New inhomogeneous boson realizations and inhomogeneous differential realizations of Lie algebras

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The inhomogeneous boson realizations (IHBR) and the corresponding inhomogeneous differential realizations (IHDR) of Lie algebras, which play an important role in the search of quasi-exactly solvable problems (QESP) of quantum mechanics, are studied. All possible IHDR of semisimple Lie algebras can be obtained in this way. As examples, the IHBR and the corresponding IHDR of Lie algebras SU(2) and SU(3) are studied in detail.

I. INTRODUCTION

Recently discovered quasi-exactly solvable problems (QESP)[1–4] of quantum mechanics have been proved to be related to the inhomogeneous differential realizations (IHDR) of Lie algebras.[2–4] Turbiner studied the one-dimensional QESP by making use of the IHDR of Lie algebra sl(2),[2] and pointed out that one could study the multi-dimensional QESP by using the IHDR of sl(m) algebra.[2] Shifman and Turbiner studied the two-dimensional QESP by making use of the IHDR of Lie algebras su(2)$\times$su(2), so(3), and su(3).[3,4] Construction of IHDR of Lie algebras is very important in the search of QESP. By extending Shifman's discussions, the authors of this paper have obtained the IHDR of any Lie algebras.[5] However, as pointed in Ref. 5, this IHDR is trivial, i.e., it does not include all possible representations of Lie algebras. In this paper, we will continue studying the IHDR of Lie algebras.

On the universal enveloping algebra $\mathscr{U}(L)$ of Lie algebra L, or on its quotient space $\mathscr{U}(L)/J$, where J is a left ideal generated through algebraic relations in $\mathscr{U}(L)$, one can obtain all the representations of L, including indecomposable and irreducible representations. These representations are well-defined. Especially, for some concrete Lie algebras, Gruber *et al.* have given the explicit expressions of these representations.[6–9] In this paper, from a representation on $\mathscr{U}(L)$ or on $\mathscr{U}(L)/J$, we define a representation of Lie algebra L on the Fock space, which is automorphic to $\mathscr{U}(L)$ or $\mathscr{U}(L)/J$, then give the IHBR of Lie algebra L. By making use of the corresponding relation between creation and annihilation operators of boson states and differential operators, we obtain the IHDR of Lie algebras.

For semisimple Lie algebras, further discussions are given. On the Fock space that corresponds to the finite-dimensional irreducible standard cyclic module $V(\lambda)$[10] as the quotient space of $\mathscr{U}(L)$, all possible IHBR of semisimple Lie algebras marked by rank L non-negative integers, where rank L is the rank of semisimple Lie algebra L, are obtained. The IHDR of L obtained from the corresponding IHBR will use $\frac{1}{2}$(dim L-rank L) independent variables, which is in accord with Shifman's inference.

As examples, we discuss varieties of IHBR and IHDR of Lie algebras SU(2) and SU(3) by making use of their representations on their universal enveloping algebras or on their quotient spaces given by Gruber *et al.*[6,7] The IHDR of SU(3) marked by two non-negative integers can be obtained.

It is worth noticing that Doebner *et al.* have studied the IHBR and IHDR of SU(2) and SU(1,1) from the representations of SU(2) and SU(1,1) on their universal enveloping algebras or on their quotient spaces.[11] Their results are the Hermitian conjugate of our results, combined with the index change. However, their method is different from the method used in this paper. One will find that our method is straightforward and simpler than that used in Ref. 11.

This paper is organized as follows. After studying in Sec. II the general procedure to construct the IHBR and IHDR of any Lie algebra, we will further show this procedure with SU(2) as an example in Sec. III. In Sec. IV, the further discussions for the semisimple Lie algebras are given. In Sec. V, the IHBR and IHDR of Lie algebra SU(3) are studied in detail.

The symbol $\mathbb{Z}^+$ denotes the set of non-negative integers. The symbol $\mathbb{C}$ denotes the complex number field.

II. GENERAL PROCEDURE

Let the basis for Lie algebra L be $\{T_a | a = 1,2,...,M, \dim L = M\}$. According to PBW theorem, the basis for the universal enveloping algebra $\mathscr{U}(L)$ of Lie algebra L can be chosen as

$$\{X(i_1,i_2,...,i_M) \equiv T_1^{i_1} T_2^{i_2} \cdots T_M^{i_M} | i_1,i_2,...,i_M \in \mathbb{Z}^+\}. \tag{2.1}$$

If we regard $\mathscr{U}(L)$ as the left L module, namely,

$$\begin{aligned}\rho(T_a)X(i_1,i_2,...,i_M) &= T_a T_1^{i_1} T_2^{i_2} \cdots T_M^{i_M} \\ &= \sum_{i'_1 \cdots i'_M} \rho(T_a)^{i'_1 i'_2 \cdots i'_M}_{i_1 i_2 \cdots i_M} X(i'_1,i'_2,...,i'_M),\end{aligned} \tag{2.2}$$

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one can obtain a representation ρ of L on $\mathscr{U}(L)$, called master representation. The matrix elements $\rho(T_a)_{i_1 i_2\cdots i_M}^{i_1' i_2'\cdots i_M'}$ are determined by the Lie product of L and are related to the M non-negative integers $i_1, i_2,...,i_M$.

Let J be a left ideal of $\mathscr{U}(L)$ generated through algebraic relations in $\mathscr{U}(L)$. On the quotient space $\mathscr{U}(L)/J$, which basis is also marked by certain of non-negative integers, (2.2) induces a representation. The different choice of J enables us to obtain all possible representations of L. For some concrete Lie algebras, these representations have been written as the explicit expressions.[6-9] Now we construct the IHBR of L from these representations. However, for the sake of convention we only construct the IHBR of L from the master representation even though the arguments work also for the representation on $\mathscr{U}(L)/J$.

We first define a Fock space $\mathscr{F}$ with basis

$$\{|i_1,i_2,...,i_M\rangle \equiv a_1^{+\,i_1} a_2^{+\,i_2}\cdots a_M^{+\,i_M}|0\rangle \,|a_i|0\rangle = 0,\ i_k \in \mathbb{Z}^+\}. \tag{2.3}$$

Then the mapping φ: $\mathscr{U}(L) \to \mathscr{F}$ defined by

$$\varphi(X(i_1,i_2,...,i_M)) = |i_1,i_2,...,i_M\rangle \tag{2.4}$$

is an associative algebraic isomorphism. Let

$$\Gamma(T_a) = \varphi\rho(T_a)\varphi^{-1}. \tag{2.5}$$

Then $\Gamma(T_a)$ is a linear operator on $\mathscr{F}$ and satisfies the relation

$$[\Gamma(T_a),\Gamma(T_b)] = \Gamma([T_a,T_b]). \tag{2.6}$$

Therefore Eq. (2.5) defines a representation of L on $\mathscr{F}$, called Fock representation. It is easy to see that

$$\Gamma(T_a)|i_1,i_2,...,i_M\rangle = \sum_{i_1' i_2'\cdots i_M'} \rho(T_a)_{i_1 i_2\cdots i_M}^{i_1' i_2'\cdots i_M'} |i_1',i_2',...,i_M'\rangle. \tag{2.7}$$

From (2.7) it follows that

$$\begin{aligned}\Gamma(T_a) &= \sum_{\substack{i_1' i_2'\cdots i_M' \\ j_1 j_2\cdots j_M}} \rho(T_a)_{j_1 j_2\cdots j_M}^{i_1' i_2'\cdots i_M'} \frac{1}{j_1! j_2!\cdots j_M!} |i_1',...,i_M'\rangle\langle j_1 j_2,...j_M| \\ &= \sum_{\substack{i_1' i_2'\cdots i_M' \\ j_1 j_2\cdots j_M}} \rho(T_a)_{j_1 j_2\cdots j_M}^{i_1' i_2'\cdots i_M'} \frac{1}{j_1! j_2!\cdots j_M!} a_1^{+\,i_1'} a_2^{+\,i_2'}\cdots a_M^{+\,i_M'}|0,0,...,0\rangle\langle 0,0,...,0| a_1^{j_1} a_2^{j_2}\cdots a_M^{j_M} \\ &= \sum_{\substack{i_1' i_2'\cdots i_M' \\ j_1 j_2\cdots j_M}} \rho(T_a)_{j_1 j_2\cdots j_M}^{i_1' i_2'\cdots i_M'} \frac{1}{j_1! j_2!\cdots j_M!} a_1^{+\,i_1'} a_2^{+\,i_2'}\cdots a_M^{+\,i_M'} {:}\exp\Big(-\sum_{i=1}^{M} a_i^+ a_i\Big){:}\, a_1^{j_1} a_2^{j_2}\cdots a_M^{j_M},\end{aligned} \tag{2.8}$$

where the formula

$$|0,0,...,0\rangle\langle 0,0,...,0| = {:}\exp\Big(-\sum_{i=1}^{M} a_i^+ a_i\Big){:} \tag{2.9}$$

is vacuum projective operator, $:\cdots:$ is normal product. Equation (2.8) is the required IHBR of L.

However, for a concrete Lie algebra, it is inconvenient to achieve the IHBR by use of Eq. (2.8). In fact, when we know the explicit form of $\rho(T_a)_{j_1\cdots j_M}^{i_1'\cdots i_M'}$, we can easily obtain the IHBR of L. In this case the following formulas usually used:

$$\begin{aligned}&a_k^+|i_1,i_2,...,i_M\rangle = |i_1,i_2,...,i_k+1,...,1_M\rangle, \\ &a_k|i_1,i_2,...,i_M\rangle = i_k|i_1,...,i_k-1,...,i_M\rangle, \\ &a_k^+ a_k|i_1,i_2,...,i_M\rangle = i_k|i_1,i_2,...,i_M\rangle, \\ &a_k a_k^+|i_1,i_2,...,i_M\rangle = (i_k+1)|i_1,i_2,...,i_M\rangle, \\ &e^{a_k^+}|i_1,i_2,...,i_M\rangle \\ &\quad = \sum_{j=0}^{\infty} \frac{1}{j!}|i_1,i_2,...,i_k+j,...,i_M\rangle, \\ &e^{a k}|i_1,i_2,...,i_M\rangle \\ &\quad = \sum_{j=0}^{i_k} \frac{i_k!}{j!(i_k-j)!}|i_1,i_2,...j,...,i_M\rangle.\end{aligned} \tag{2.10}$$

In the next section this technique will be shown by virtue of SU(2) as an example, and in Sec. V with SU(3) as an example.

From the IHBR of L, we can immediately write the IHDR of L by making use of the following corresponding relations:

$$\xi_i \Leftrightarrow a_i^+, \quad \frac{\partial}{\partial\xi_i} \Leftrightarrow a_i \quad (i = 1,2,...,M). \tag{2.11}$$

III. IHBR AND IHDR OF LIE ALGEBRA SU(2)

Let the basis for Lie algebra SU(2) be $\{T_+, T_-, T_0\}$ with the following commutation relations:

$$[T_0,T_\pm] = \pm 2T_\pm, \quad [T_+,T_-] = T_0. \tag{3.1}$$

The basis for the universal enveloping algebra $\mathscr{U}(\mathrm{SU}(2))$ of SU(2) can be chosen as

$$\{X(m,n,r) \equiv T_+^m\, T_-^n\, T_0^r \,|m,n,r \in \mathbb{Z}^+\}. \tag{3.2}$$

(1) From the master representation of SU(2) on $\mathscr{U}(\mathrm{SU}(2))$ given in Ref. 6

$$\begin{aligned}\rho(T_0)X(m,n,r) &= X(m,n,r+1) + 2(m-n)X(m,n,r), \\ \rho(T_+)X(m,n,r) &= X(m+1,n,r), \\ \rho(T_-)X(m,n,r) &= X(m,n+1,r) - mX(m-1,n,r+1) \\ &\quad + m(2n-m+1)X(m-1,n,r),\end{aligned} \tag{3.3}$$

the corresponding Fock representation is obtained as

$$\Gamma(T_0)|m,n,r\rangle = |m,n,r+1\rangle + 2(m-n)|m,n,r\rangle,$$
$$\Gamma(T_+)|m,n,r\rangle = |m+1,n,r\rangle,$$
$$\Gamma(T_-)|m,n,r\rangle = |m,n+1,r\rangle - m|m-1,n,r+1\rangle + m(2n-m+1)|m-1,n,r\rangle. \quad (3.4)$$

From (3.4) we can immediately obtain the IHBR as

$$\Gamma(T_0) = a_3^+ + 2a_1^+a_1 - 2a_2^+a_2,$$
$$\Gamma(T_+) = a_1^+, \quad (3.5)$$
$$\Gamma(T_-) = a_2^+ - a_3^+a_1 + 2a_2^+a_2a_1 - 2a_1^+a_2^2.$$

The corresponding IHDR is obtained as

$$D(T_0) = \xi_3 + 2\xi_1\frac{\partial}{\partial\xi_1} - 2\xi_2\frac{\partial}{\partial\xi_2},$$
$$D(T_+) = \xi_1, \quad (3.6)$$
$$D(T_-) = \xi_2 - \xi_3\frac{\partial}{\partial\xi_1} + 2\xi_2\frac{\partial^2}{\partial\xi_2\,\partial\xi_1} - 2\xi_1\frac{\partial^2}{\partial\xi_1^2}.$$

In comparison with the result in Ref. 11, we will find that (3.5) and (3.6) are the Hermitian conjugate of Doebner's result, combined with an index change $1\leftrightarrow 2$.

(2) From the representation[6] of SU(2) on the quotient space $\mathscr{U}(\mathrm{SU}(2))/I_1$, where I_1 is a left ideal of $\mathscr{U}(\mathrm{SU}(2))$ generated by $T_0 - \Lambda\cdot 1$ ($\Lambda\in\mathbb{C}$), with basis

$$\{X(m,n)\equiv X(m,n,0)\,\mathrm{Mod}\; \mathrm{I}|m,n\in\mathbb{Z}^+\}, \quad (3.7)$$
$$\rho(T_0)X(m,n) = [\Lambda + 2(m-n)]X(m,n),$$
$$\rho(T_+)X(m,n) = X(m+1,n),$$
$$\rho(T_-)X(m,n) = X(m,n+1) + m(-\Lambda+2n-m+1)\times X(m-1,n), \quad (3.8)$$

we can obtain the Fock representation. From this Fock representation the IHBR of SU(2) is obtained as

$$\Gamma(T_0) = \Lambda + 2a_1^+a_1 - 2a_2^+a_2,$$
$$\Gamma(T_+) = a_1^+, \quad (3.9)$$
$$\Gamma(T_-) = a_2^+ - \Lambda a_1 + 2a_2^+a_2a_1 - a_1^+a_1^2.$$

The corresponding IHDR is

$$D(T_0) = \Lambda + 2\xi_1\frac{\partial}{\partial\xi_1} - 2\xi_2\frac{\partial}{\partial\xi_2},$$
$$D(T_+) = \xi_1, \quad (3.10)$$
$$D(T_-) = \xi_2 - \Lambda\frac{\partial}{\partial\xi_1} + 2\xi_2\frac{\partial^2}{\partial\xi_2\,\partial\xi_1} - \xi_1\frac{\partial^2}{\partial\xi_1^2}.$$

(3) The master representation ρ induces on the quotient space $\mathscr{U}(\mathrm{SU}(2))/I_2$, where I_2 is a left ideal generated by $T_- - \lambda\mathbb{1}$ ($\lambda\in\mathbb{C}$), with basis

$$\{X(m,r)\equiv X(m,0,r)\,\mathrm{Mod}\; I_2|m,r\in\mathbb{Z}^+\} \quad (3.11)$$

a representation[6]

$$\rho(T_0)X(m,r) = X(m,r+1) + 2mX(m,r),$$
$$\rho(T_+)X(m,r) = X(m+1,r),$$
$$\rho(T_0)X(m,r) = \lambda\sum_{k=0}^{r}\frac{r!}{(r-k)!k!}X(m,k) - mX(m-1,r+1) - m(m-1)X(m-1,r). \quad (3.12)$$

From the Fock representation that corresponds to (3.12) one obtains the IHBR of SU(2) as

$$\Gamma(T_0) = a_2^+ + 2a_1^+a_1,$$
$$\Gamma(T_+) = a_1^+, \quad (3.13)$$
$$\Gamma(T_-) = \lambda e^{a_2} - a_2^+a_1 - a_1^+a_1^2,$$

The corresponding IHDR of SU(2) is obtained as

$$D(T_0) = \xi_2 + 2\xi_1\frac{\partial}{\partial\xi_1},$$
$$D(T_+) = \xi_1, \quad (3.14)$$
$$D(T_-) = \lambda e^{\partial/\partial\xi_2} - \xi_2\frac{\partial}{\partial\xi_1} - \xi_1\frac{\partial^2}{\partial\xi_1^2},$$

(4) Let I_3 be a left ideal generated by T_-, $T_0 - \Lambda$ ($\Lambda\in\mathbb{C}$). Then on the quotient space $V\equiv\mathscr{U}(\mathrm{SU}(2))/I_3$ with basis

$$\{X(m)\equiv X(m,0,0)\,\mathrm{Mod}\; I_3|m\in\mathbb{Z}^+\}, \quad (3.15)$$

the master representation ρ induces a representation[6]

$$\rho(T_0)X(m) = (\Lambda+2m)X(m),$$
$$\rho(T_+)X(m) = X(m+1), \quad (3.16)$$
$$\rho(T_-)X(m) = m(-\Lambda-m+1)X(m-1).$$

From corresponding Fock representation we obtain the IHBR and IHDR as

$$\Gamma(T_0) = \Lambda + 2a^+a, \quad D(T_0) = \Lambda + 2\xi\frac{\partial}{\partial\xi},$$
$$\Gamma(T_+) = a^+, \quad D(T_+) = \xi, \quad (3.17)$$
$$\Gamma(T_-) = -\Lambda a - a^+a^2, \quad D(T_-) = -\Lambda\frac{\partial}{\partial\xi} - \xi\frac{\partial^2}{\partial\xi^2}.$$

(5) When $\Lambda = -N$, where $N\in\mathbb{Z}^+$, the subspace V_N of V spanned by

$$V_N:\{X(m)\in V|m\geqslant N+1, m\in\mathbb{Z}^+\} \quad (3.18)$$

is invariant under the action of (3.16). On the quotient space $\widetilde{V}_N\equiv V/V_N$ with basis

$$\widetilde{V}_N:\{X_N(m)\equiv(N-m)!X(m)\,\mathrm{Mod}\; V_N|m=0,1,2,\ldots,N\},$$
$$\dim\widetilde{V}_N = N+1, \quad (3.19)$$

(3.16) subduces a finite-dimensional representation[6]

$$\rho(T_0)X_N(m) = (-N/2+m)2X_N(m),$$
$$\rho(T_+)X_N(m) = (N-m)X_N(m+1), \quad (3.20)$$
$$\rho(T_-)X_N(m) = mX_N(m-1).$$

The corresponding Fock representation is obtained as

$$\Gamma(T_0')|m\rangle = (-N/2+m)|m\rangle,$$
$$\Gamma(T_+)|m\rangle = (N-m)|m+1\rangle, \quad (3.21)$$
$$\Gamma(T_-)|m\rangle = m|m-1\rangle \quad (T_0' = \tfrac{1}{2}T_0).$$

It should be noted that the space carrying the representation (3.21) is a $(N+1)$-dimensional invariant subspace $\mathscr{F}_N$ spanned by $\{|m\rangle | m=0,1,...,N\}$ of Fock space with basis

$$\mathscr{F}:\{|m\rangle = a^{+m}|0\rangle | m\in\mathbb{Z}^+,\ a|0\rangle = 0\}. \qquad (3.22)$$

From the finite-dimensional Fock representation (3.21) we obtain the finite-dimensional IHBR and IHDR as

$$\Gamma(T_0') = -N/2 + a^+a,\quad D(T_0') = -\frac{N}{2} + \xi\frac{\partial}{\partial\xi},$$
$$\Gamma(T_+) = Na^+ - a^{+2}a,\quad D(T_+) = N\xi - \xi^2\frac{\partial}{\partial\xi},$$
$$\Gamma(T_-) = a,\quad D(T_-) = \frac{\partial}{\partial\xi}, \qquad (3.23)$$

which are just the result obtained in Ref. 5.

IV. COMMENTS ON IHDR OF SEMISIMPLE LIE ALGEBRAS

Let L be a semisimple Lie algebra with Cartan decomposition $L = H\oplus\Sigma_{\beta\in\Phi}L_\beta$, where H is Cartan subspace $(\dim H = l = \operatorname{rank} L)$, Φ is root system, L_β is the root space of root β. Let $\Phi^+ = \{\beta_1,\beta_2,...,\beta_m\}$ be a set of positive roots and $\Delta = \{\alpha_1,\alpha_2,...,\alpha_l\}\subset\Phi$ the simple root system. Choose the basis for L as

$$\{y_{\beta_1},y_{\beta_2},...,y_{\beta_m},h_1,h_2,...,h_l,x_{\beta_1},x_{\beta_2},...,x_{\beta_m}\}, \qquad (4.1)$$

where $y_{\beta_i}\in L_{-\beta_i}$, $x_{\beta_i}\in L_{\beta_i}$, $h_i\equiv h_{\alpha_i}\in H$, with the following Lie product:

$$[h_i,h_j] = 0\quad (i,j=1,2,...,l),\quad [h_i,x_{\beta_j}] = \beta_j(h_i)x_{\beta_j},$$
$$[h_i,y_{\beta_j}] = -\beta_j(h_i)y_{\beta_j},\quad [x_{\beta_i},y_{\beta_j}] = \delta_{ij}h_{\beta_j}. \qquad (4.2)$$

If we regard the universal enveloping algebra $\mathscr{U}(L)$ of L with PBW basis

$$\{y_{\beta_1}^{i_1}y_{\beta_2}^{i_2}\cdots y_{\beta_m}^{i_m}h_1^{k_1}h_2^{k_2}\cdots h_l^{k_l}x_{\beta_1}^{j_1}x_{\beta_2}^{j_2}\cdots x_{\beta_m}^{j_m}|i_p,k_p,j_p\in\mathbb{Z}^+\} \qquad (4.3)$$

as the left L module, we obtain the master representation of L.

Let $I(\lambda)$ $(\lambda\in H^*)$ be a left ideal generated by $\{x_{\beta_i},h_\alpha - \lambda(h_\alpha)1|\beta_i\in\Phi^+,\alpha\in\Phi\}$. Then the quotient space $Z(\lambda) = \mathscr{U}(L)/I(\lambda)$ with basis

$$\{X(i_1,i_2,...,i_m)\equiv[y_{\beta_1}^{i_1},y_{\beta_2}^{i_2}\cdots y_m^{i_m}]\operatorname{Mod} I(\lambda)|ik\in\mathbb{Z}^+\} \qquad (4.4)$$

is a indecomposable standard cyclic module with the highest weight λ.[10]

If λ is a dominant integral linear function, i.e., $\lambda(h_i)\equiv\Lambda_i\in\mathbb{Z}^+$, the left ideal $Y(\lambda)$ generated by $\{y_{\alpha_c}^{\Lambda_i+1}\operatorname{Mod}I(\lambda)|i=1,2,...,l\}$ is the unique maximal proper submodule and the quotient module $V(\lambda)\equiv Z(\lambda)/Y(\lambda)$ spanned by

$$\{B(i_1,i_2,...,i_m)$$
$$\equiv X(i_1,i_2,...,i_m)\operatorname{Mod} Y(\lambda)|i_1,i_2,...,i_m\in\mathbb{Z}^+\} \qquad (4.5)$$

is a finite-dimensional irreducible L module,[10] on which the representation is marked by $l = \operatorname{rank} L$ non-negative integers $(\Lambda_1,\Lambda_2,...,\Lambda_l)$.

From the representation on $V(\lambda)$, we can obtain the IHBR and the corresponding IHDR of L by use of the procedure given in Sec. II, which are marked by $(\Lambda_1,\Lambda_2,...,\Lambda_l)$. These IHDR include all possible representations of L.

Notice that the basis for $V(\lambda)$ is marked by m non-negative integers, where

$$m = \tfrac{1}{2}(\dim L - \operatorname{rank} L). \qquad (4.6)$$

Thus the IHBR obtained from the representation on $V(\lambda)$ uses creation and annihilation operators of m boson states and the corresponding IHDR uses m independent variables. Because $V(\lambda)$ is the irreducible module, m is the minimal number to construct all possible inhomogeneous differential representations. This fact is in accord with Shifman's arguments.

V. IHBR AND IHDR OF SU(3)

We choose the ordered basis for Lie algebra SU(3) as

$$\{e_{31},e_{32},e_{21},\ h_1 = e_{11}-e_{22},\ h_2 = e_{22}-e_{33},\ e_{13},e_{23},e_{12}\}, \qquad (5.1)$$

where e_{ij} is a 3×3 matrix with matrix element $(e_{ij})_{kl} = \delta_{ik}\delta_{jl}$. The basis for the universal enveloping algebra $\mathscr{U}(\mathrm{SU}(3))$ of SU(3) can be chosen as

$$\{e_{31}^m e_{32}^n e_{21}^p h_1^k h_2^l e_{13}^r e_{32}^s e_{21}^t|m,n,p,k,l,r,s,t\in\mathbb{Z}^+\}. \qquad (5.2)$$

The basis for the standard cyclic module $Z(\lambda) = \mathscr{U}(\mathrm{SU}(3))/I$, where I is a left ideal generated by $\{e_{13},e_{33},e_{12},\ h_1-\lambda(h_1)1,\ h_2-\lambda(h_2)1\}$, can be chosen as

$$\{X(m,n,p)\equiv(e_{31}^m e_{32}^n e_{21}^p)\operatorname{Mod} I|m,n,p\in\mathbb{Z}^+\}. \qquad (5.3)$$

The representation on $Z(\lambda)$ is obtained as[7]

$$\rho(h_1)X(m,n,p) = (\Lambda_1+n-m-2p)X(m,n,p),$$
$$\rho(h_2)X(m,n,p) = (\Lambda_2-2n+p-m)X(m,n,p),$$
$$\rho(e_{12})X(m,n,p) = p(\Lambda_1-p+1)X(m,n,p-1)$$
$$-mX(m-1,n+1,p),$$
$$\rho(e_{23})X(m,n,p) = n(\Lambda_2-m+p-n+1)X(m,n-1,p)$$
$$+mX(m-1,n,p+1),$$
$$\rho(e_{13})X(m,n,p) = np(\Lambda_1-p+1)X(m,n-1,p-1)$$
$$+m(\Lambda_1+\Lambda_2+p-m-n+1)$$
$$\times X(m-1,n,p), \qquad (5.4)$$
$$\rho(e_{31})X(m,n,p) = X(m+1,n,p),$$
$$\rho(e_{32})X(m,n,p) = X(m,n+1,p),$$
$$\rho(e_{21})X(m,n,p) = X(m,n,p+1) - nX(m+1,n-1,p),$$
$$[\Lambda_1 = \lambda(h_1),\quad \Lambda_2 = \lambda(h_2)].$$

From the Fock representation that corresponds to (5.4) the IHBR is obtained as

$$\rho(h_1) = \Lambda_1 - a_1^+a_1 + a_2^+a_2 - 2a_3^+a_3,$$
$$\rho(h_2) = \Lambda_2 - a_1^+a_1 - 2a_2^+a_2 + a_3^+a_3,$$
$$\rho(e_{12}) = \Lambda_1a_1 - a_3^+a_3^2 - a_2^+a_1,$$
$$\rho(e_{23}) = \Lambda_2a_2 - a_1^+a_1a_2 + a_3^+a_3a_2 - a_2^+a_2^2,$$

$$\rho(e_{13}) = \Lambda_1 a_2 a_3 - a_3^+ a_3^2 a_2 + (\Lambda_1+\Lambda_2)a_1 + a_3^+ a_3 a_1 - a_1^+ a_1^2 - a_2^+ a_2 a_1, \tag{5.5}$$
$$\rho(e_{31}) = a_1^+,$$
$$\rho(e_{32}) = a_2^+,$$
$$\rho(e_{31}) = a_3^+ - a_1^+ a_2.$$

The corresponding IHDR is obtained as

$$D(h_1) = \Lambda_1 - \xi\frac{\partial}{\partial\xi} + \eta\frac{\partial}{\partial\eta} - 2\zeta\frac{\partial}{\partial\zeta},$$
$$D(h_2) = \Lambda_2 - \xi\frac{\partial}{\partial\xi} - 2\eta\frac{\partial}{\partial\eta} + \zeta\frac{\partial}{\partial\zeta},$$
$$D(e_{12}) = \Lambda_1\frac{\partial}{\partial\xi} - \zeta\frac{\partial^2}{\partial\zeta^2} - \eta\frac{\partial}{\partial\xi},$$
$$D(e_{23}) = \Lambda_2\frac{\partial}{\partial\eta} - \xi\frac{\partial^2}{\partial\xi\,\partial\eta} + \zeta\frac{\partial^2}{\partial\zeta\,\partial\eta} - \eta\frac{\partial^2}{\partial\eta^2},$$
$$D(e_{13}) = \Lambda_1\frac{\partial^2}{\partial\eta\,\partial\zeta} - \zeta\frac{\partial^2}{\partial\zeta^2}\frac{\partial}{\partial\eta} + (\Lambda_1+\Lambda_2)\frac{\partial}{\partial\xi} + \zeta\frac{\partial^2}{\partial\zeta\,\partial\xi} - \xi\frac{\partial^2}{\partial\xi^2} - \eta\frac{\partial^2}{\partial\eta\,\partial\xi}, \tag{5.6}$$
$$D(e_{31}) = \xi,$$
$$D(\epsilon_{32}) = \eta,$$
$$D(e_{21}) = \zeta - \xi\frac{\partial}{\partial\eta},$$

which is a realization on the infinite-dimensional space of polynomials

$$\{\xi^m\eta^n\zeta^p | m,n,p\in\mathbb{Z}^+\}. \tag{5.7}$$

If Λ_1, $\Lambda_2\in\mathbb{Z}^+$, the left ideal $Y(\Lambda_1,\Lambda_2)\equiv Y(\lambda)$ generated by $\{X(0,0,1)^{\Lambda_1+1}, X(0,1,0)^{\Lambda_2+1}\}$ is the maximal proper module and the quotient module $V(\Lambda_1,\Lambda_2) = Z(\lambda)/Y(\Lambda_1,\Lambda_2)$ spanned by

$$\{\widetilde{X}(m,n,p)\equiv X(m,n,p)\,\mathrm{Mod}\; Y(\Lambda_1,\Lambda_2) | m,n,p\in\mathbb{Z}^+\} \tag{5.8}$$

is the finite-dimensional irreducible module. But it is difficult to determine the linear independent basis for $V(\Lambda_1,\Lambda_2)$ from the set (5.8), and the resulting expressions of the representations on $V(\Lambda_1,\Lambda_2)$ are complicated and unenlightening.[7] However, for some special cases we can give concrete discussions.

A. Triplet

When $\Lambda_1 = 1$, $\Lambda_2 = 0$, the basis for $V(1,0)$ can be chosen as

$$\{\widetilde{X}(0,0,0),\widetilde{X}(0,0,1),\widetilde{X}(0,1,1)\equiv\widetilde{X}(1,0,0)\}, \tag{5.9}$$

where $e_{31} = e_{32}e_{21}$ is used. The representation induced on $V(1,0)$ is just the triplet of SU(3). From this triplet we can obtain the IHDR on the space of polynomials with basis.

$$\{1,\zeta,\eta\zeta\}. \tag{5.10}$$

B. Symmetrized IHDR of SU(3)

We start from (5.4).

It is observed that the subspace $J(\Lambda_1,\Lambda_2)$ with basis

$$\{X(m,n,p)\in Z(\lambda) | \Lambda_1\in\mathbb{Z}^+,\Lambda_2\in\mathbb{C},p\geqslant\Lambda_1+1\} \tag{5.11}$$

is invariant. On the quotient space $W(\Lambda_1,\Lambda_2) \equiv Z(\lambda)/J(\Lambda_1,\Lambda_2)$ with basis

$$\{\overline{X}(m,n,p)\equiv(\Lambda_1-p)!X(m,n,p)\,\mathrm{Mod}\; J(\Lambda_1,\Lambda_2) | m,n\in\mathbb{Z}^+, p\leqslant\Lambda_1\}, \tag{5.12}$$

Eq. (5.3) induces a representation as

$$\rho(h_1)\overline{X}(m,n,p) = (\Lambda_1+n-m-2p)\overline{X}(m,n,p),$$
$$\rho(h_2)\overline{X}(m,n,p) = (\Lambda_2-2n+p-m)\overline{X}(m,n,p),$$
$$\rho(e_{12})\overline{X}(m,n,p) = p\overline{X}(m,n,p-1) - m\overline{X}(m-1,n+1,p),$$
$$\rho(e_{23})\overline{X}(m,n,p) = n(\Lambda_2-m+p-n+1)\overline{X}(m,n-1,p) + m(\Lambda_1-p)\overline{X}(m-1,n,p+1),$$
$$\rho(e_{13})\overline{X}(m,n,p) = np\overline{X}(m,n-1,p-1) + m(\Lambda_1+\Lambda_2+p-n-m+1)\overline{X}(m-1,n,p), \tag{5.13}$$
$$\rho(e_{31})\overline{X}(m,n,p) = \overline{X}(m+1,n,p),$$
$$\rho(e_{32})\overline{X}(m,n,p) = \overline{X}(m,n+1,p),$$
$$\rho(e_{21})\overline{X}(m,n,p) = (\Lambda_1-p)\overline{X}(m,n,p+1) - n\overline{X}(m+1,n-1,p).$$

The IHDR is obtained as

$$D(h_1) = \Lambda_1 - \xi\frac{\partial}{\partial\xi} + \eta\frac{\partial}{\partial\eta} - 2\zeta\frac{\partial}{\partial\zeta}$$
$$D(h_2) = \Lambda_2 - \xi\frac{\partial}{\partial\xi} - 2\eta\frac{\partial}{\partial\eta} + \zeta\frac{\partial}{\partial\zeta}$$
$$D(e_{12}) = \frac{\partial}{\partial\zeta} - \eta\frac{\partial}{\partial\xi}$$
$$D(e_{23}) = \Lambda_2\frac{\partial}{\partial\eta} - \xi\frac{\partial^2}{\partial\xi\,\partial\eta} - \zeta\frac{\partial^2}{\partial\zeta\,\partial\eta} - \eta\frac{\partial^2}{\partial\eta^2} + \Lambda_1\frac{\partial}{\partial\xi} - \zeta\frac{\partial}{\partial\xi} \tag{5.14}$$
$$D(e_{13}) = \frac{\partial^2}{\partial\eta\partial\zeta} + (\Lambda_1+\Lambda_2)\frac{\partial}{\partial\xi} + \zeta\frac{\partial^2}{\partial\zeta\,\partial\xi} - \eta\frac{\partial^2}{\partial\eta\,\partial\xi} - \xi\frac{\partial^2}{\partial\xi^2}$$
$$D(e_{31}) = \xi, \quad D(e_{32}) = \eta, \quad D(e_{21}) = \Lambda_1 - \zeta - \xi\frac{\partial}{\partial\eta},$$

which is a realization on the space of polynomials

$$\{\xi^m\eta^n\zeta^p | m,n,p\in\mathbb{Z}^+, 0\leqslant p\leqslant\Lambda_1\}. \tag{5.15}$$

Particularly, when $\Lambda_1 = 0$, and $p = 0$, the representation (5.13) becomes

$$\rho(h_1)\overline{X}(m,n) = (n-m)\overline{X}(m,n),$$
$$\rho(h_2)\overline{X}(m,n) = (\Lambda_2-m-2n)\overline{X}(m,n),$$
$$\rho(e_{23})\overline{X}(m,n) = n(\Lambda_2-m-n+1)\overline{X}(m,n-1),$$
$$\rho(e_{13})\overline{X}(m,n) = m(\Lambda_2-m-n+1)\overline{X}(m-1,n),$$
$$\rho(e_{31})\overline{X}(m,n) = \overline{X}(m+1,n),$$
$$\rho(e_{32})\overline{X}(m,n) = \overline{X}(m,n+1),$$
$$\rho(e_{21})\overline{X}(m,n) = -n\overline{X}(m+1,n-1), \tag{5.16}$$

where $\overline{X}(m,n)\equiv\overline{X}(m,n,0)$.

When $\Lambda_2 \in \mathbb{Z}^+$, it is easy to see that the subspace $Q(0,\Lambda_2)$ spanned by

$$\{\bar{X}(m,n) \in W(0,\Lambda_2) | m+n \geqslant \Lambda_2 + 1, m,n \in \mathbb{Z}^+\} \quad (5.17)$$

is invariant. On the quotient space $H(\Lambda_2) = W(0,\Lambda_2)/Q(0,\Lambda_2)$ with basis

$$\{H(m,n) \equiv (\Lambda_2 - m - n)! \bar{X}(m,n) \operatorname{Mod} Q(0,\Lambda_2) | 0 \leqslant m+n \leqslant \Lambda_2\} \quad (5.18)$$

$$\dim H(\Lambda_2) = \tfrac{1}{2}(\Lambda_2+1)(\Lambda_2+2),$$

(5.16) induces a representation on $H(\Lambda_2)$ as

$$\begin{aligned}
&\rho(h_1)H(m,n) = (n-m)H(m,n),\\
&\rho(h_2)H(m,n) = (\Lambda_2 - m - 2n)H(m,n),\\
&\rho(e_{12})H(m,n) = -mH(m-1,n+1),\\
&\rho(e_{23})H(m,n) = nH(m,n-1),\\
&\rho(e_{13})H(m,n) = mH(m-1,n),\\
&\rho(e_{31})H(m,n) = (\Lambda_2 - m - n)H(m+1,n),\\
&\rho(e_{32})H(m,n) = (\Lambda_2 - m - n)H(m,n+1),\\
&\rho(e_{21})H(m,n) = -nH(m+1,n-1).
\end{aligned} \quad (5.19)$$

From the Fock representation that corresponds to (5.19) the IHBR is obtained as

$$\begin{aligned}
&\rho(h_1) = a_2^+ a_2 - a_1^+ a_1, \quad \rho(h_2) = \Lambda_2 - a_1^+ a_1 - 2a_2^+ a_2,\\
&\rho(e_{12}) = -a_2^+ a_1, \quad \rho(e_{23}) = a_2,\\
&\rho(e_{13}) = a_1, \quad \rho(e_{31}) = \Lambda_2 a_1^+ - a_1^{+2} a_1 - a_1^+ a_2^+ a_2,\\
&\rho(e_{32}) = \Lambda_2 a_2^+ - a_2^+ a_1^+ a_1 - a_2^{+2} a_2, \quad \rho(e_{21}) = -a_1^+ a_2.
\end{aligned} \quad (5.20)$$

The corresponding IHDR is obtained as

$$\begin{aligned}
&D(h_1) = \eta\frac{\partial}{\partial\eta} - \xi\frac{\partial}{\partial\xi}, \quad \rho(h_2) = \Lambda_2 - \xi\frac{\partial}{\partial\xi} - 2\eta\frac{\partial}{\partial\eta},\\
&D(e_{12}) = -\eta\frac{\partial}{\partial\xi}, \quad D(e_{23}) = \frac{\partial}{\partial\eta},\\
&D(e_{13}) = \frac{\partial}{\partial\xi}, \quad D(e_{31}) = \Lambda_2\xi - \xi^2\frac{\partial}{\partial\xi} - \xi\eta\frac{\partial}{\partial\eta},\\
&D(e_{32}) = \Lambda_2\eta - \eta\xi\frac{\partial}{\partial\xi} - \eta^2\frac{\partial}{\partial\eta}, \quad D(e_{21}) = -\xi\frac{\partial}{\partial\eta},
\end{aligned} \quad (5.21)$$

which is a realization on the finite dimensional space of polynomials with basis

$$\{\xi^m\eta^n | 0 \leqslant m+n \leqslant \Lambda_2, m,n \in \mathbb{Z}^+\}. \quad (5.22)$$

In fact, if we choose the representation D in Ref. 5 as the antitriplet, we just obtain the realization (5.21) by making use of the general formula in Ref. 5. Therefore, the realization (5.21) marked by a non-negative integer is commensurate with the symmetrized direct product of Λ_2 antitriplets of SU(3). So we call the realization (5.21) the symmetrized IHDR.

C. Antitriplet

Let $\Lambda_2 = 1$. Then we obtain the antitriplet from (5.21) on the three dimensional space of polynomials with basis

$$\{\xi,\eta,1\}. \quad (5.23)$$

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