

Long-term effect of inter-mode transitions in quantum Markovian process

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Abstract. We study the Markovian process of a multi-mode open system connecting with a non-equilibrium environment, which consists of several heat baths with different temperatures. As an illustration, we study the steady state of three linearly coupled harmonic oscillators in long time evolution, two of which contact with two independent bosonic heat baths with different temperatures respectively. We show that the inter-mode transitions mediated by the environment is responsible for the long time behavior of the dynamics evolution, which is usually considered to take effect only in short time dynamics of the system immersed in a equilibrium heat bath with a single temperature. These inter-mode transitions are essential to the non-equilibrium flux between subsystems, thus they cannot be neglected.

1 Introduction

When a small system contacts with a simple environment, i.e., a heat bath in canonical equilibrium with a temperature T , it would approach its canonical state with the same temperature T . This dynamics process is called *thermalization* [1–5]. However, non-equilibrium systems are more general in nature, and exhibit more rich physics. A typical example is a composite system connecting with more than one heat baths with different temperatures. For a long-term evolution, the open composite system would not approach its canonical thermal state, but still it would be stabilized to a certain steady state. We call this process *non-thermal stabilization*.

Such composite system coupling with multiple independent heat baths appear in many artificial systems, like the superconducting circuit and quantum dots, and also natural systems, like the excitons in photon-synthesis system [6–8]. In these composite systems, the interaction between the subsystems is always on, and that may affect the response of the system to the environment.

A rigorous treatment of the interacting composite system should be based on the normal modes of the system. In an open system, both the equilibrium and non-equilibrium cases as we mentioned above, the energy exchange with environment would mediate the transitions between these normal modes of the total system, which we call the *inter-mode transition*. It was usually believed that this inter-mode transition only takes effect on the dynamics of transient evolution within the time scale determined

by the time-energy uncertainty [1,9–12], and averagely it has no effect to the steady state behavior after a long time evolution. This is also known as secular approximation or rotating-wave approximation (RWA).

However, in this paper, we find that indeed such inter-mode transitions have long-term effect in non-equilibrium system even for Markovian process. As an example, we study the steady state of three linearly coupled harmonic oscillators (HOs), two of which contact with two independent bosonic heat baths with different temperatures respectively. We find that if the inter-mode transition were ignored, there would be some counter-intuitive results in the long time steady state. We show that the inter-mode transitions are essential to the non-equilibrium flux inside the composite system. As a comparison, we also show that such effect does not appear in equilibrium environments. We emphasize that the omission of these inter-mode transitions is consistent with conventional equilibrium reservoirs as studied in previous works [1,9–12].

The paper is arranged as follows. In Section 2, we setup the model of the coupled system and give a master equation. In Section 3, we give the stabilization result and make some analytical discussion by eliminating the degree of freedom of the mediating data bus. We show that the omission of the inter-mode transition is consistent with the equilibrium reservoirs, and give a physical explanation. In Section 4, we propose a possible implementation. The calculation is assisted by some properties of the characteristic description of Wigner function and Fokker-Planck equation. We leave these tricks in the appendices. Finally, summary is drawn in Section 5.

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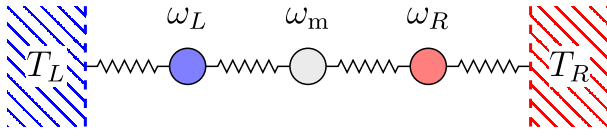


Fig. 1. Demonstration of the coupled oscillators system. Two remotely located HOs are indirectly coupled by another mediating one. The HOs at the two ends contact with independent heat baths with different temperatures $T_{L/R}$.

2 Model setup

To study the long-term dynamics of a composite system coupled to a complicated environment, we use the coupled HOs system as an illustration. The system we study here is illustrated in Figure 1. Two HOs with frequencies $\omega_{L,R}$ contact with two independent heat baths with different temperatures. In experiments, microscopic devices with mutual interactions are separated from each other for only several micrometers, thus it is unclear to discuss their local temperatures. Here we introduce a third HO as a data bus to mediate their coupling, which makes it possible to separate the two HOs for a certain distance and we can discuss their local temperatures clearly. Effectively, we suppose the mediating HO does not contact with any environment.

The three oscillators system can be described by a quadratic coupled Hamiltonian $H_S = H_0 + V$, where

$$\begin{aligned} H_0 &= \omega_L \hat{a}_L^\dagger \hat{a}_L + \omega_R \hat{a}_R^\dagger \hat{a}_R + \omega_m \hat{b}^\dagger \hat{b}, \\ V &= g_L (\hat{a}_L^\dagger \hat{b} + \hat{a}_L \hat{b}^\dagger) + g_R (\hat{a}_R^\dagger \hat{b} + \hat{a}_R \hat{b}^\dagger), \end{aligned} \quad (1)$$

and H_0 describes the free Hamiltonian with local modes respectively defined by annihilation operators \hat{a}_L , \hat{a}_R and \hat{b} ; V describes the coupling among the local modes.

We assume the two oscillators locate remotely at different places, thus they may suffer from independent baths. We also assume each bath stays at a canonical thermal state with a temperature $T_{L/R}$. The whole system can be described by the total Hamiltonian $\mathcal{H} = H_S + H_B + V_{SB}$, where

$$\begin{aligned} H_B &= \sum_{\mathbf{k}_L} \omega_{\mathbf{k}_L} \hat{c}_{\mathbf{k}_L}^\dagger \hat{c}_{\mathbf{k}_L} + \sum_{\mathbf{k}_R} \omega_{\mathbf{k}_R} \hat{c}_{\mathbf{k}_R}^\dagger \hat{c}_{\mathbf{k}_R}, \\ V_{SB} &= \sum_{\sigma=L,R} \hat{a}_\sigma^\dagger \Gamma_\sigma + \hat{a}_\sigma \Gamma_\sigma^\dagger, \end{aligned} \quad (2)$$

and $\Gamma_\sigma = \sum_{\mathbf{k}_\sigma} g_{\mathbf{k}_\sigma} \hat{c}_{\mathbf{k}_\sigma}$. H_B is the free Hamiltonian of the two boson heat baths, each of which is modeled as a collection of boson modes, described by the boson annihilation operators $\hat{c}_{\mathbf{k}_\sigma}$. V_{SB} represents the linear coupling between the system and the environment.

We need to derive a master equation to study the dynamics of the open system. Actually for the coupled HO system, a correct treatment of the master equation should be based on the normal modes of H_S , but not the local modes $\hat{a}_{L/R}$ and \hat{b} . Otherwise, it may give rise to some counter-intuitive results. Thus, we diagonalize the

Hamiltonian H_S as:

$$\begin{aligned} H_S &= \left(\hat{a}_L^\dagger, \hat{b}^\dagger, \hat{a}_R^\dagger \right) \begin{bmatrix} \omega_L & g_L & \\ g_L & \omega_m & g_R \\ & g_R & \omega_R \end{bmatrix} \begin{pmatrix} \hat{a}_L \\ \hat{b} \\ \hat{a}_R \end{pmatrix} \\ &\equiv \mathbf{a}^\dagger \Omega \mathbf{a} = \sum_{i=1}^3 \varepsilon_i \hat{A}_i^\dagger \hat{A}_i, \end{aligned} \quad (3)$$

where $\mathbf{a} = (\hat{a}_L, \hat{b}, \hat{a}_R)^T$, and we also denote it as $(\hat{a}_1, \hat{a}_2, \hat{a}_3)^T$ hereafter (with redefined indices 1, 2, 3 for $\hat{a}_L, \hat{b}, \hat{a}_R$ respectively). $\mathbf{A} = U \mathbf{a} = (\hat{A}_1, \hat{A}_2, \hat{A}_3)^T$ for A_i 's being the normal modes. $U \Omega U^\dagger = \text{diag} \{ \varepsilon_1, \varepsilon_2, \varepsilon_3 \}$ gives the eigen frequencies of the normal modes. Although the normal modes are decoupled from each other in the isolated H_S , we can see below that the environment could mediate induce some effective couplings between these normal modes.

With the above notations, in Appendix A we derive a master equation to describe the long-term dynamics of the open system via Born-Markovian approximation [1]. In Schrödinger's picture, it reads as:

$$\begin{aligned} \partial_t \rho &= i \left[\rho, \sum \varepsilon_i \hat{A}_i^\dagger \hat{A}_i \right] \\ &+ \sum_{ij} \frac{A_{ij}^-}{2} \left(2 \hat{A}_i \rho \hat{A}_j^\dagger - \left\{ \hat{A}_j^\dagger \hat{A}_i, \rho \right\}_+ \right) \\ &+ \frac{A_{ij}^+}{2} \left(2 \hat{A}_j^\dagger \rho \hat{A}_i - \left\{ \hat{A}_j \hat{A}_i^\dagger, \rho \right\}_+ \right), \end{aligned} \quad (4)$$

where

$$\begin{aligned} A_{ij}^+ &= \frac{\gamma_L}{2} U_{i1} U_{j1}^* [N_L(\varepsilon_i) + N_L(\varepsilon_j)] \\ &+ \frac{\gamma_R}{2} U_{i3} U_{j3}^* [N_R(\varepsilon_i) + N_R(\varepsilon_j)], \\ A_{ji}^- &= \frac{\gamma_L}{2} U_{i1} U_{j1}^* [N_L(\varepsilon_i) + N_L(\varepsilon_j) + 2] \\ &+ \frac{\gamma_R}{2} U_{i3} U_{j3}^* [N_R(\varepsilon_i) + N_R(\varepsilon_j) + 2]. \end{aligned} \quad (5)$$

Here, $\gamma_\sigma(\varepsilon_i) = 2\pi J_\sigma(\varepsilon_i)$ characterizes the coupling strength with each bath, and $J_\sigma(\omega) = \sum_{\mathbf{k}_\sigma} |g_{\mathbf{k}_\sigma}|^2 \delta(\omega - \omega_{\mathbf{k}_\sigma})$ is the coupling distribution. For the usual case, we can assume that $\gamma_\sigma(\omega) \simeq \gamma_\sigma$ does not depend too much on ω and can be treated as constant. $N_\sigma(\omega) = [\exp(\omega/kT_\sigma) - 1]^{-1}$ is the Planck distribution for $\sigma = L, R$.

It is observed from the above master equation that the environment indeed induces an effective coupling between two normal modes A_i and A_j . A_{ij}^\pm measure the transitions of the normal modes. This environment-mediating effect can be understood in the following way. The coupled system exchanges energy with environment through the interaction V_{SB} . Immediately after the normal mode A_i of the system emits an energy quanta ε_i to the environment, another process may happen in succession that the normal mode A_j absorbs back ε_j from the environment. Also, there is another possibility for the reversed process. Thus, different normal modes A_i 's of H_S are coupled with

the mediation of the environment. In fact, this effect of environment mediated coupling can be also found for two modes coupled to a common heat bath [13,14].

The transition terms with $i \neq j$ describes the effective coupling between different normal modes. In the interaction picture, these terms would contribute an oscillating factor $\exp[\pm i\delta\epsilon_{ij}t]$ resulted from the energy difference of the modes \hat{A}_i and \hat{A}_j . This transition effect would be ignored if we apply RWA by dropping these terms. However, as can be seen in the following, such ignorance would give rise to counter-intuitive results for non-equilibrium system.

3 Long-term stabilization dynamics

Comparing with the long time Markovian thermalization process in a heat bath with a single temperature, the present environment with two temperatures $T_{L/R}$ cannot stabilize the whole system into a canonical thermal state. In this section, we first calculate the steady state of the open quantum system by straightforwardly solving the above master equation equation (4). Then we consider the mediating HO as a quantum data bus in the large detuning limit. The adiabatic elimination of this oscillator can formally induce a direct coupling between the left and right HOs. In this case, the analytical results about the stabilization can be obtained explicitly.

3.1 Steady state in long time limit

We now consider the indirect coupling case with a mediating data bus. The master equation without RWA can be solved with the help of the characteristic function of Wigner representation, which is defined as (see Appendix B),

$$\begin{aligned}\chi(\boldsymbol{\mu}) &\equiv \text{Tr} [\rho \exp(\mathbf{A}^\dagger \boldsymbol{\mu} - \boldsymbol{\mu}^\dagger \mathbf{A})] \\ &= \text{Tr} [\rho \exp(\mathbf{a}^\dagger \boldsymbol{\kappa} - \boldsymbol{\kappa}^\dagger \mathbf{a})].\end{aligned}\quad (6)$$

Here, $\boldsymbol{\mu} = (\mu_1, \mu_2, \mu_3)^T$ and $\boldsymbol{\kappa} = (\kappa_1, \kappa_2, \kappa_3)^T$ are complex vectors with respect to the normal and local modes, and $\boldsymbol{\mu} = U\boldsymbol{\kappa}$. The corresponding Wigner function with three modes is defined as the Fourier transform of $\chi(\boldsymbol{\kappa})$,

$$W(\boldsymbol{\alpha}, \boldsymbol{\alpha}^*) = \frac{1}{(\pi^2)^3} \int d^2\boldsymbol{\kappa} e^{-\boldsymbol{\alpha}^\dagger \boldsymbol{\kappa} + \boldsymbol{\kappa}^\dagger \boldsymbol{\alpha}} \chi(\boldsymbol{\kappa}).$$

With this definition, we obtain the equation of $\chi(\boldsymbol{\mu})$ as:

$$\partial_t \chi + \mathbf{zT} \frac{\partial}{\partial \mathbf{z}^T} \chi = \mathbf{zDz}^T \chi, \quad (7)$$

where $\mathbf{z} = (\mu_1, \mu_2, \mu_3, \mu_1^*, \mu_2^*, \mu_3^*)$, and

$$\mathbf{T} = \begin{bmatrix} T^- & \mathbf{0} \\ \mathbf{0} & T^+ \end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix} \mathbf{0} & P \\ P^T & \mathbf{0} \end{bmatrix}.$$

\mathbf{T} and \mathbf{D} are 6×6 matrices with 3×3 blocks T^\pm and P defined by:

$$\begin{aligned}P_{ij} &= -\frac{1}{4} (A_{ij}^- + A_{ji}^+), \\ T_{ij}^- &= \frac{1}{2} (A_{ij}^- - A_{ji}^+) - i\varepsilon_i \delta_{ij}, \\ T_{ij}^+ &= \frac{1}{2} (A_{ji}^- - A_{ij}^+) + i\varepsilon_i \delta_{ij}.\end{aligned}\quad (8)$$

The equation (7) is the Fourier transformation of the Fokker-Planck equation about the Wigner function [15]. Formally we give the analytical solution for the steady state in Appendix C. Its expression is given as:

$$\chi(\boldsymbol{\mu}) = \chi(U\boldsymbol{\kappa}) = \exp[\mathbf{zV}^{-1}\mathbf{D}'(\mathbf{zV}^{-1})^T], \quad (9)$$

where V diagonalizes the matrix \mathbf{T} , i.e., $V\mathbf{T}V^{-1} = \text{diag}\{\mathbf{d}_1, \dots, \mathbf{d}_6\}$, and $\mathbf{D}'_{ij} = [V\mathbf{D}V^T]_{ij}/(\mathbf{d}_i + \mathbf{d}_j)$.

All the steady state properties of the composite system can be obtained from this formal solution equation (9). Specially, we are interested in the steady state of the HOs at the two ends. We can obtain $\chi_\sigma(\kappa_\sigma)$ for each local oscillator just by setting $\kappa_i = 0$ for all $i \neq \sigma$. Notice that V^{-1} and \mathbf{D}' in the exponent of equation (9) are block diagonal and anti-diagonal respectively, thus it can be verified that $\chi_\sigma(\kappa_\sigma)$ is always of the following Gaussian form,

$$\chi(\kappa_\sigma, \kappa_\sigma^*) = \exp\left[-\left(N_\sigma^{\text{eff}} + \frac{1}{2}\right)|\kappa_\sigma|^2\right], \quad (10)$$

where N_σ^{eff} is a positive constant. In Appendix B, we show that if $\chi(\kappa_\sigma, \kappa_\sigma^*)$ has the Gaussian form like equation (10), the state of the oscillator is a canonical state, and there is no squeezing. Since each HO can finally reach a canonical steady state, we can treat it as an equivalent thermal state and define an effective temperature from its average occupation $N_\sigma^{\text{eff}} = \langle \hat{a}_\sigma^\dagger \hat{a}_\sigma \rangle$,

$$T_\sigma^{\text{eff}} = \omega_\sigma / \ln\left(1 + \frac{1}{N_\sigma^{\text{eff}}}\right). \quad (11)$$

In Figures 2a and 2b we show the effective temperatures of the oscillators at the two ends calculated from equation (9), changing with the detuning $\Delta = \omega_L - \omega_R$ and the coupling strengths $g_{L/R}$. Here we set $\bar{\omega} = (\omega_L + \omega_R)/2 \equiv 1$ as the energy scale and $\omega_m = 2$.

When the detuning Δ becomes large or when their coupling strength $g_{L/R}$ becomes small, the two oscillators tend to be thermalized with their own heat bath respectively. The effective temperatures get to the closest point around the resonance regime $\Delta \simeq 0$. This observation means that they are affected by the reservoir at the opposite side and heat transfer happens greatly. When $g_L = g_R$, the extremum points locate exactly at $\omega_L = \omega_R$, while they shift aside when $g_L \neq g_R$. When the interaction becomes strong, the effective temperatures of the two oscillators tend to get closer and closer, away from that of each heat bath.

As comparison, we also show some counter-intuitive observation resulting from the improper omission of the

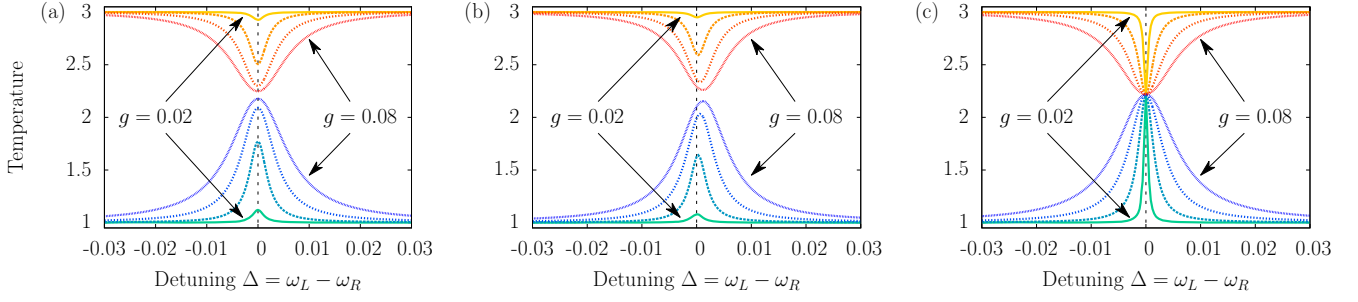


Fig. 2. The effective temperatures T_L^{eff} (lower blue ones) and T_R^{eff} (upper red ones) of the oscillators at the two ends calculated from the result without (a) and (b) and with (c) RWA. We set $\bar{\omega} = (\omega_L + \omega_R)/2 = 1$ as the unit, and $T_L = 1$, $T_R = 3$, $\gamma_L = 0.002$, $\gamma_R = 0.003$, $\omega_m = 2$. We set $g_L = g_R = g$ in (a) and (c), and $g_L = g$, $g_R = 0.8g$ in (b). We plot four groups of curves according to $g = 0.02, 0.04, 0.06, 0.08$, distributed from outside to inside in these figures. At the small regime around the degeneracy point, RWA cannot give us a good enough result, especially when g is weak. The extremum points are shifted aside when $g_L \neq g_R$.

transition terms like $2\hat{A}_i\rho\hat{A}_j^\dagger - \{\hat{A}_j^\dagger\hat{A}_i, \rho\}$ with $i \neq j$ (see Fig. 2c). In the large detuning area, this approximation shows well consistence with previous result in Figure 2a. But the effective temperatures always equals at the degeneracy point even when the coupling strength g is quite weak, i.e., when the oscillators tend to be decoupled from each other. A similar problem was also studied in reference [5], where they considered two interacting two-level systems respectively contacting two independent heat baths with different temperatures, and they obtained a result similar to ours shown in Figure 2c, which is valid only when the coupling strength is quite large.

With the above comparison, we look back at the master equation carefully, the transition terms like $e^{-i\delta\epsilon_{ij}t} \times (2\hat{A}_i\rho\hat{A}_j^\dagger - \{\hat{A}_j^\dagger\hat{A}_i, \rho\}_+)$ with $i \neq j$ contribute to the energy exchange of different modes \hat{A}_i and \hat{A}_j , which can be characterized by $\langle\hat{A}_i^\dagger\hat{A}_j\rangle$. The oscillating factor at the front describes the phase of this transition. The transition rate $\delta\epsilon_{ij} = \epsilon_i - \epsilon_j$ is determined by the detuning and coupling strength. When Δ and $g_{L,R}$ are quite small, the omission of these terms seems doubtful.

Intuitively, the only reason why these inter-mode transition terms cannot be dropped is that they rotate too slowly. However, remember that we only focus on the steady behavior $t \rightarrow \infty$. In this case, even a quite slowly rotating term should be averaged to zero. Indeed, in the following we would see that in equilibrium reservoirs, ignorance of such transitions does give the correct result even when the transition rate $\delta\epsilon_{ij}$ is small, and the real reason lies in the non-equilibrium environment.

3.2 Effective coupling in adiabatic limit

When the detuning of ω_m to $\omega_{L/R}$ is large, we can eliminate the mediating degree of freedom adiabatically to simplify our analysis. We apply Fröhlich-Nakajima transformation [16–18], and obtain the following simplified Hamiltonian, which describes a system of two directly coupled oscillators,

$$H_S = \omega'_L \hat{a}_L^\dagger \hat{a}_L + \omega'_R \hat{a}_R^\dagger \hat{a}_R + g \left(\hat{a}_L^\dagger \hat{a}_R + \hat{a}_L \hat{a}_R^\dagger \right), \quad (12)$$

where

$$\omega'_{L,R} = \omega_{L,R} + \frac{g_{L,R}^2}{\omega_{L,R} - \omega_m},$$

$$g = \frac{1}{2} \left(\frac{g_L g_R}{\omega_L - \omega_m} + \frac{g_L g_R}{\omega_R - \omega_m} \right). \quad (13)$$

The coupling strengths g_L and g_R contribute a correction to the renormalized frequencies $\omega'_{L,R}$.

For this simplified Hamiltonian, we can write down the analytical expression of eigen frequencies ϵ_i and the transformation U for the normal modes \hat{A}_\pm . Denoting $\omega'_L = \bar{\omega} - \Delta/2$, $\omega'_R = \bar{\omega} + \Delta/2$, we have

$$\epsilon_\pm = \bar{\omega} \pm \tilde{\Delta}_g, \quad \tilde{\Delta}_g = \left(\frac{\Delta^2}{4} + g^2 \right)^{\frac{1}{2}},$$

$$U = \begin{bmatrix} \alpha & \beta \\ \beta & -\alpha \end{bmatrix}, \quad \frac{2\alpha\beta}{\alpha^2 - \beta^2} = \frac{2g}{\Delta}. \quad (14)$$

It follows from equation (14) that the energy difference $\delta\epsilon_{ij} = 2\tilde{\Delta}_g$ depends on the detuning Δ and coupling strength g . When Δ and g are small, the factors $\exp[\pm 2i\tilde{\Delta}_g t]$ of the transition terms between the two normal modes oscillate quite slowly.

We carry out the similar calculation for this simplified two oscillators system as previously, which gives an explicit expression for the steady state of each oscillator, described by the characteristic function $\chi_\sigma(\kappa_\sigma)$,

$$\chi_\sigma(\kappa_\sigma) = \exp \left[- \left(N_\sigma^{\text{eff}} + \frac{1}{2} \right) |\kappa_\sigma|^2 \right],$$

$$N_\sigma^{\text{eff}} = \left[A_\sigma N_L(\epsilon_-) + B_\sigma N_L(\epsilon_+) \right. \\ \left. + C_\sigma N_R(\epsilon_-) + D_\sigma N_R(\epsilon_+) \right] / \Phi. \quad (15)$$

Here N_σ^{eff} is the occupation number of the effective thermal distribution ($\sigma = L, R$) determined by the linear

combination of $N_{L,R}(\varepsilon_{\pm})$, and the coefficients are,

$$\begin{aligned}\Phi &= \gamma_L \gamma_R (\gamma_L + \gamma_R)^2 \\ &+ 16\tilde{\Delta}_g^2 (\alpha^2 \gamma_L + \beta^2 \gamma_R) (\beta^2 \gamma_L + \alpha^2 \gamma_R), \\ A_L &= \alpha^2 \left[\gamma_L \gamma_R (\gamma_L + \gamma_R)^2 + 16\alpha^2 \tilde{\Delta}_g^2 (\beta^2 \gamma_L + \alpha^2 \gamma_R) \gamma_L \right], \\ B_L &= \beta^2 \left[\gamma_L \gamma_R (\gamma_L + \gamma_R)^2 + 16\beta^2 \tilde{\Delta}_g^2 (\alpha^2 \gamma_L + \beta^2 \gamma_R) \gamma_L \right], \\ C_L &= 16\alpha^2 \beta^2 \tilde{\Delta}_g^2 (\beta^2 \gamma_L + \alpha^2 \gamma_R) \gamma_R, \\ D_L &= 16\alpha^2 \beta^2 \tilde{\Delta}_g^2 (\alpha^2 \gamma_L + \beta^2 \gamma_R) \gamma_R,\end{aligned}$$

and

$$\begin{aligned}A_R &= 16\alpha^2 \beta^2 \tilde{\Delta}_g^2 (\beta^2 \gamma_L + \alpha^2 \gamma_R) \gamma_L, \\ B_R &= 16\alpha^2 \beta^2 \tilde{\Delta}_g^2 (\alpha^2 \gamma_L + \beta^2 \gamma_R) \gamma_L, \\ C_R &= \beta^2 \left[\gamma_L \gamma_R (\gamma_L + \gamma_R)^2 + 16\beta^2 \tilde{\Delta}_g^2 (\alpha^2 \gamma_L + \beta^2 \gamma_R) \gamma_R \right], \\ D_R &= \alpha^2 \left[\gamma_L \gamma_R (\gamma_L + \gamma_R)^2 + 16\alpha^2 \tilde{\Delta}_g^2 (\beta^2 \gamma_L + \alpha^2 \gamma_R) \gamma_R \right].\end{aligned}$$

From equation (15) we see that each oscillator achieves a canonical state. Especially, at the degeneracy point $\omega'_L = \omega'_R$, we have $\alpha^2 = \beta^2 = 1/2$, and the difference of the populations is,

$$\begin{aligned}N_L^{\text{eff}} - N_R^{\text{eff}} &= \frac{[N_L(\varepsilon_+) - N_R(\varepsilon_+)] + [N_L(\varepsilon_-) - N_R(\varepsilon_-)]}{2(1 + 4g^2/\gamma_L \gamma_R)}.\end{aligned}\quad (16)$$

The above equation (16) explicitly shows that the effective temperatures of the two oscillators are not equal at the degeneracy point when $T_L \neq T_R$.

In the equilibrium case, we have $T_L = T_R = \bar{T}$ and $N_L(\varepsilon) = N_R(\varepsilon) \equiv \bar{N}(\varepsilon)$. The above result equation (15), which is obtained without RWA, still holds. And we can explicitly obtain N_{σ}^{eff} as:

$$\begin{aligned}N_L^{\text{eff}} &= \alpha^2 \bar{N}(\varepsilon_-) + \beta^2 \bar{N}(\varepsilon_+), \\ N_R^{\text{eff}} &= \beta^2 \bar{N}(\varepsilon_-) + \alpha^2 \bar{N}(\varepsilon_+).\end{aligned}\quad (17)$$

However, a simple calculation by omitting the inter-mode transitions also gives exactly the same analytical result as equation (17), even when the transition rate $\tilde{\Delta}_g = [\Delta^2/4 + g^2]^{1/2}$ is small. Both calculations, with and without RWA, give the steady state of the two oscillators, i.e.,

$$\rho_s = \frac{1}{\mathcal{Z}} \exp \left[-\frac{1}{k\bar{T}} \left(\varepsilon_- \hat{A}_-^\dagger \hat{A}_- + \varepsilon_+ \hat{A}_+^\dagger \hat{A}_+ \right) \right], \quad (18)$$

no matter how slowly the transition coefficients rotate.

3.3 Inter-mode transition and flux

Here we give a physical explanation why the omission of the inter-mode transitions is consistent with equilibrium

system but not allowed for non-equilibrium system. We still consider the model of three oscillators. If we omit all the inter-mode transitions in equation (4), we obtain the following master equation,

$$\begin{aligned}\partial_t \rho &= i \left[\rho, \sum_i \varepsilon_i \hat{A}_i^\dagger \hat{A}_i \right] \\ &+ \sum_i \frac{A_{ii}^-}{2} \left(2\hat{A}_i \rho \hat{A}_i^\dagger - \left\{ \hat{A}_i^\dagger \hat{A}_i, \rho \right\}_+ \right) \\ &+ \frac{A_{ii}^+}{2} \left(2\hat{A}_i^\dagger \rho \hat{A}_i - \left\{ \hat{A}_i \hat{A}_i^\dagger, \rho \right\}_+ \right).\end{aligned}\quad (19)$$

In this equation with RWA, all the normal modes \hat{A}_i are decoupled from each other. It can be verified that the steady state of this equation is:

$$\begin{aligned}\rho_s &= \frac{1}{\mathcal{Z}} \exp \left[-\sum_i \tilde{\beta}_i \hat{A}_i^\dagger \hat{A}_i \right], \\ \tilde{\beta}_i &= \ln [A_{ii}^- / A_{ii}^+].\end{aligned}\quad (20)$$

Such steady solution has a property that for $i \neq j$, we have $\text{Tr}[\rho_s \hat{A}_i^\dagger \hat{A}_j] = 0$, which is also consistent with the fact that there is no inter-mode transition.

Recall that $\hat{A}_i = U_{in} \hat{a}_n$, generally we can write down the transition amplitudes for the local modes as:

$$\langle \hat{a}_m^\dagger \hat{a}_n \rangle = \sum_i U_{mi} U_{ni}^* \langle \hat{A}_i^\dagger \hat{A}_i \rangle + \sum_{i \neq j} U_{mi} U_{nj}^* \langle \hat{A}_i^\dagger \hat{A}_j \rangle.$$

Since we can always choose a proper phase to guarantee that all U_{mi} 's are real, if all the inter-mode transitions are omitted, i.e., $\langle \hat{A}_i^\dagger \hat{A}_j \rangle = 0$ for $i \neq j$, immediately we obtain

$$\langle \hat{a}_m^\dagger \hat{a}_n \rangle - \langle \hat{a}_n^\dagger \hat{a}_m \rangle = 0.\quad (21)$$

Indeed, $\langle \hat{a}_m^\dagger \hat{a}_n \rangle - \langle \hat{a}_n^\dagger \hat{a}_m \rangle$ is proportional to the energy or particle flux between the local sites. For example, we consider the particle exchange of the mediating mode \hat{b} shown in Figure 1. By Heisenberg equation, we have

$$\begin{aligned}\partial_t \langle \hat{b}^\dagger \hat{b} \rangle &= ig_L \left(\langle \hat{a}_L^\dagger \hat{b} \rangle - \langle \hat{a}_L \hat{b}^\dagger \rangle \right) \\ &+ ig_R \left(\langle \hat{a}_R^\dagger \hat{b} \rangle - \langle \hat{a}_R \hat{b}^\dagger \rangle \right).\end{aligned}\quad (22)$$

From this equation, we can define the particle flux from \hat{b} to the left/right site as $J_\sigma \equiv ig_\sigma (\langle \hat{a}_\sigma^\dagger \hat{b} \rangle - \langle \hat{a}_\sigma \hat{b}^\dagger \rangle)$, where $\sigma = L, R$. Thus, the omission of the inter-mode transitions would always give $J_\sigma = 0$, which means that there is no net flux between the local sites.

For equilibrium systems, there is no net flux between the subsystems, thus the omission of these inter-mode transitions is consistent, even when the transition rate is quite small. That is, as we mentioned before, when we focus on the steady behavior $t \rightarrow \infty$, even a quite slowly rotating term should be averaged to zero. However, the existence of a steady flux is an essential element of non-equilibrium systems. Therefore, we conclude that for the non-equilibrium case, the inter-mode transitions

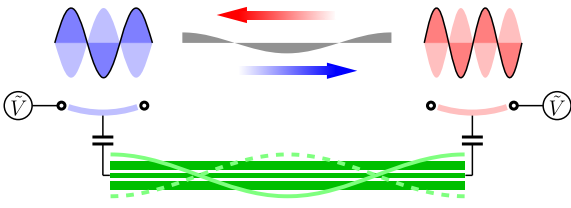


Fig. 3. Two remotely located NAMRs are indirectly coupled via a superconducting TLR. The bias voltage provides a difference between the NAMR and the TLR, so they can be coupled by a capacitance. At the same time, the voltage noise provides each NAMR with independent heat bath which has different effective temperature.

would contribute to long-term effect even in Markovian systems. This is different from the previous viewpoints that the inter-mode transitions only have transient effect within the time scale determined by the time-energy uncertainty, which usually applies in conventional thermalization process [1,9–12].

4 Physical implementation

In this section, we discuss a possible implementation scheme, which is composed of two nano-mechanical oscillators (NAMR) connected via a superconducting transmission line (TLR), in order to test the theoretical results we have got here, as shown in Figure 3.

The electromagnetic field inside the TLR may be treated as several boson modes whose frequencies are discretely distributed [19,20]. Only one of the TLR modes, which is nearly resonant to the NAMRs, can couple with the NAMRs effectively. For example, the voltage distribution of the lowest even mode along the TLR is,

$$V(x) = \sqrt{\frac{\omega_m}{cL}} \cos \frac{2\pi x}{L} (\hat{b} + \hat{b}^\dagger), \quad (23)$$

where $\omega_m = 2\pi/L\sqrt{lc}$ is the frequency of this mode, L is the length of the TLR, and l, c are the inductance and capacitance per unit length.

The voltage gets maximum at the two ends, where the NAMRs are coupled with the TLR via a displacement dependent capacitance [21–25]. The vibration mode of each NAMR may be also treated as a single boson. To the lowest order, C_x depends linearly on the movement of the NAMR, $C_x \simeq C_x^0(1 + \hat{x}/d_0)$. Applying a voltage bias V_g to the NAMRs, we have the interaction as

$$H_{\text{int}} = \frac{1}{2}C_x^0 \left(1 + \frac{\hat{x}}{d_0}\right) (V(x) - V_g)^2. \quad (24)$$

Quantizing the coordinate of the NAMR as $\hat{x} = \delta x_0(\hat{a} + \hat{a}^\dagger)$, we obtain an interaction term as $H_{\text{int}} = g(\hat{a} + \hat{a}^\dagger)(\hat{b} + \hat{b}^\dagger)$. For typical parameters, $C_x^0 \simeq 0.65$ fF, $V_g = 4$ V, $d_0 \simeq 50$ nm, $\delta x_0 \simeq 5$ fm, $lc \simeq 4$ fF, $\omega_m/2\pi \simeq 5$ GHz, the coupling strength is estimated as $g/2\pi \simeq 6$ MHz [24,26]. In

this regime, it is appropriate to apply J-C approximation to have $H_{\text{int}} = g(\hat{a}\hat{b}^\dagger + \hat{a}^\dagger\hat{b})$.

For the mediating TLR, $\omega_m/2\pi \simeq 5$ GHz, $Q > 10^4$, and the lifetime of the photon inside the resonator is $\tau > 1$ μ s [20]. The NAMR with $\omega/2\pi > 1$ GHz usually has a lower mechanics quality, $Q \simeq 500$ [27], and it depends on the fabrication techniques [28]. Thus the relaxation time of the NAMR is much shorter than the TLR, and we can neglect the dissipation of the TLR. The voltage noises applied to the NAMRs at the two sides, which arise from resistance and bring in the Joule heat, may provide the NAMRs with independent heat baths with different effective temperatures, and this can be controlled and measured in experiment [29–31].

5 Summary

In summary, we have studied the long-term behavior of a coupled HO system connecting with a complicated environment which consists of two independent heat baths with different temperatures. We derived a master equation with respect to the normal modes. With the help of the characteristic description of Wigner function, we obtained the numerical and analytical results for the steady state of each local oscillator.

These results show that the inter-mode transitions mediated by the environment are essential to non-equilibrium flux between the interacting subsystems, thus they would contribute to long-term effect even in Markovian systems. This is different from the case in conventional thermalization problems, where only one canonical heat bath is involved. The non-thermal stabilization process is determined by the competition between the rate of the inter-mode transition and that of the energy exchange with each private heat bath.

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Appendix A: Derivation of master equation

We show the derivation of the master equation equation (4) here. In the interaction picture of $H_S + H_B$, the interaction with the environment becomes,

$$\begin{aligned} V_I(t) &= V_I^L(t) + V_I^R(t), \\ V_I^\sigma(t) &= \hat{a}_\sigma^\dagger(t)\Gamma_\sigma(t) + \hat{a}_\sigma(t)\Gamma_\sigma^\dagger(t), \end{aligned} \quad (\text{A.1})$$

where

$$\begin{aligned} \hat{a}_\sigma(t) &= \sum_j U_{j\sigma}^* \hat{A}_j e^{-i\varepsilon_j t}, \\ \Gamma_\sigma(t) &= \sum_{\mathbf{k}_\sigma} g_{\mathbf{k}_\sigma} \hat{c}_{\mathbf{k}_\sigma} e^{-i\omega_{\mathbf{k}_\sigma} t}. \end{aligned} \quad (\text{A.2})$$

Here $\hat{A}_i = U_{ij}\hat{a}_j$ are the normal modes of the interacting oscillators system.

We take Born-Markovian approximation and put these interaction terms into the following equation [1],

$$\begin{aligned} \partial_t \rho &= - \int_0^\infty d\tau \mathbf{Tr}_B [V_I(t), [V_I(t-\tau), \rho(t) \otimes \rho_B]] \\ &= - \int_0^\infty d\tau \mathbf{Tr}_B [V_I^L(t), [V_I^L(t-\tau), \rho(t) \otimes \rho_B]] \\ &\quad - \int_0^\infty d\tau \mathbf{Tr}_B [V_I^R(t), [V_I^R(t-\tau), \rho(t) \otimes \rho_B]]. \end{aligned} \quad (\text{A.3})$$

Here we assume that the state of each bath is a canonical thermal one and does not change with time, i.e., $\rho_B = \rho_B^L \otimes \rho_B^R$, $\rho_B^\sigma \propto \exp[-H_B^\sigma/kT_\sigma]$ and $H_B^\sigma = \omega_{\mathbf{k}_\sigma} \hat{c}_{\mathbf{k}_\sigma}^\dagger \hat{c}_{\mathbf{k}_\sigma}$, where $T_{L/R}$ is the temperature of the left/right heat bath. Thus, terms like $\mathbf{Tr}_B [V_I^L(t)V_I^R(t-\tau)\rho(t) \otimes \rho_B]$ always vanish, because they only contain the first moment of each bath.

The rightside of equation (A.3) contains two integrals of the same form. Each integral gives four terms, one of which is calculated below as an example,

$$\begin{aligned} &\int_0^\infty d\tau \mathbf{Tr}_B [\hat{a}_\sigma^\dagger(t) \Gamma_\sigma(t) \rho(t) \otimes \rho_B \hat{a}_\sigma(t-\tau) \Gamma_\sigma^\dagger(t-\tau)] \\ &= \int_0^\infty d\tau \hat{a}_\sigma^\dagger(t) \rho(t) \hat{a}_\sigma(t-\tau) \langle \Gamma_\sigma^\dagger(t-\tau) \Gamma_\sigma(t) \rangle_B, \end{aligned} \quad (\text{A.4})$$

where

$$\begin{aligned} \hat{a}_\sigma^\dagger(t) \rho(t) \hat{a}_\sigma(t-\tau) &= \sum_{i,j} U_{i\sigma} U_{j\sigma}^* \hat{A}_i^\dagger \rho \hat{A}_j e^{i(\varepsilon_i - \varepsilon_j)t} e^{i\varepsilon_j \tau}, \\ \langle \Gamma_\sigma^\dagger(t-\tau) \Gamma_\sigma(t) \rangle_B &= \sum_{\mathbf{k}_\sigma} |g_{\mathbf{k}_\sigma}|^2 \langle \hat{c}_{\mathbf{k}_\sigma}^\dagger \hat{c}_{\mathbf{k}_\sigma} \rangle_{\text{th}} e^{-i\omega_{\mathbf{k}_\sigma} \tau} \\ &= \int_0^\infty d\omega J_\sigma(\omega) N_\sigma(\omega) e^{-i\omega \tau}. \end{aligned} \quad (\text{A.5})$$

Here, $J_\sigma(\omega) = \sum_{\mathbf{k}_\sigma} |g_{\mathbf{k}_\sigma}|^2 \delta(\omega - \omega_{\mathbf{k}_\sigma})$ is the coupling spectrum, and $N_\sigma(\varepsilon_i) = [\exp(\varepsilon_i/kT_\sigma) - 1]^{-1}$ is the Planck distribution with temperature T_σ . The integral equation (A.4) gives

$$\begin{aligned} &\sum_{i,j} U_{i\sigma} U_{j\sigma}^* \hat{A}_i^\dagger \rho \hat{A}_j e^{i(\varepsilon_i - \varepsilon_j)t} \\ &\quad \times \frac{\gamma_\sigma(\varepsilon_j)}{2} N(\varepsilon_j) + i\mathbf{P} \int_0^\infty d\omega \frac{J_\sigma(\omega) N_\sigma(\omega)}{\varepsilon_j - \omega}. \end{aligned} \quad (\text{A.6})$$

Here, we denote $\gamma_\sigma(\varepsilon_i) = 2\pi J_\sigma(\varepsilon_i)$, which characterizes the coupling strength with each heat bath. The principle integral contributes to Lamb shift, and we omit this term in this paper.

The physical meaning of equation (A.4) may be understood in the following way. At time $t - \tau$, the coupled HO system emits energy to the environment, and then absorbs back at time t . However, the energy exchange with the environment during this process is done by the total normal

modes \hat{A}_i but not the local modes \hat{a}_i . Thus, when the emission and absorption modes are not the same one, there is an oscillating factor $\exp[i(\varepsilon_i - \varepsilon_j)t]$ left. $\delta\varepsilon_{ij} \equiv \varepsilon_i - \varepsilon_j$ characterizes the splitting amplitude resulting from the coupling. By the mediation of the environment, the different normal modes \hat{A}_i of the system are coupled together.

Other terms of equation (A.3) can be also obtained as above. Each of the two integrals gives the following Lindblad-like form with an extra oscillating factor,

$$\begin{aligned} &\sum_{ij} \frac{1}{2} U_{i\sigma} U_{j\sigma}^* [\gamma_\sigma(\varepsilon_i) [N_\sigma(\varepsilon_i) + 1] + \gamma_\sigma(\varepsilon_j) [N_\sigma(\varepsilon_j) + 1]] \\ &\quad \times \left(\hat{A}_j \rho \hat{A}_i^\dagger - \frac{1}{2} \left\{ \hat{A}_i^\dagger \hat{A}_j, \rho \right\}_+ \right) e^{i\delta\varepsilon_{ij}t} \\ &+ \sum_{ij} \frac{1}{2} U_{i\sigma} U_{j\sigma}^* [\gamma_\sigma(\varepsilon_j) N_\sigma(\varepsilon_i) + \gamma_\sigma(\varepsilon_j) N_\sigma(\varepsilon_j)] \\ &\quad \times \left(\hat{A}_i^\dagger \rho \hat{A}_j - \frac{1}{2} \left\{ \hat{A}_j \hat{A}_i^\dagger, \rho \right\}_+ \right) e^{i\delta\varepsilon_{ij}t}. \end{aligned} \quad (\text{A.7})$$

For simplicity, we assume $\gamma_\sigma(\varepsilon_i) \simeq \gamma_\sigma$ does not depend too much on ω and can be treated as constant. In sum of equations (A.3) and (A.7), we get the following master equation in Schrödinger's picture, and the oscillating factors do not appear,

$$\begin{aligned} \partial_t \rho &= i \left[\rho, \sum \varepsilon_i \hat{A}_i^\dagger \hat{A}_i \right] \\ &\quad + \sum_{ij} \frac{\Lambda_{ij}^-}{2} \left(2\hat{A}_i \rho \hat{A}_j^\dagger - \left\{ \hat{A}_j^\dagger \hat{A}_i, \rho \right\}_+ \right) \\ &\quad + \frac{\Lambda_{ij}^+}{2} \left(2\hat{A}_i^\dagger \rho \hat{A}_j - \left\{ \hat{A}_j \hat{A}_i^\dagger, \rho \right\}_+ \right), \end{aligned} \quad (\text{A.8})$$

where

$$\begin{aligned} \Lambda_{ij}^+ &= \frac{\gamma_L}{2} U_{i1} U_{j1}^* [N_L(\varepsilon_i) + N_L(\varepsilon_j)] \\ &\quad + \frac{\gamma_R}{2} U_{i3} U_{j3}^* [N_R(\varepsilon_i) + N_R(\varepsilon_j)], \\ \Lambda_{ji}^- &= \frac{\gamma_L}{2} U_{i1} U_{j1}^* [N_L(\varepsilon_i) + N_L(\varepsilon_j) + 2] \\ &\quad + \frac{\gamma_R}{2} U_{i3} U_{j3}^* [N_R(\varepsilon_i) + N_R(\varepsilon_j) + 2]. \end{aligned} \quad (\text{A.9})$$

The terms with $i \neq j$ describes the transition between different normal modes. These terms are often omitted by RWA.

Appendix B: Characteristic function of Wigner representation

The Wigner representation often give us great convenience to study properties of quantum oscillators. It can be defined from a characteristic function [32],

$$\chi_w(\kappa, \kappa^*) = \mathbf{Tr} \left[e^{\kappa \hat{a}^\dagger - \kappa^* \hat{a}} \rho \right]. \quad (\text{B.1})$$

The Wigner function is defined as the Fourier transform of $\chi_w(\kappa, \kappa^*)$,

$$W(\alpha, \alpha^*) = \frac{1}{\pi^2} \int d^2\kappa e^{-\kappa\alpha^* + \kappa^*\alpha} \chi_w(\kappa, \kappa^*). \quad (\text{B.2})$$

For a system that consists of two oscillators, the characteristic function can be defined as:

$$\chi_{12}(\kappa_1, \kappa_2) = \text{Tr}_{12} \left[e^{\kappa_1 \hat{a}_1^\dagger - \kappa_1^* \hat{a}_1} e^{\kappa_2 \hat{a}_2^\dagger - \kappa_2^* \hat{a}_2} \rho_{12} \right].$$

From this definition, immediately we can find that if we had known $\chi_{12}(\kappa_1, \kappa_2)$ for the whole system explicitly, it would be quite easy to get the description of the subsystems $\chi_{1(2)}$, just by setting $\kappa_{2(1)} = 0$ in $\chi_{12}(\kappa_1, \kappa_2)$, without having to calculate the reduced density matrix of subsystems $\rho_{1(2)}$. This provides a simple method for us to study the state of subsystems.

Besides, for the thermal state of an oscillator $\rho_T = Z^{-1} \exp[-\frac{\omega}{kT} \hat{a}^\dagger \hat{a}]$, the characteristic function is,

$$\chi_T(\kappa, \kappa^*) = \exp \left[- \left(N + \frac{1}{2} \right) |\kappa|^2 \right]. \quad (\text{B.3})$$

Here $N = [\exp(\omega/kT) - 1]^{-1}$ is the Planck distribution.

As seen from the definition, $\chi_w(\kappa, \kappa^*)$ and $W(\alpha, \alpha^*)$ can be mapped into each other through Fourier transformation. It is also well-known that there is one-to-one correspondence between a physical Wigner function and a density matrix. Therefore, there is one and only one density matrix ρ decided by a legal $\chi_w(\kappa, \kappa^*)$.

Thus, if we have a characteristic function $\chi(\kappa, \kappa^*)$ which has a Gaussian form like equation (B.3), with $N \geq 0$, we can always come into the fact that the corresponding density matrix is:

$$\rho = \frac{1}{Z} \sum_{n=0}^{\infty} e^{-n\beta^{\text{eff}}\omega} |n\rangle\langle n|, \quad (\text{B.4})$$

where β^{eff} comes from $N = [\exp(\beta^{\text{eff}}\omega) - 1]^{-1}$. This is a canonical state for the oscillator with an effective temperature $1/\beta^{\text{eff}}$. A more rigorous proof lies below.

Proof: From the definition of $\chi_w(\kappa, \kappa^*)$, we have

$$\begin{aligned} \chi_w(\kappa, \kappa^*) &= e^{\frac{1}{2}|\kappa|^2} \text{Tr} \left[e^{\kappa \hat{a}^\dagger} \rho e^{-\kappa^* \hat{a}} \right] \\ &= e^{\frac{1}{2}|\kappa|^2} \int \frac{d^2\alpha}{\pi} e^{\kappa\alpha^* - \kappa^*\alpha} \langle \alpha | \rho | \alpha \rangle. \end{aligned} \quad (\text{B.5})$$

If we have a Gaussian formed characteristic function like equation (B.3), we can correspondingly get $\langle \alpha | \rho_T | \alpha \rangle$ by reversed transformation of the equation above,

$$\begin{aligned} \langle \alpha | \rho_T | \alpha \rangle &= \int \frac{d^2\kappa}{\pi} e^{-\kappa\alpha^* + \kappa^*\alpha} \exp \left[-(N+1) |\kappa|^2 \right] \\ &= \frac{1}{N+1} \exp \left[-\frac{|\alpha|^2}{N+1} \right]. \end{aligned} \quad (\text{B.6})$$

On the other hand, we can also expand $\langle \alpha | \rho_T | \alpha \rangle$ as:

$$\begin{aligned} \langle \alpha | \rho_T | \alpha \rangle &= \sum_{m,n} \langle \alpha | m \rangle \langle m | \rho_T | n \rangle \langle n | \alpha \rangle \\ &= \sum_{m,n} \langle m | \rho_T | n \rangle \frac{(\alpha^*)^m \alpha^n}{\sqrt{m!n!}} e^{-|\alpha|^2}. \end{aligned} \quad (\text{B.7})$$

Comparing with the expansion of equation (B.6), we can get the matrix elements of ρ_T ,

$$\langle m | \rho_T | n \rangle = \frac{(1 + \frac{1}{N})^{-n}}{N+1} \delta_{mn}. \quad (\text{B.8})$$

Denote $\exp[\beta^{\text{eff}}\omega] = 1 + \frac{1}{N}$, we can see that ρ_T is a canonical state.

Appendix C: Steady solution of Fokker-Planck equation

The standard form of Fokker-Planck equation and its characteristic equation are as follows,

$$\begin{aligned} \frac{\partial P}{\partial t} + \sum_i \lambda_i \frac{\partial}{\partial y_i} (y_i P) &= \frac{1}{2} \sum_{ij} \sigma_{ij} \frac{\partial^2 P}{\partial y_i \partial y_j}, \\ \frac{\partial f}{\partial t} - \sum_i \lambda_i \xi_i \frac{\partial f}{\partial \xi_i} &= -\frac{1}{2} f \sum_{ij} \sigma_{ij} \xi_i \xi_j, \quad \text{Re} \lambda_i < 0. \end{aligned} \quad (\text{C.1})$$

$f(\boldsymbol{\xi}, t)$ is the Fourier transformation of $P(\mathbf{y}, t)$,

$$f(\boldsymbol{\xi}, t) = \int d^n y P(\mathbf{y}, t) e^{-i\boldsymbol{\xi} \cdot \mathbf{y}}.$$

The equation of $f(\boldsymbol{\xi}, t)$ is a first-order quasi-linear partial differential one. It can be solved analytically [33], and the solution is,

$$f(\boldsymbol{\xi}, t) = \Phi(\xi_i e^{\lambda_i t}, \dots) \exp \left[-\frac{1}{2} \sum_{ij} \sigma_{ij} \frac{\xi_i \xi_j}{\lambda_i + \lambda_j} \right]. \quad (\text{C.2})$$

$\Phi(\dots)$ is determined according to the initial condition, and $\Phi(t \rightarrow \infty) = 1$.

In our problem, the equation of the characteristic function is

$$\partial_t \chi + \mathbf{z} \mathbf{T} \frac{\partial}{\partial \mathbf{z}^T} \chi = \mathbf{z} \mathbf{D} \mathbf{z}^T \chi. \quad (\text{C.3})$$

The only difference with the standard form is that \mathbf{T} is not diagonal here. We first diagonalize it and make it a standard form. Denoting $\mathbf{V} \mathbf{T} \mathbf{V}^{-1} = \text{diag}\{\mathbf{d}_1, \dots, \mathbf{d}_n\}$ and $\mathbf{z} = \boldsymbol{\xi} \mathbf{V}$, we can transform our equation into the standard Fokker-Planck form,

$$\partial_t \chi + \boldsymbol{\xi} \mathbf{d} \frac{\partial}{\partial \boldsymbol{\xi}^T} \chi = \boldsymbol{\xi} \mathbf{V} \mathbf{D} \mathbf{V}^T \boldsymbol{\xi}^T \chi. \quad (\text{C.4})$$

Now we could write down the steady solution as

$$\chi(\mathbf{z}) = \exp \left[\mathbf{z} \mathbf{V}^{-1} \mathbf{D}' (\mathbf{z} \mathbf{V}^{-1})^T \right], \quad (\text{C.5})$$

where

$$\mathbf{D}'_{ij} = \frac{[\mathbf{V} \mathbf{D} \mathbf{V}^T]_{ij}}{\mathbf{d}_i + \mathbf{d}_j}.$$

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