained on certain quotient spaces. Finally, the q-deformed differential realization of quantum group given by Alvarez-Gaume, Gomez, and Sierra (Preprint CERN-Th 5369/89) is derived from the boson realization.

I. INTRODUCTION

Recently, there is much interest in the Yang–Baxter equation(YBE),[1,2] which plays a crucial role in exactly solvable models in statistical mechanics and integrable models in low-dimensional field theories. Drinfeld[3] and Jimbo[4–6] showed that a q-deformation $U(L)_q$ of the universal enveloping algebra $U(L)$ of a simple Lie algebra L has a Hopf algebraic structure and there is a solution of the YBE to each representation of $U(L)_q$. Now, the associative algebra $U(L)_q$ is loosely called quantum group (QG). Thus it is significant to develop the representation theory of QG.

The q-deformed boson realization (also called q-analog of Swhinger–Jordan mapping) for QG has independently been achieved by different authors,[7–9] which simplifies the manipulations constructing representations of QG. However, this realization is not carried out on the true physics space, the Fock space, but on a q-deformation of the usual Fock space constructed with the q-deformed boson operators.

In this paper, by analyzing the properties of the boson algebra, the q-deformed boson operators used in Refs. 7–9 are realized on the Fock space, i.e., these operators are expressed in terms of the usual creation operators and annihilation operators of boson states. The representations of QG are then obtained on the Fock space with $(C_n)_q$ as an example. From the algebraic point of view, we also obtain infinite-dimensional indecomposable representations and finite-dimensional representations of a q-analog of the Heisenberg–Weyl (HW) algebra, which is a subalgebra of the boson algebra and also called q-HW algebra. We also show that the q-deformed differential realization of QG given by Alvarez-Gaume *et al.*[10] can be derived by the boson realization of this paper. The works in this paper can be regarded as continuation of the previous works.[11–15]

II. REFORMULATION OF SOME WELL-ESTABLISHED FACTS AND GENERALIZATION

The one-state HW algebra H_1 is a Lie algebra spanned by the operators b^+, b, and e that satisfy the commutation relations

$$[b,b^+]=e, \quad [e,b^+]=0=[e,b]. \tag{1}$$

According to the PBW theorem, the basis for the universal enveloping algebra $U(H_1)$ of H_1 can be chosen as

$$u(m,n,r)=b^{+m}\cdot b^n\cdot e^r, \quad m,n,r\in\mathbb{N}=\{0,1,2,...\}.$$

With respect to the left ideal L_e generated by the element e-1 the quotient algebra is $U(H_1)/L_e\equiv\Omega(H_1)$:

$$u(m,n)=u(m,n,0)\,\mathrm{Mod}\,L, \quad m,n\in\mathbb{N}.$$

Then, we can identify operator e with real number 1 on $\Omega(H_1)$ as a linear space. Here, $\Omega(H_1)$ also has a quotient space $F_1(\lambda)=\Omega(H_1)/L_b$:

$$u(m)=u(m,0)\,\mathrm{Mod}\,L_b, \quad m\in\mathbb{N},$$

corresponding to a left ideal L_b generated by the element $b-\lambda$ ($\lambda\in$ of the field $\mathbb{C}$ of a complex number).

As associative algebras, $\Omega(H_1), U(H_1)$, and $F_1(\lambda)$ are naturally linear spaces. As operators, the elements of $\Omega(H_1)$ and $U(H_1)$ act on these spaces in a natural way:

$$\rho(x)\cdot\bar{u}=x\cdot\bar{u}, \quad x\in U(H_1)$$

or

$$\Omega(H_1); \quad \forall\bar{u}\in U(H),\Omega(H_1),F_1(\lambda).$$

Using the associative mutiplication and (1), we have

$$b\cdot u(0)=\lambda u(0), \quad b^+\cdot b\cdot u(n)=n\cdot u(n),$$
$$b^+\cdot u(n)=u(n+1), \quad b\cdot u(n)=n\cdot u(n-1), \tag{2}$$

The vectors $u(n)(n\in\mathbb{N})$ for $\lambda=0$ are denoted by $v(n)$ and then $b\cdot v(0)=0$. Thus $v(n)$ is just an unnormalized basis for the Fock space

$$F_1=F_1(\lambda=0):\{|n\rangle=v(n)|n\in\mathbb{N}\},$$

with a vacuum state $v(0)=|0\rangle$. In this Fock space, $|\lambda\rangle=u(0)$ is a coherent state satisfying $b|\lambda\rangle=\lambda|\lambda\rangle$, and b^+ and b are, respectively, called creation operator and annihilation operator. The associative algebra $\Omega(H_1)$ is an operator algebra on the Fock space F_1 with the definition of action (2). The above-mentioned facts can be seen in Ref. 16.

The boson algebra B_1 is an associative algebra generated

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by b, b^+, and N. Here, N is an abstract number operator on $\Omega(H_1)$ satisfying

$$[N,b^+] = b^+, \quad [N,b] = -b. \tag{3}$$

In general, N is not necessarily $b^+ \cdot b$. Considering a correspondence between the basis of B_1,

$$f(m,n,r) = b^{+m} \cdot b^n \cdot N^r, \quad m,n,r \in \mathbb{N}$$

and the element

$$b^{+m} \cdot b^n \cdot (b^+ \cdot b)^r = \sum_{k=1}^{m+r} \sum_{l=1}^{n+r} C_{kl} \cdot b^{+k} \cdot b^l \in \Omega(H_1), \tag{4}$$

of $\Omega(H_1)$, we can observe that B_1 is an isomorphism of a subalgebra $\{b^{+m} \cdot b^n \cdot (b^+ \cdot b)^r | m,n,r \in \mathbb{N}\}$ of $\Omega(H_1)$. Here, the coefficients C_{kl} are determined by the basic commutation relations (1).

Corresponding to b and b^+, respectively, the q-deformed boson operators are denoted by a and a^+ and satisfy

$$[N,a] = -a, [N,a^+] = a^+,$$
$$a \cdot a^+ - q^{-1} \cdot a^+ \cdot a = q^N = Q, \quad q \in \mathbb{C}. \tag{5}$$

The operators a, a^+, and N generate an associative algebra $B_1(q)$, which is a q-analog of the boson algebra and called q-boson algebra. This algebra has been used to obtain the representations of the quantum groups $SU(2)_q = U(su(2))_q$ and $SU(N)_q = U(su(N))_q$ on the q-deformed Fock spaces.[9] As an example of these spaces, a one-state q-deformed Fock space is defined by

$$\widetilde{F}_1(q) : \{a^{+n}|\tilde{0}\rangle \,|\, a|\tilde{0}\rangle = 0\}.$$

It is useful to note that the abstract number operator N is not $a^+ \cdot a$ even in isomorphic significance. As follows we will give a and a^+ a boson realization, that is to say, we will express a and a^+ in terms of the usual boson operators b and b^+.

III. REALIZATION OF $B_1(q)$ ON THE FOCK SPACE F_1

It can be directly proved that there exists such a pair of operators a and a^+ on the Fock space F_1 that

$$a^+|n\rangle = |n+1\rangle, \quad a|n\rangle = [n]|n-1\rangle, \tag{6}$$

where the definition $[f] = (q^f - q^{-f})/(q - q^{-1})$ holds for any complex number f or operator f. In fact, in terms of the vacuum projective operator

$$|0\rangle\langle 0| = :e^{-b^+ \cdot b}: = \sum_{k=0}^{\infty} \left\{ \frac{(-1)^k}{k!} \right\} b^{+k} \cdot b^k, \tag{7}$$

the q-deformed boson operators a^+ and a are expressed as

$$a^+ = b^+;$$
$$a = \sum_{n=0}^{\infty} \frac{[n]|n-1\rangle\langle n|}{n!}$$
$$= \sum_{n=0}^{\infty} [n] \cdot b^{+n-1} |0\rangle\langle 0| \left(\frac{b^n}{n!} \right)$$
$$= \sum_{k,n=0}^{\infty} \left\{ \frac{(-1)^k \cdot [n]}{(n! \cdot k!)} \right\} b^{+n+k-1} \cdot b^{n+k}, \tag{8}$$

and the number operator N is expressed as $N = b^+ \cdot b$. Through (6), it is easy to check that the operators a and a^+ given by (8) satisfy the basic relations (5). Therefore, (8) gives a boson realization of the q-analog of the boson algebra B_1.

For the applications in the following discussion, we give some useful relations:

$$a \cdot a^{+m} = q^{-m} \cdot a^{+m} \cdot a + [m] \cdot a^{+m-1} \cdot Q, \tag{9a}$$
$$Q \cdot a^+ = q \cdot a^+ \cdot Q, \quad Q \cdot a = q^{-1} \cdot a \cdot Q, \tag{9b}$$
$$Q \cdot a^{+m} = q^m \cdot a^{+m} \cdot Q, \quad Q \cdot a^m = q^{-m} \cdot a^m \cdot Q, \tag{9c}$$
$$a^+ \cdot Q^m = q^{-1} \cdot Q^m \cdot a^+, \quad a \cdot Q^m = q^m \cdot Q^m \cdot a, \tag{9d}$$

from (5) by induction. It should be pointed out that the above relations also hold for the abstract operators a, a^+, and Q, which do not need to be on the Fock space and only need to satisfy (5).

From (5) and (9b), we observe that the operators a, a^+, and $Q = q^N$ generate a subalgebra $H_1(q)$ of the associative algebra $B_1(q)$. As $q \to 1$, the generating relations

$$a \cdot a^+ - q^{-1} \cdot a^+ \cdot a = Q, \quad Q \cdot a^+ = q \cdot a^+ \cdot Q,$$
$$Q \cdot a = q^{-1} \cdot a \cdot Q, \tag{10}$$

of $H_1(q)$ become the commutation relations of the Lie algebra H_1 on the quotient space $\Omega(H_1)$, where $e = 1$. Therefore, $H_1(q)$ is a q-analog of the associative algebra $\Omega(H_1)$ and we call it the q-HW algebra.

The above discussion can naturally be extended to the case of the n-state HW algebra $H_n : \{b_i, b_i^+, e | i = 1,2,...,n\}$. Correspondingly, we have an n-state boson algebra B_k with number operators N_i $(i = 1,2,...,n)$, its q-analog $B_n(q)$, and the subalgebra $H_n(q)$ with $Q_i = q^{N_i}$.

IV. REPRESENTATIONS OF THE q-HW ALGEBRA

We consider the representation of the associative algebra $H_1(q)$ on itself defined by

$$\rho(g) \cdot x = g \cdot x \begin{cases} \text{for any } g \in H_1(q) \text{ as an associative algebra,} \\ \text{for any } x \in H_1(q) \text{ as a linear space.} \end{cases}$$

The above defined representation is called left regular representation. On the basis

$$X(m,k,r) = a^{+m} \cdot a^k \cdot Q^r, \quad m,k,r \in \mathbb{N}$$

for $H_1(q)$, the explicit expression of the representation is

$$\rho(a^+) X(m,k,r) = X(m+1,k,r),$$
$$\rho(a) X(m,k,r) = q^{-m} X(m,k+1,r) + q^{-k}[m] X(m-1,k,r+1), \tag{11}$$
$$\rho(Q) X(m,k,r) = q^{m-n} X(m,k,r+1).$$

Like the master representation of the Lie algebra, the representation (11) is indecomposable in the label r as well as label k. This follows from the fact that the values of k and r do not decrease under action of this representation. For given M and $R \in \mathbb{N}$ the subspace $V(M,R)$:

$$\{X(m, M+k, R+r) | m,k,r \in \mathbb{N}\}$$

transforms invariantly under the action of ρ, but its complement space $\overline{V}(M,R)$:

$$\{X(m,k,r)|m\in\mathbb{N},\quad k=0,1,2,...,M-1, r=0,1,2,...,r-1\}$$

is not invariant under this action. Thus (11) gives an infinite-dimensional indecomposable representation of $H_1(q)$ and subduces some new indecomposable representations on certain invariant subspaces $V(M,R)$.

On the quotient space $V_1=H_1(q)/I_a$:$\{X(m,r)=X(m,0,r)\mathrm{Mod}\cdot I_a\}$ with respect to the left ideal I_a generated by $a-\mu$ ($\mu\in\mathbb{C}$), the representation (10) induces a new representation

$$\rho(a^+)X(m,r)=X(m+1,r),$$

$$\rho(a)X(m,r)=\mu\cdot q^{r-m}X(m,r)+[m]X(m-1,r+1), \tag{12}$$

$$\rho(Q)X(m,r)=q^mX(m,r+1).$$

Corresponding to an invariant subspace S_M: $\{X(m,r)|m+r>\text{ a given } M\in\mathbb{N}\}$, the quotient space $Q_M=V_1/S_M$:$\{W(m,r)=X(m,r)\mathrm{Mod}\cdot S_M\}$ is a finite-dimensional space with the dimension

$$\dim\cdot Q_M=\tfrac{1}{2}(M+1)(M+2). \tag{13}$$

On the space Q_M, the representation (12) induces a finite-dimensional representation

$$\rho(a)W(m,r)=\begin{cases}W(m+1,r), & \text{for } m+r<M,\\ 0, & \text{for } m+r=M;\end{cases}$$

$$\rho(Q)W(m,r)=\begin{cases}q^mW(m,r+1), & \text{for } m+r<M,\\ 0, & \text{for } m+r=M;\end{cases}$$

$$\rho(a)W(m,r)=\mu q^{-m}W(m,r)+[m]W(m-1,r+1). \tag{14}$$

For example, on the space Q_1:$\{W(0,0),W(1,0),W(0,1)\}$, (14) gives a three-dimensional representation,

$$\rho(a^+)=\begin{pmatrix}0&0&0\\1&0&0\\0&0&0\end{pmatrix},\quad \rho(Q)=\begin{pmatrix}0&0&0\\0&0&0\\1&0&0\end{pmatrix},$$

$$\rho(a)=\begin{pmatrix}\mu&0&0\\0&\mu q^{-1}&0\\0&1&\mu q\end{pmatrix}. \tag{15}$$

It is easy to check that the representations (11), (12), (14), and (15) indeed satisfy the generating relation (10).

V. BOSON REALIZATION OF THE QUANTUM GROUP $(C_n)_q$ IN THE FOCK SPACE

The quantum group $U(L)_q$ is an associative algebra generated by the elements h_i, e_i, and f_i ($i=1,2,...,s$) that satisfy the q-deformed commutation relations[3,17,18]

$$\begin{aligned}&[h_i,h_j]=0,\quad [h_i,e_j]=\alpha_{ij}e_j,\\ &[h_i,f_j]=-\alpha_{ij}f_j,\\ &[e_i,f_j]=\delta_{ij}[h]_{q_i},\end{aligned} \tag{16}$$

and the Serre relations

$$\sum_{0\leqslant k\leqslant 1-\alpha_{ij}}(-1)^k\begin{bmatrix}1-\alpha_{ij}\\ k\end{bmatrix}_{q_i}\cdot G_i^{1-\alpha_{ij}-k}\cdot G_j\cdot G_i^k=0, \tag{17}$$

$$G=f,e,$$

where α_{ij} is the matrix element of the Cartan matrix $\alpha=(\alpha_{ij})$ corresponding to a classical Lie algebra L and we have defined

$$\begin{bmatrix}m\\ n\end{bmatrix}_{q_i}=[m]!_{q_i}\{[m-n]!_{q_i}[n]!_{q_i}\}^{-1}$$

$$\begin{aligned}&[m]!_{q_i}=[m]_{q_i}[m-1]_{q_i}\cdots[2]_{q_i}[1]_{q_i},\\ &[\tilde{f}]_{q_i}=(q_i^{\tilde{f}}-q_i^{-\tilde{f}})/(q_i-q_i^{-1})\\ &q_i=q^{d_i},\quad d_i\alpha_{ij}=d_j\alpha_{ji},\quad d_i\in\{\pm1,\pm2,...\}.\end{aligned} \tag{18}$$

The boson realization of a quantum group is an isomorphic mapping B of $U(L)_q$ to the operator algebra $\Omega(H_n)$ on the Fock space F_n. Now, we consider the case associated with Lie algebra C_n. Since the Cartan matrix $\alpha(C_n)$ of C_n is related to the Cartan matrix $\alpha(A_{n-1})$ of the Lie algebra A_{n-1} in the following way:[18]

$$\alpha(C_n)=\begin{pmatrix}\alpha(A_{n-1}) & & 0\\ & & \vdots\\ & & 0\\ & & -2\\ 0\cdots0, & -1 & 2\end{pmatrix},$$

$$\alpha(A_{n-1})=\begin{pmatrix}2,-1,0,...,0\\ -1,2,0,...,0\\ \cdots\cdots\cdots\\ 0,0,...,2,-1\\ 0,0,...,-1,2\end{pmatrix},$$

we can generalize the q-deformed boson realization of $(A_{n-1})_q$ (Ref. 9) to obtain the boson realization of $(C_n)_q$:

$$\begin{aligned}&\hat{e}_i=B(e_i)=b_i^+a_{i+1},\quad \hat{f}_i=B(f_i)=b_{i+1}^+a_i,\\ &\hat{h}_i=B(h_i)=b_i^+b_i-b_{i+1}^+b_{i+1}\quad (i=1,2,...,n-1)\\ &\hat{e}_n=B(e_n)=(q+q^{-1})^{-1}b_n^{+2},\\ &\hat{f}=B(f_n)=-(q+q^{-1})^{-1}a_n^2,\\ &\hat{h}_n=B(h_n)=b_n^+b_n-\tfrac{1}{2},\end{aligned} \tag{19}$$

where a_i is defined as

$$a_i=\sum_{k,m=0}^{\infty}(-1)^k[m]_q(k!m!)^{-1}b_i^{+m+k-1}b_i^{m+k}, \tag{20}$$

$$i=1,2,...,n.$$

It is easy to check that the commutation relations (16) are satisfied by this realization (19). Using (5) and (8), we calculate

$$\hat{G}_j^2\hat{G}_{j\pm1}-(q+q^{-1})\hat{G}_j\hat{G}_{j\pm1}\hat{G}_j+\hat{G}_{j\pm1}\hat{G}_j^2=0 \qquad (j=1,2,...,n-1), \tag{21}$$

$$\sum_{m=0}^{3}(-1)^m\begin{bmatrix}3\\ m\end{bmatrix}_q\hat{G}_{n-1}^{3-m}\hat{G}_n\hat{G}_{n-1}^m=0,\quad G=e,f, \tag{22}$$

that is to say, the Serre relations are also satisfied by this realization. However, as Biedenharn pointed out in his own case of SU(2),[7] this realization only holds on the Fock space, but does not hold on other spaces such as $\Omega(H_n)$ and $\mathrm{U}(H_n)$.

From this realization, we easily obtain an infinite-dimensional representation of $(C_n)_q$ on the Fock space F_n as

$$\Gamma(h_j)|k_i\rangle = (k_i - k_{i+1})|k_i\rangle,$$
$$\Gamma(e_j)|k_i\rangle = [k_{j+1}]|k_i - \delta_{i,j+1} + \delta_{ij}\rangle,$$
$$\Gamma(f_j)|k_i\rangle = [k_i]|k_i + \delta_{i,j+1} - \delta_{ij}\rangle, \quad j = 1,2,...,n-1;$$
$$\Gamma(e_n)|k_i\rangle = (q+q^{-1})^{-1}|k_i + 2\delta_{ni}\rangle,$$
$$\Gamma(f_n)|k_i\rangle = -(q+q^{-1})^{-1}[k_n][k_n - 1]|k_i - 2\delta_{ni}\rangle, \tag{23}$$
$$\Gamma(h_n)|k_i\rangle = (k_n - \tfrac{1}{2})|k_i\rangle;$$
$$|k_i\rangle = b_1^{+k_1} b_2^{+k_2} \cdots b_n^{+k_n}|0\rangle,$$
$$k_1, k_2, ..., k_n \in \mathbb{N}.$$

Because the sum $K = \Sigma_{i=1}^n k_i$ is changed by two or zero under the action of Γ, namely, $(-1)^K$ is invariant under this action, the representation Γ is reducible and correspondingly the representation space F_n is decomposed into a direct sum of two invariant subspaces

$$F_n^+ : \{|k_i\rangle | \sum_{i=1}^n k_i = 2p, p\in\mathbb{N}\}$$

and

$$F_n^- : \{|k_i\rangle | \sum_{i=1}^n k_i = 2p+1, p\in\mathbb{N}\}.$$

On a space F_n^+ or F_n^-, Γ subduces an irreducible representation Γ^+ or Γ^-, i.e., $\Gamma = \Gamma^+ \oplus \Gamma^-$.

It should be pointed out that each representation obtained here is only a symmetrized one and we will need to use q-analogs of multiboson operators[19] to obtain other type representations according to Biedernharn.[8]

VI. FROM THE BOSON REALIZATION TO THE *Q*-ANALOG OF DIFFERENTIAL REALIZATION

There obviously exists the 1–1 correspondences

$$b^+ \leftrightarrow z, \quad b \leftrightarrow \frac{\partial}{\partial z}, |m\rangle \leftrightarrow z^m,$$

between the Fock space $F_1 : \{|m\rangle = b^{+m}|0\rangle\}$ and the Bargmenn space $\tilde{S}_1$ $\{f(z)|f(z)$ is a holomorphic function of one variable z on the complex plane $\mathbb{C}\}$ with the basis $z^n (n\in\mathbb{N})$. We define a q-analog D_z of usual differential operator ∂/z as the correspondence of b by

$$D_z \cdot z^m = [m] \cdot z^{m-1}. \tag{24}$$

Then, we have

$$D_z(z-\lambda)^m = ((qz-\lambda)^m - (q^{-1}z - \lambda)^m)/((q-q^{-1})z) \tag{25}$$

and then

$$D_z f(z) = \sum_{m=0}^{\infty} \left(\frac{f^{(m)}(\lambda)}{m!}\right) D_z (z-\lambda)^m = \frac{f(qz) - f(q^{-1}z)}{(q-q^{-1})z}, \tag{26}$$

where

$$f^{(m)}(\lambda) = \left(\frac{\partial^m}{\partial z^m}\right) f(z)\Big|_{z=\lambda}.$$

In fact, the operator D_z is an integration operator on $\tilde{S}_1$ with the integration kernal

$$\mathscr{K}(\xi,z) = ((\xi - qz)(\xi - q^{-1}z))^{-1}, \tag{27}$$

i.e.,

$$D_z f(z) = \left(\frac{1}{(2\pi i)}\right) \oint_c \mathscr{K}(\xi,z) f(\xi) d\xi, \tag{28}$$

where c is a close curve around the points qz and $q^{-1}z$ on the complex plane $\mathbb{C}$. As $q\to 1, D_z \to \partial/\partial z$. The q-analog operator D_z of $\partial/\partial z$, which is derived in terms of boson realization here, has been given by Alvarez-Gaume *et al.* in Ref. 10. They applied it to construct the representations of $\mathrm{SU}(2)_q$ as follows:

$$J_+ = z_1 D_{z_2}, J_- = z_2 D_{z_1},$$
$$J_3 = \tfrac{1}{2}\left(z_1 \frac{\partial}{\partial z_1} - z_2 \frac{\partial}{\partial z_2}\right),$$
$$|j,m\rangle = ([j+m]![j-m]!)^{-1/2} z_1^{j+m} z_2^{j-m},$$
$$J\pm |j,m\rangle = ([j\mp m][j\pm m+1])^{-1/2}|j,m\rangle, \tag{29}$$
$$J_3|j,m\rangle = m|j,m\rangle,$$
$$j = 0,\tfrac{1}{2},1,\tfrac{3}{2},2...; \quad m = j,j-1,j-2,...,-j \text{ for a given } j.$$

Thus the discussion in this section links up the boson realization and the differential realization for the quantum group.

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[1] C. N. Yang, Phys. Rev. Lett. **19**, 1312 (1967).
[2] R. J. Baxter, *Exactly Solved Models in Statistical Mechanics* (Academic, London, 1982).
[3] V. G. Drinfel'd, *Proceedings of the IMC* (Berkeley, CA, 1986), p. 798.
[4] M. Jimbo, Lett. Math. Phys. **10**, 63 (1985).
[5] M. Jimbo, Lett. Math. Phys. **11**, 247 (1986).
[6] M. Jimbo, Commun. Math. Phys. **102**, 537 (1986).
[7] L. C. Biedenharn, J. Phys. A. **22**, L873 (1989).
[8] A. J. Macfarlane, J. Phys. A **22** (1989).
[9] C.-P. Sun and H.-C. Fu, J. Phys. A **22**, L983 (1989).
[10] L. Alveraz-Gaume *et al.* preprint CERN-Th 5369/89.
[11] C.-P. Sun, J. Phys. A **20**, 4551 (1987).
[12] C.-P. Sun, J. Phys. A **20**, 5823 (1987).
[13] C.-P. Sun, J. Phys. A **20**, L1157 (1987).
[14] C.-P. Sun and H.-C. Fu, Nuovo Cimento B **115**, 1 (1990).
[15] H.-C. Fu and C.-P. Sun, J. Math. Phys. **31**, 287 (1990).
[16] B. Gruber, H. D. Doebner, and Ph. Feinsilver, KINAM **4**, 241 (1982).
[17] M. Rosso, Commun. Math. Phys. **117**, 581 (1988).
[18] T. Hayashi, Commun. Math. Phys. **127**, 129 (1990).
[19] L. C. Biedenharm, in *Group Theory and its Applications*, edited by E. H. Lieb (Academic, New York, 1971), p. 1.