A new q-deformed boson realization of quantum algebra $sl_q(2)$ and nongeneric $sl_q(2)$ R-matrices

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In this paper a new q-deformed boson realization of quantum algebra $sl_q(n+1)$ is presented for the first time, and its representations are obtained in the nongeneric case that q is a root of unity. A new type of $sl_q(2)$ R-matrices are systematically constructed through the universal R-matrix.

I. INTRODUCTION

It is well known that R-matrices for Yang–Baxter equation play a crucial role in nonlinear physics such as low-dimensional integrable field theory, exactly solvable models in statistical mechanics, and conformal field theory.[1-3] The standard way of obtaining an R-matrix is substituting the representation of a quantum group (sometimes strictly called q-analog of universal enveloping algebra of a classical Lie algebra or quantum algebra) into the so-called universal R-matrix.[4-6] The R-matrices obtained in this way are called standard R-matrices. Recently, many new R-matrices, that are sometimes called nonstandard R-matrices, have been constructed by means of the extended Kauffman's diagram technique.[7] These new R-matrices associated with classical Lie algebras provide new representations of braid group and have been Yang–Baxterized to satisfy Yang–Baxter equation with a spectral parameter.[8,9] The possible relations between the nonstandard R-matrices and quantum algebras have also been discussed with the $sl_q(2)$ case as an example.[10]

The main purpose of this paper is to construct a new type of R-matrix essentially different from the standard and nonstandard ones in the standard way. To this end, we first establish a nongeneric q-deformed boson realization of the finite-dimensional representations of quantum algebra $sl_q(2)$ based on the previous works,[11-14] then we generally construct a new class of R-matrices associated with $sl_q(2)$ through the universal R-matrix. Because of the indecomposition property of the new representations used in this paper, the obtained R-matrices possess the following nongeneric properties: (1) They satisfy the Yang–Baxter equation only when q is a root of unity and cannot be obtained from the corresponding standard or nonstandard R-matrices by letting $q^p=1$. (2) They no longer have the eigenvalue structure that the standard and nonstandard R-matrices possess, thus the scheme to Yang–Baxterize them may be completely new and need further investigation.

II. A NEW q-DEFORMED BOSON REALIZATION OF $sl_q(n+1)$

According to the discussion about the q-deformed boson operators independently presented by different authors, we can define a q-deformed boson algebra $\mathscr{B}_q(l)$ as an associative algebra over the complex number field. It is generated by a_i^+, $a_i=a_i^-$, and N_i $(i=1,2,\ldots,l)$ that satisfy

$$\begin{aligned}&a_ia_i^+=[N_i+1],\quad a_i^+a_i=[N_i],\quad a_i^+a_j=a_ja_i^+\\&(i\neq j),\quad [N_i,a_j^\pm]=\pm\delta_{ij}a_j^\pm,\\&[a_i^\pm,a_j^\pm]=0,\quad [N_i,N_j]=0.\end{aligned}\tag{2.1}$$

The so-called q-deformed boson realization of a quantum algebra $G_q:\{g\}$ is the image $B(G_q)=\widehat{G}_q:\{\hat{g}=B(g)\}$ of a homomorphic mapping $B:G_q\to\mathscr{B}_q(l)$. Now let us turn to quantum algebra $sl_q(n+1)$, which is generated by X_i^+ and H_i $(i=1,2,3,\ldots,n)$ that satisfy

$$\begin{aligned}&[H_i,X_j^\pm]=\pm\alpha_{ij}X_j^\pm,\\&[X_i^+,X_j^-]=\delta_{ij}[H_j],\\&X_i^{\pm 2}X_{i\pm 1}^\pm-(q+q^{-1})X_i^\pm X_{i\pm 1}^\pm X_i^\pm+X_{i\pm 1}^\pm X_i^{\pm 2}=0,\end{aligned}\tag{2.2}$$

where $\alpha_{ij}=2\delta_{ij}-\delta_{ij+1}-\delta_{ij-1}$. From (2.1) we can easily prove that the following elements of $\mathscr{B}_q(n+1)$

$$\begin{aligned}&X_i^+=a_i^+a_{i+1}^+,\quad X_1^-=-a_ia_{i+1},\\&H_i=(N_i+N_{i+j}+j),\\&\qquad i=1,3,5,\ldots,n-1+\tfrac{1}{2}(1-(-1)^n)\\&X_j^+=-a_ja_{j+1},\quad X_j^-=a_j^+a_{j+1}^+,\\&H_j=-(N_j+N_{j+1}+1),\\&\qquad j=2,4,6,\ldots,n-\tfrac{1}{2}(1-(-1)^n),\end{aligned}\tag{2.3}$$

indeed define a q-deformed boson realization of $sl_q(n+1)$, that is to say, (2.3) satisfy (2.2).

Before going further we would like to point out that there is an essential difference between the above realization and the previous one. The previous q-deformed boson realization $B(\mathrm{sl}_q(n+1))$:$\{\hat{g}=B(g)\}$ satisfies

$$[\hat{g},N]=0, \quad N=\sum_{i=1}^{n+1} N_i \quad \forall g\in \mathrm{sl}_q(n+1).$$

In other words, "the number of particles" remains unchanged under the action of $\hat{g}$, but it is not the case now. In contrast, there exist some $g\in \mathrm{sl}_q(n+1)$ such that $[\hat{g},N]\neq 0$, so it is no longer possible to find a finite-dimensional representation space for $\mathrm{sl}_q(n+1)$ when q is generic. Fortunately, when q is a root of unity we can find some $\mathrm{sl}_q(n+1)$-invariant subspaces and finite-dimensional representation spaces for $\mathrm{sl}_q(n+1)$. This is done in the next section.

III. FINITE-DIMENSIONAL REPRESENTATIONS OF $\mathrm{sl}_q(n+1)$

For the q-deformed boson algebra $\mathscr{B}_q(n+1)$, we define a q-deformed Fock space $\mathscr{F}_q(n+1)$

$$\begin{aligned}\{|m_i\rangle &= |m_1 m_2\cdots m_{n+1}\rangle = a_1^{+m_1} a_2^{+m_2}\cdots a_{n+1}^{+m_{n+1}}|0\rangle \\ &\quad \times |m_1\cdots m_{n+1}\in \mathbb{Z}^+ \\ &= \{0,1,2,\ldots\}\},\end{aligned}$$

where the q-vacuum state $|0\rangle$ satisfies

$$a_i|0\rangle = N_i|0\rangle = 0, \quad i=1,2,\ldots,n+1,$$

then we obtain a natural representation ρ of $\mathrm{sl}_q(n+1)$ on $\mathscr{F}_q$:

$$\begin{aligned} g_i|m_j\rangle &= |m_j+\delta_{ij}+\delta_{i+1\,j}\rangle, \\ f_i|m_j\rangle &= -[m_i][m_{i+1}]|m_i-\delta_{ij}-\delta_{i+1\,j}\rangle \qquad (3.1)\\ h_i|m_j\rangle &= \pm(m_i+m_{i+1}+1)|m_j\rangle,\end{aligned}$$

where $g_i = a_i^+ a_{i+1}^+$, $f_i = -a_i a_{i+1}$, and $h_i = \pm(N_i+N_{i+1}+1)$ are the elements of the set $\{X_i^{\pm}, H_i | i=1,2,\ldots,n+1\}$. From (3.1) it is easily seen that there is a ρ-invariant subspace $V_{n+1}^{[N]}$ (for a given $N\in\mathbb{Z}^+$)

$$\Big\{|m_j\rangle\in\mathscr{F}_q(n+1)\,|\sum_k m_{2k}-\sum_k m_{2k-1}=N\Big\}.$$

When q is nongeneric, i.e., $q^p=1$, $V^{[N]}$ has ρ-invariant subspaces

$$\begin{aligned}&W_{n+1}^{[N]}(\alpha_i,p)\,[i=1,2,3,\ldots,\tfrac{1}{2}(n-\tfrac{1}{2}(1-(-1)^n)),\alpha_i\in\mathbb{Z}^+]\\ &\quad =\Big\{|m_j\rangle\in V^{[N]}|m_{2i}=N+\sum_k m_{2k-1}+\sum_{k\neq i} m_{2k}\geqslant\alpha_i p\Big\}.\end{aligned}$$

It will be shown that in some cases, the quotient spaces $V_{n+1}^{[N]}/\Sigma_i W_{n+1}^{[N]}(\alpha_i,p)$ are finite dimensional.

Now we consider $\mathrm{sl}_q(3)$ as an explicit example. In this special case, the q-deformed boson realization is

$$\begin{aligned} H_1 &= (N_1+N_2+1), \quad e_1 = X_1^+ = a_1^+ a_2^+, \\ f_1 &= X_1^- = -a_1 a_2 \\ H_2 &= -(N_2+N_3+1), \quad e_2 = X_2^+ = -a_2 a_3, \\ f_2 &= X_2^- = a_2^+ a_3^+.\end{aligned}$$

Correspondingly, we denote the basis of $V_3^{[N]}$ by $F(m,n)=|m,N+m+n,n\rangle$. On this basis we obtain a representation $\rho^{[N]}$:

$$\begin{aligned} e_1F(m,n) &= F(m+1,n) \\ f_1F(m,n) &= -[m][m+n+N]F(m-1,n), \\ e_2F(m,n) &= -[m+n+N][n]F(m,n-1), \\ f_2F(m,n) &= F(m,n+1), \\ H_1F(m,n) &= (2m+n+N+1)F(m,n), \\ H_2F(m,n) &= -(2n+m+N+1)F(m,n),\end{aligned}$$

which is an infinite-dimensional irreducible representation in the generic case. But when q is a root of unity, we have a series of $\rho^{[N]}$-invariant subspaces $W_3^N(\alpha,p)$:$\{F(m,n)|m+n+N\geqslant\alpha p\}$, which is marked by such extreme vectors V:$\{F(m,n)|m+n=\alpha p-N-1\}$ that

$$f_1V=e_2V=0.$$

It is easy to prove that for a given N, α, the quotient space

$$\begin{aligned} Q_3^N(\alpha,p) &= V_3^{[N]}/W_3^N(\alpha,p):\{\overline{F}(m,n)=F(m,n)\\ &\quad \times \mathrm{Mod}\; W_3^{[N]}(\alpha,p)|m+n\\ &\quad \leqslant \alpha p-N-1, \quad m,n\geqslant 0\}\end{aligned}$$

is finite dimensional and its dimension is

$$D_3(\alpha,N,p)=\tfrac{1}{2}(\alpha p-N+1)(\alpha p-N).$$

Therefore, $\rho^{[N]}$ naturally induces a $\frac{1}{2}(\alpha p-N+1)(\alpha p-N)$-dimensional representation on $Q_3^N(\alpha,p)$. Explicitly, we have

$$\begin{aligned} e_1\overline{F}(m,n) &= \theta(\alpha p-N-1-m-n)\overline{F}(m+1,n), \\ f_1\overline{F}(m,n) &= -[m][m+n+N]\overline{F}(m-1,n), \\ e_2\overline{F}(m,n) &= -[m+n+N][n]\overline{F}(m,n-1), \\ f_2\overline{F}(m,n) &= \theta(\alpha p-N-1-m-n)\overline{F}(m,n+1), \\ H_1\overline{F}(m,n) &= (2m+n+N+1)\overline{F}(m,n), \\ H_2\overline{F}(m,n) &= -(2n+m+N+1)\overline{F}(m,n),\end{aligned}$$

where $\theta(x)=1$ for $x>0$ and $\theta(x)=0$ for $x\leqslant 0$. For example, when $\alpha p-N=3$ we have a six-dimensional representation

$$\begin{aligned} e_1 &= E_{21}+E_{42}+E_{53}, \\ f_1 &= [2]E_{12}+[2]E_{24}+E_{35}, \\ e_2 &= [2]E_{13}+E_{25}+[2]E_{36}, \\ f_2 &= E_{31}+E_{52}+E_{63}, \\ H_1 &= (N+1)E_{11}+(N+3)E_{22}+(N+2)E_{33} \\ &\quad +(N+5)E_{44}+(N+4)E_{55}+(N+3)E_{66}, \\ H_2 &= -[(N+1)E_{11}+(N+2)E_{22}+(N+3)E_{33} \\ &\quad +(N+3)E_{44}+(N+4)E_{55}+(N+5)E_{66}].\end{aligned}$$

IV. NONGENERIC REPRESENTATIONS OF $\mathrm{sl}_q(2)$

In order to construct nongeneric R-matrices associated with quantum algebra $\mathrm{sl}_q(2)$, in this section we will study the nongeneric representations of $\mathrm{sl}_q(2)$ in detail.

Using the new q-deformed boson realization of $\mathrm{sl}_q(2)$,

$$J_3=(N_1+N_2+1), \quad J_+=a_1^+a_2^+, \quad J_-=-a_1a_2,$$

we can obtain a representation

$$J_+ f_N(m) = f_N(m+1),$$
$$J_- f_N(m) = -[m][N+m] f_N(m-1), \tag{4.1}$$
$$J_3 f_N(m) = (2m+N) f_N(m),$$

on the invariant subspace $W_\alpha^N(\alpha, N \in \mathbb{Z}^+)$:

$$\{f_N(m) = (a_1^+)^m (a_2^+)^{N+m}|0\rangle \in \mathscr{F}_q^N(2) \,|\, m+N \geqslant \alpha p\},$$

of the q-deformed Fock space $\mathscr{F}_q(2)$:

$$\{|m_1 m_2\rangle = a_1^{+m_1} a_2^{+m_2}|0\rangle \,|\, N_1|0\rangle = a_i|0\rangle = 0,$$
$$i = 1,2, m_1 m_2 \in \mathbb{Z}^+\}.$$

It is obvious that this representation is infinite dimensional. To obtain a finite-dimensional representation let us consider the quotient space $Q_\alpha(n,N) = \mathscr{F}_q^N(2)/W_\alpha^N$ $(N = \alpha p - n)$:

$$\{f(J,M) = |J+M, J+M+\alpha p - n\rangle \text{Mod } W_\alpha^N | M$$
$$= -J, -J+1,\ldots,J\},$$

where $J = (n-1)/2$. Now one can easily see that $Q_\alpha(n,N)$ carries an n-dimensional representation, which can be written as

$$J_+ \bar{f}(J,M) = \bar{f}(J,M+1)\theta(J-M),$$
$$J_- \bar{f}(J,M) = -[J+M][J+M-n]\bar{f}(J,M-1), \tag{4.2}$$
$$J_3 \bar{f}(J,M) = (2M+\alpha p)\bar{f}(J,M).$$

Besides W_α^N, there exist other $\mathrm{sl}_q(2)$-invariant subspaces of $\mathscr{F}_q^N(2)$ from which we can construct new representations. It is easy to check that S_α^N $(\alpha, N \in \mathbb{Z}^+)$

$$\{f_N(m) = |m, m+N\rangle \in \mathscr{F}_q^N(2) \,|\, m \geqslant \alpha p\}$$

are also $\mathrm{sl}_q(2)$-invariant. As a result, we have the quotient spaces $Q_\alpha^N = \mathscr{F}_q^N(2)/S_\alpha^N$:

$$\{|m, m+N\rangle \text{Mod } S_\alpha^N | 0 \leqslant m \leqslant \alpha p - 1\}$$

on which αp-dimensional representations can be constructed as follows.

First, we choose a set of bases for Q_α^N:

$$\{f_N(J,M) = (a_1^+)^{J+M} (a_2^+)^{J+M+N}|o\rangle \text{Mod } S_\alpha^N | M$$
$$= -J, -J+1,\ldots,J\},$$

where $J = (\alpha P - 1)/2$, and $N = 1,2,\ldots,\alpha p - 1$. Then, we give the explicit form of the representation on Q_α^N:

$$J_+ \bar{f}_N(J,M) = \bar{f}_N(J,M+1)\theta(J-M),$$
$$J_- \bar{f}_N(J,M) = -[J+M][J+M+N]\bar{f}_N(J,M-1), \tag{4.3}$$
$$J_3 \bar{f}_N(J,M) = (2M+\alpha p + N)\bar{f}_N(J,M).$$

It should be pointed out that because in the nongeneric case $[M+p] = [M]$, we cannot obtain any new representations on Q_α^N $(N \geqslant \alpha p)$ Moreover, we can prove that all of the above representations are indecomposable, so we can expect that they will give rise to a new type of R-matrix.

V. NONGENERIC R-MATRICES ASSOCIATED WITH $\mathrm{sl}_q(2)$

Having constructed the nongeneric representations of $\mathrm{sl}_q(2)$, now we are prepared to obtain the nongeneric R-matrices associated with them. First, we rewrite (4.2) and (4.3) as

$$(J_+)_m^{m'} = \delta_{m+1}^{m'}\theta(J-m),$$
$$(J_-)_m^{m'} = [J+m][J-m+1]\delta_{m-1}^{m'},$$
$$(J_3)_m^{m'} = (2m'+\alpha p)\delta_m^{m'}, \quad m,m' = -J, -J+1,\ldots,J$$

and

$$(J_+)_m^{m'} = \delta_{m+1}^{m'}\theta(J-m),$$
$$(J_-)_m^{m'} = [J+m][J-m+1-N]\delta_{m-1}^{m'},$$
$$(J_3)_m^{m'} = (2m'+\alpha p + N)\delta_m^{m'}, \quad m,m' = -J,\ldots,J,$$

which are convenient to use. Then, we substitute them into the universal R-matrix

$$\mathscr{R} = q^{(1/2)(J_3 \otimes J_3)} \sum_{n=0}^{\infty} \frac{(1-q^{-2})^n}{[n]_q!} \times (q^{(1/2)J_3} J_+ \otimes q^{-(1/2)J_3} J_-)^n q^{(1/2)n(n-1)}$$

to obtain

$$(R^{JJ})_{ml}^{m'l'} = q^{(1/2)(2m'+\alpha p)(2l'+\alpha p)}\delta_m^{m'}\delta_l^{l'} + q^{(1/2)(2m'+\alpha p)(2l'+\alpha p)} \sum_{n=1}^{2J} \frac{(1-q^{-2})^n}{[n]_q!}$$
$$\times q^{-n(n-1)/2} q^{(m'-l')n} \times \prod_{i=1}^{n} [J+l'+i]_q [J+1-(l'+i)]_q \delta_{m+n}^{m'}\delta_{l-n}^{l'} \tag{5.1}$$

and

$$(R^{JJ})_{ml}^{m'l'} = q^{(1/2)(2m'+\alpha p+N)(2l'+\alpha p+N)}\delta_m^{m'}\delta_l^{l'} + q^{(1/2)(2m'+\alpha p+N)(2l'+\alpha p+N)} \sum_{n=1}^{2J} \frac{(1-q^{-2})^n}{[n]_q!}$$
$$\times q^{-n(n-1)/2} q^{(m'-l')n} \times \prod_{i=1}^{n} [J+l'+i]_q [J+1-N-(l'+i)]_q \delta_{m+n}^{m'}\delta_{l-n}^{l'}, \tag{5.2}$$

respectively. The R-matrices concerning two different representation spaces can be obtained in the same way.

To show that (5.1) and (5.2) can provide some new R-matrices essentially different from the standard and nonstandard ones already known to us, we give an example here. Taking $\alpha = 1, p = 3$, and $N = 2$ in (5.2), we have

$$R^{11}=\begin{bmatrix} q^{1/2} &&&&&&&& \\ & q^{-1/2} & q^{-1/2}(q-q^{-1}) &&&&&& \\ & 0 & q^{-1/2} &&&&&& \\ &&& q^{-3/2} & 0 & 0 &&& \\ &&& 0 & q^{1/2} & q^{-1/2}(q-q^{-1}) &&& \\ &&& 0 & 0 & q^{-3/2} &&& \\ &&&&&& q^{-3/2} & 0 & \\ &&&&&& 0 & q^{-3/2} & \\ &&&&&&&& q^{-3/2} \end{bmatrix}, \quad q^3=1.$$

Finally, we would like to point out that the new R-matrices possess nongeneric properties as mentioned in the Introduction, so many problems naturally arise. Further discussions about them are beyond the scope of this paper, and will be presented in forthcoming papers.

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